Design of Fractional Order Digital Differentiator Using Inverse Multiquadric Radial Basis Function

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Abstract. In this paper, a fractional order digital differentiator is designed by using Inverse multiquadric radial basis function (RBF). First, the RBF interpolation approach is described. Then, the non-integer delay sample estimation is derived by using RBF approach. Next, the Grünwald-Letnikov derivative and non-integer delay sample delay are applied to obtain the transfer function of the proposed method i.e. fractional order digital differentiator. The design accuracy of the proposed method is better then the conventional methods like examples Time domain least squares method, Fractional sample delay method and Frequency response approximation method.

Keywords: Digital differentiator, fractional derivative, non-integer delay, radial basis function.

1 Introduction

During the past three decades, fractional calculus has received a great deal of attention in many engineering applications and science including fluid flow, automatic control, electrical networks, electromagnetic theory and image processing [1]-[4]. Fractional dimension is used to measure some real-world data such as coastline, clouds dust in the air and network of neurons in the body [5]-[6]. We aim out interests at the digital realization of fractional derivative, which named as digital fractional order differentiator (FOD). Because digital FOD can determine and estimate the more characteristic of a given digital signal than integral order differentiator (IOD), it has been being an especial and useful tool in many increasing application, such as fractional order controls, radar and sonar processing, nonlinear or chaos time series processing and forecasting, geological signal detecting and processing, image processing etc. Fractional sample delay has become an important device in the applications of time adjustment in the digital receiver, antenna array processing, speech coding and synthesis, modelling of musical instrument, and comb-filter design etc [7]-[10].

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The integer order n of derivative $D^n f(x) = (d^n f(x)/dx^n)$ of function is generalized to fractional order $D^v f(x)$, where v is a real number. One of the important research topics in fractional calculus is to implement the fractional operator D^v in continuous and discrete time domain. An excellent survey of this implementation has been presented in [11]. Some techniques have been used already for the rational function approximation of continuous-time case i.e. curve fitting, evaluation and interpolation. These methods include Carlson's method, Roy's method, Chareff's method and Oustaloup's method [12]-[15]. For discrete-time case, there have been several methods presented to design finite-impulse-response (FIR) and infinite-impulse-response (IIR) filters for implementing operator D^v, including fractional differencing formula or Euler method, Tustin method, continued fraction method, least square method and Prony's method [16]-[22].

On the other hand, the radial basis function (RBF) has been widely used in multivariate interpolation, neural network, time series prediction, control of nonlinear systems, mesh-free approximation, and target tracking in voice data [23]-[26]. The early work has been done on the designing of fractional order differentiator using Gaussian radial basis function (RBF) [28]. But in this paper we are using Inverse multiquadric basis function (RBF). The theory and implementation of radial basis function(RBF) is surveyed in the book [27]. A radial basis function is defined as a real valued function $\phi(t)$, whose value depends only on the distance from the origin. The notation $|\cdot|$ denotes the absolute value. Generally the radial basis function are used i.e. Gaussian, Inverse multiquadric, Raised-Cosine.

$$Gaussian : \phi(t) = \exp -t^2/\sigma^2 \tag{1}$$

$$InverseMultiquadric : \phi(t) = \sigma/\sqrt{t^2+\sigma^2} \tag{2}$$

$$Raised - Cosine : \phi(t) = 1/2 * \sigma(1 + \cos(\pi * t/\sigma)) \tag{3}$$

where σ is known as the shape parameter, which is used to change the shape of function $\phi(t)$. The purpose of this paper is to use RBF interpolation approach to design the fractional order digital differentiator. The design error can be reduced using RBF interpolation approach by varying the shape parameter of radial basis function.

This paper is organized as follows: In section II, the radial basis interpolation method is described. By using the radial basis function interpolation the values of non-integer delay sample estimation of discrete-time sequence is obtained. In section III, apply the definition of fractional derivative i.e. Grünwald-Letnikov and non-integer delay sample estimation for obtaining the transfer function of fractional order differentiator. And, some numerical examples have given in this paper which show the effectiveness of this design approach. Finally, a conclusion is made.

2 Radial Basis Function

In this section the RBF interpolation method is first described [28]. Then, this interpolation method is applied to solve the non-integer delay sample estimation problem which is used in the design of fractional order differentiator in next section.

Radial Basis Function Interpolation The details of Radial basis function interpolation method can be found in [26]. Now this method is described briefly below: Given a set of N+1 different points $t_0, t_1, t_2, t_3, \cdots, t_N$ and a corresponding set of $N+1$ real numbers $s_0, s_1, s_2, \cdots, s_N$, the interpolation problem is to find a function $s(t)$ that satisfies the interpolation condition

$$s(t_k) = s_k \qquad k = 0, 1, 2, \cdots, N \tag{4}$$

The RBF interpolation method consists of choosing a function $s(t)$ that has the following term

$$s(t) = \sum_{k=0}^{N} w_k \phi(|(t - t_k)|) \tag{5}$$

The above equation can be written in the matrix form as

$$\begin{bmatrix} \phi(|(t_0-t_0)|) & \phi(|(t_0-t_1)|) & \phi(|(t_0-t_2)|) & \cdots & \phi(|(t_0-t_N)|) \\ \phi(|(t_1-t_0)|) & \phi(|(t_1-t_1)|) & \phi(|(t_1-t_2)|) & \cdots & \phi(|(t_1-t_N)|) \\ \phi(|(t_2-t_0)|) & \phi(|(t_2-t_1)|) & \phi(|(t_2-t_2)|) & \cdots & \phi(|(t_2-t_N)|) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi(|(t_N-t_0)|) & \phi(|(t_N-t_1)|) & \phi(|(t_N-t_2)|) & \cdots & \phi(|(t_N-t_N)|) \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix} = \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_N \end{bmatrix}$$

Function $s(t)$ represent a sum of $N+1$ radial basis function, each associated with a different center t_k and weighted coefficient is w_k. substituting interpolation condition of (4) into (5), we get the following simultaneous linear equation

$$\phi(|(t_m - t_k)|) = \phi_{mk}$$

$$\begin{bmatrix} \phi_{00} & \phi_{01} & \phi_{02} & \cdots & \phi_{0N} \\ \phi_{10} & \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{20} & \phi_{21} & \phi_{22} & \cdots & \phi_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{N0} & \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix} = \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_N \end{bmatrix} \tag{6}$$

Let vectors S and W be

$$S = \begin{bmatrix} s_0 \; s_1 \; s_2 \cdots s_N \end{bmatrix}^T \tag{7}$$

$$W = \begin{bmatrix} w_0 \; w_1 \; w_2 \cdots w_N \end{bmatrix}^T \tag{8}$$

Where φ denotes an $(N+1) \times (N+1)$ matrix with the element ϕ_{mk}, then (6) can be written as

$$\varphi W = S \tag{9}$$

if $t_0, t_1, t_2, t_3, \cdots, t_N$ are distinct points, then the matrix φ is non-singular matrix. Thus the unknown vector W is given by

$$W = \varphi^{-1} S \tag{10}$$

$s(t)$ is computable for the given t and it can be obtained only if the value of W is known.

2.1 Non-integer Delay Sample Estimation

In the following, we will use the RBF interpolation method to solve the non-integer delay sample estimation problem because the proposed fractional order differentiator design method is based on this estimation method. The problem to be studied is how to estimate the non-integer delay sample $s(n-d)$ from the given integer delay samples $s(n), s(n-1), s(n-2), \cdots, s(n-N)$, where N is an integer and d is a real number in the interval $d \in [0, N]$. In this paper we use weighted average approach that is to find the non-integer delay samples is estimated by

$$s(n-d) = \sum_{m=0}^{N} g(m,d) s(n-m) \tag{11}$$

Now, the remaining problem is how to use the RBF interpolation method to determine the weights $g(m,d)$. To solve this problem, let us choose $t_k = n-k$ and $s_k = s(n-k)$, then the RBF interpolation in (5) becomes

$$s(t) = \sum_{k=0}^{N} w_k \phi(|(t-t_k)|)$$

$$s(t) = \sum_{k=0}^{N} w_k \phi(|(t-(n-k))|) \tag{12}$$

Because $t_k = n-k$ and $t_m = n-m$, are chosen we have

$$\phi(|(t_m - t_k)|) = \phi(|k-m|) = \phi_{mk} \tag{13}$$

Using the above expression and $s_k = s(n-k)$, the new simultaneous linear equation in (6) reduces to

$$\begin{bmatrix} \phi(0) & \phi(1) & \phi(2) & \cdots & \phi(N) \\ \phi(1) & \phi(0) & \phi(1) & \cdots & \phi(N-1) \\ \phi(2) & \phi(1) & \phi(0) & \cdots & \phi(N-2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi(N) & \phi(N-1) & \phi(N-2) & \cdots & \phi(0) \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix} = \begin{bmatrix} s(n) \\ s(n-1) \\ s(n-2) \\ \vdots \\ s(n-N) \end{bmatrix} \tag{14}$$

This equation can be shortened as the form of $\varphi W = S$ as described in (9). Clearly, φ is an Symmetric matrix and Toeplitz matrix. Let the inverse of matrix φ be denoted by

$$\varphi^{-1} = \begin{bmatrix} \alpha_{00} & \alpha_{01} & \alpha_{02} & \cdots & \alpha_{0N} \\ \alpha_{10} & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1N} \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{N0} & \alpha_{N1} & \alpha_{N2} & \cdots & \alpha_{NN} \end{bmatrix} \tag{15}$$

We know that, $W = \varphi^{-1} S$

$$\begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix} = \varphi^{-1} \begin{bmatrix} s(n) \\ s(n-1) \\ s(n-2) \\ \vdots \\ s(n-N) \end{bmatrix} = \begin{bmatrix} \sum_{m=0}^{N} \alpha_{0m} s(n-m) \\ \sum_{m=0}^{N} \alpha_{1m} s(n-m) \\ \sum_{m=0}^{N} \alpha_{2m} s(n-m) \\ \vdots \\ \sum_{m=0}^{N} \alpha_{Nm} s(n-m) \end{bmatrix} \tag{16}$$

Above expression implies that

$$w_k = \sum_{m=0}^{N} \alpha_{km} s(n-m) \qquad k = 0, 1, 2, \cdots, N \tag{17}$$

substituting (17) into (12)

$$s(t) = \sum_{k=0}^{N} w_k \phi(|(t-(n-k))|)$$

$$s(t) = \sum_{k=0}^{N} \left(\sum_{m=0}^{N} \alpha_{km} s(n-m)) \phi(|(t-(n-k))| \right)$$

$$s(t) = \sum_{m=0}^{N} \left(\sum_{k=0}^{N} \alpha_{km} \phi(|(t-(n-k))|) \right) s(n-m) \tag{18}$$

Taking $t = n - d$ for the expression in the discrete form

$$s(n-d) = \sum_{m=0}^{N} \left(\sum_{k=0}^{N} \alpha_{km} \phi(|(k-d)|) \right) s(n-m) \tag{19}$$

After comparing the eq.(19) with (11),we get the weights g(m,d)

$$g(m,d) = \sum_{k=0}^{N} \alpha_{km} \phi(|(k-d)|) \tag{20}$$

Finally, given the radial basis function $\phi(t)$ and estimate the non-integer delay sample $s(n-d)$ from the given integer delay samples $s(n), s(n-1), s(n-2), \cdots, s(n-N)$ is summarized below:

Step 1) Compute the matrix φ whose matrix elements are given by $\phi_{mk} = \phi(|k-m|)$.

Step 2) Calculate the inverse matrix φ^{-1} with element α_{km}.

Step 3) Use (20) to compute the weights g(m,d).

Step 4) The non-integer delayed sample is estimated by

$$s(n-d) = \sum_{m=0}^{N} g(m,d)s(n-m)$$

In the next section with the help RBF based non-integer delay estimation method to design the fractional order differentiator.

3 Design of Fractional Order Differentiator

In this section fractional derivative will be explained and then apply RBF based non-integer delay sample estimation method to obtain the transfer function of the fractional order differentiator.

3.1 Fractional Derivative

There are several definiton for fractional integral and fractional derivative to obtain the transfer function of the fractional order differentiator such as the Riemann-Liouville, the Grünwald-Letnikov and Caputo definitions [1]-[4]. But in this paper we will use the Grünwald-Letnikov definition which is given by

$$D^v s(t) = \lim_{h \to 0} \sum_{k=0}^{\infty} \frac{(-1)^k C_k^v}{h^v} s(t-kh) \tag{21}$$

Where coefficient C_k^v is given by

$$\begin{aligned} C_k^v &= \frac{\Gamma(v+1)}{\Gamma(k+1)\Gamma(n-k+1)} \\ &= \begin{cases} 1 & k=0 \\ \frac{v(v-1)(v-2)\cdots(v-k+1)}{1.2.3\cdots k} & k \geq 1 \end{cases} \end{aligned} \tag{22}$$

The above notation $\Gamma(.)$ is gamma function. Based on this definition, the fractional derivative of exponential and sinusoidal signals are given by

$$D^v e^{\alpha t} = \alpha^v e^{\alpha t} \tag{23}$$

$$D^v A \sin(wt+\phi) = Aw^v \cos(wt+\phi) = Aw^v \sin(wt+\phi+\frac{\pi}{2}v) \tag{24}$$

The fourier transform of $D^v s(t)$ is $(jw)^v S(w)$. This means that when a signal $s(t)$ passes through a differentiator with frequency response $(jw)^v$, then the output of the differentiator is the fractional derivative $D^v s(t)$. Thus the ideal frequency response of fractional order differentiator is $(jw)^v$. So we will use the Grünwald-Letnikov derivative method in (21) and RBF-based non-integer delay sample estimation method to design fractional order differentiator.

3.2 Design of Fractional Order Differentiator

Now we will use the RBF interpolation method and Grünwald-Letnikov derivative to design a fractional order digital differentiator that approximates the following frequency domain specification as well as possible:

$$H_d(w) = (jw)^v e^{-jwI} \tag{25}$$

Where I is a prescribed delay value. First, let us define coefficient $a(k)$ below

$$a(k) = (-1)^k C_k^v \tag{26}$$

Eq.(21) can be written as

$$D^v s(t) = \lim_{h \to 0} \sum_{k=0}^{\infty} \frac{(-1)^k C_k^v}{h^v} s(t - kh)$$

$$D^v s(t) = \lim_{h \to 0} \sum_{k=0}^{\infty} \frac{a(k)}{h^v} s(t - kh) \tag{27}$$

Fig.1 show the coefficient response of $a(k)$ for various order. Figure shows that $a(k)$ is rapidly decaying sequence for various order v. Thus after truncation the eq.(27) can be approximated by

$$D^v s(t) \approx \lim_{h \to 0} \sum_{k=0}^{L} \frac{a(k)}{h^v} s(t - kh) \tag{28}$$

Where L is the truncation length. Moreover by removing the limit, the $D^v s(t)$ can be approximated by

$$D^v s(t) \approx \sum_{k=0}^{L} \frac{a(k)}{h^v} s(t - kh) \tag{29}$$

Where h is the smaller and the better approximation in (29). By taking $t = n - I$, the discrete-time derivative signal $D^v s(n - I)$ can be obtained as

$$D^v s(n - I) \approx \sum_{k=0}^{L} \frac{a(k)}{h^v} s(n - I - kh) \tag{30}$$

because $s(n-I-kh)$ are non-integer delay samples of signal $s(n)$, the $s(n-I-kh)$ needs to be estimated by using the formula (11):

$$s(n-I-kh) = \sum_{m=0}^{N} g(m, I+kh)s(n-m) \tag{31}$$

Substitute the value of $s(n-I-kh)$ in eq.(30)

$$D^v s(n-I) \approx \sum_{k=0}^{L} \frac{a(k)}{h^v} \sum_{m=0}^{N} g(m, I+kh)s(n-m)$$

$$D^v s(n-I) = \sum_{m=0}^{N} \left[\frac{1}{h^v} \sum_{k=0}^{L} a(k)g(m, I+kh) \right] s(n-m) \tag{32}$$

Defining the coefficients

$$b(m) = \frac{1}{h^v} \sum_{k=0}^{L} a(k)g(m, I+kh) \tag{33}$$

then eq.(32) can be written as

$$D^v s(n-I) \approx \sum_{m=0}^{N} b(m)s(n-m)$$

$$D^v s(n-I) = b(n) * s(n) \tag{34}$$

Where $*$ denotes the convolution sum operator. Taking the z-transform at both sides of eq.(34), we get

$$Y(z) = \left(\sum_{m=0}^{N} b(m)z^{-m} \right) S(z) \tag{35}$$

Y(z) is the z-transform of $D^v s(n-I)$ using the property of z-transform and $S(z)$ is the z-transform of $s(n)$.

The definition of FIR filter can be defined as

$$B(z) = \sum_{m=0}^{N} b(m)z^{-m} \tag{36}$$

Ideally the frequency response of FIR filter is $(jw)^v e^{-jwI}$ and eq.(36) show the transfer function of fractional order digital differentiator. Now, given the radial basis function $\phi(t)$ with shape parameter σ, integer N, fractional order v, delay I, integer L and small positive number h, the procedure to design fractional order digital differentiator $B(z)$ is summarized below:

Step 1) Compute the matrix φ whose elements are given by $\phi_{mk} = \phi(|k-m|)$.
Step 2) Calculate the inverse matrix φ^{-1} with element α_{nm}.
Step 3) Use (20) to compute the weights

$$g(m, I + kh) = \sum_{n=0}^{N} \alpha_{nm}\phi(|n - I - kh|).$$

Step 4) Compute the coefficient of $a(k)$ by using (26).
Step 5) Use (33) to calculate the coefficients of $b(m)$.
Step 6) The transfer function of the designed fractional order differentiator is given by $B(z) = \sum_{m=0}^{N} b(m)z^{-m}$.

Finally, some remarks are made follows: First, a large integer L needs to be chosen for reducing errors which occur in (28). Second, a smaller positive number h needs to be chosen for reducing the approximation error which occur in (29). Third, if N is large, the designed fractional order digital differentiator is a long FIR filter. To reduce the calculation complexity and implementation complexity the Prony Method in [30] can be used to approximate the long-length FIR filter B(z) by an IIR filter below

$$\bar{B}(z) = \frac{\sum_{n=0}^{N_1} b_1(n)z^{-n}}{1 + \sum_{n=1}^{N_1} b_2(n)z^{-n}} \tag{37}$$

The coefficient of $b_1(n)$ and $b_2(n)$ can be obtained by putting $B(Z)$ equal to the $\bar{B}(z)$, then we get an expression.

$$\left(\sum_{m=0}^{N} b(m)z^{-m}\right)\left(1 + \sum_{n=1}^{N_1} b_2(n)z^{-n}\right) = \sum_{n=0}^{N_1} b_1(n)z^{-n}$$

$$\left(\sum_{m=0}^{N} b(m)z^{-m}\right) + \left(\sum_{m=0}^{N}\sum_{n=1}^{N_1} b(m)b_2(n)z^{-n}z^{-m}\right) = \sum_{n=0}^{N_1} b_1(n)z^{-n} \tag{38}$$

Using the convolution operator, this equation reduces to

$$b(n) + \sum_{k=1}^{N_1} b_2(k)b(n-k) = \begin{cases} b_1(n) & 0 \leq n \leq N_1 \\ 0 & N_1 + 1 \leq n \leq N \end{cases} \tag{39}$$

The above first $N_1 + 1$ equalities can be written in matrix form as

$$\begin{bmatrix} b(0) & 0 & 0 & \cdots & 0 \\ b(0) & b(0) & 0 & \cdots & 0 \\ b(2) & b(1) & b(0) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b(N_1) & b(N_1-1) & b(N_1-2) & \cdots & b(0) \end{bmatrix} \begin{bmatrix} 1 \\ b_2(1) \\ b_2(2) \\ \vdots \\ b_2(N_1) \end{bmatrix} = \begin{bmatrix} b_1(0) \\ b_1(1) \\ b_1(2) \\ \vdots \\ b_1(N_1)) \end{bmatrix} \tag{40}$$

Eq.(39) for $N_1 + 1 \leq n \leq N$ can be written in matrix form as

$$\begin{bmatrix} b(N_1) & b(N_1-1) & b(N_1-2) & \cdots & b(1) \\ b(N_1+1) & b(N_1) & b(N_1-1) & \cdots & b(2) \\ b(N_1+2) & b(N_1+1) & b(N_1) & \cdots & b(3) \\ \vdots & & & & \\ b(N-1) & b(N-2) & b(N-3) & \cdots & b(N-N_1) \end{bmatrix} \begin{bmatrix} b_2(1) \\ b_2(2) \\ b_2(3) \\ \vdots \\ b_2(N_1) \end{bmatrix} = - \begin{bmatrix} b(N_1+1) \\ b(N_1+2) \\ b(N_1+3) \\ \vdots \\ b(N) \end{bmatrix} \tag{41}$$

In the Prony method, we assume that $N > 2N_1 + 1$. Thus from the eq.(41) we can get the coefficient of $b_2(n)$ using least square method. Once the $b_2(n)$ obtained then we can obtained the coefficient of $b_1(n)$ from the eq.(40). Under the condition $N > 2N_1+1$, we prefer to choose a large N_1 for reducing the error. The complexity of IIR filter implementation $B(z)$ will be increased as increasing the order N_1. So N_1 must be chosen by considering the trade-off between the error and complexity. In our experience N_1 must be chosen in the interval $[5, 20]$.

4 Design Example

In this subsection we will study about the design error and compare the performance of RBF-based fractional order digital differentiator with conventional methods. To evaluate the performance of the RBF, the least squares error of frequency response is defined by

$$E = \sqrt{\int_0^{\lambda\pi} |B(e^{jw}) - H_d(w)|^2 dw} \tag{42}$$

The smaller the error E is, the better performance of the design method has.

Example 1: In this example, we will study the magnitude and phase response for the Gaussian radial basis function $\phi(t)$ in (1). The design parameters are chosen as $N = 60$, $I = 30$, $L = 620$, $h = 0.05$, and $\lambda = 0.9$. Moreover, Fig. 2(a),(b) show the magnitude and phase responses (solid line) for the Gaussian with $\sigma = 2.3$ and order $v = 0.5$. In Fig. 2(a) the dashed line show the ideal magnitude response w^v. Fig. 2(b) show the phase response $90 * [angleB(e^{jw}) + wI]/0.5\pi$. In Fig. 2(b) the dashed line show the ideal phase response $90v$.

Example 2: In this example, we will study the magnitude and phase response for the Inverse multiquadric radial basis function $\phi(t)$ in (2). The design parameters are chosen as $N = 60$, $I = 30$, $L = 620$, $h = 0.05$, and $\lambda = 0.9$. Moreover, Fig. 3(a),(b) show the magnitude and phase responses (solid line) for the Inverse multiquadric with $\sigma = 6.4$ and order $v = 0.5$. In Fig. 3(a) the dashed line show the ideal magnitude response w^v. Fig. 3(b) show the phase response $90 * [angle(B(e^{jw})) + wI]/0.5\pi$. In Fig. 3(b) the dashed line show the ideal phase response $90v$.

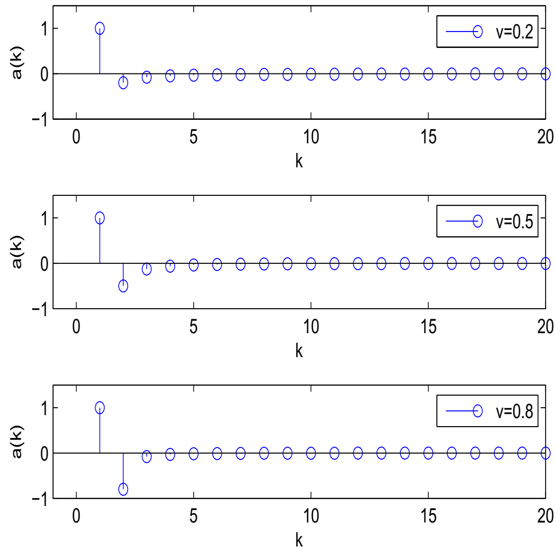

Fig. 1. The coefficient sequence $a(k)$ for various order v

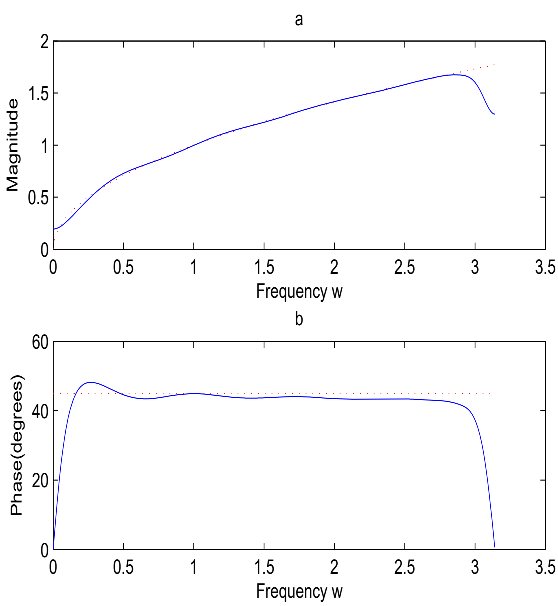

Fig. 2. (a) Magnitude response. (b) Phase response. Solid line show the designed results and dashed line show the ideal response for Gaussian RBF.

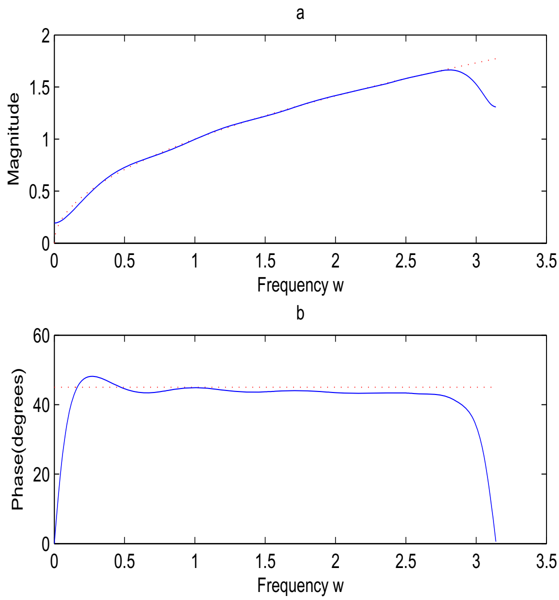

Fig. 3. (a)Magnitude response. (b) Phase response. Solid line show the designed results and dashed line show the ideal response for Inverse multiquadric RBF.

Example 3: In this example, let us compare the proposed method with the conventional time domain least-squares method in [20] whose design procedure is described below:
Step 1) Expand the fractional order Tustin differentiator $[U(z)]^v$ in [20] as the following power series form:

$$[U(z)]^v = \left(2\frac{1-z^{-1}}{1+z^{-1}}\right)^v$$

$$[U(z)]^v = 2^v \left[\sum_{k=0}^{\infty} C_k^v(-z^{-1})^k\right]\left[\sum_{k=0}^{\infty} C_k^{-v} z^{-k}\right]$$

$$[U(z)]^v = 2^v \left(1 + \sum_{k=1}^{\infty} u(k) z^{-k}\right)$$

Where filter coefficient $u(k)$ is the convolution sum of $(-1)^k C_k^v$ and C_k^{-v}. After truncating the higher order terms, $[U(z)]^v$ can be approximated by FIR filter

$$\bar{U}(z) = 2^v \left(1 + \sum_{k=1}^{N_c} u(k) z^{-k}\right) \tag{43}$$

Where N_c is the truncation length.

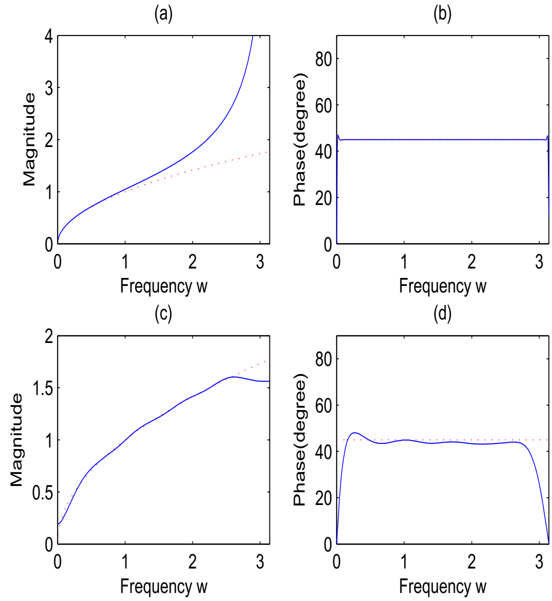

Fig. 4. Solid line show the designed results of fractional order IIR differentiator (a), (b) The results of $\hat{U}(z)$ in conventional method (c), (d) The results of $\bar{B}(z)$ in proposed method. The dashed line is the ideal response.

Step 2) Using the Prony method, the long-length FIR filter $\bar{U}(z)$ can be approximated by IIR filter:

$$\hat{U}(z) = \frac{\sum_{n=0}^{N_2} u_1(n) z^{-n}}{1 + \sum_{n=1}^{N_2} u_2(n) z^{-n}} \tag{44}$$

$$\hat{U}(z) = \bar{U}(z)$$

$$2^v \left[u(n) + \sum_{k=1}^{N_2} u_2(n) u(n-k) \right] = \begin{cases} u_1(n) & 0 \le n \le N_2 \\ 0 & N_2 + 1 \le n \le N \end{cases} \tag{45}$$

Now, one example is used to compare this conventional design method with the proposed design method in (37). The parameters in conventional design are chosen as $N_c = 60$, $N_2 = 10$, $v = 0.5$. Fig. 4(a),(b) show the magnitude and phase response (solid line) of the designed differentiator $\hat{U}(z)$. The dashed line is the ideal response. The maximum pole radius is 0.9719, so IIR filter $\hat{U}(z)$ is stable. From this result the error of phase is very small. But the magnitude error at high frequency band is very large. After $B(e^{jw})$ in (42) is changed to $\hat{U}(e^{jw})$, the error with $\lambda = 0.9$ is 13.6095 for this traditional design. For comparison, the designed result of proposed RBF method are reported below. The design

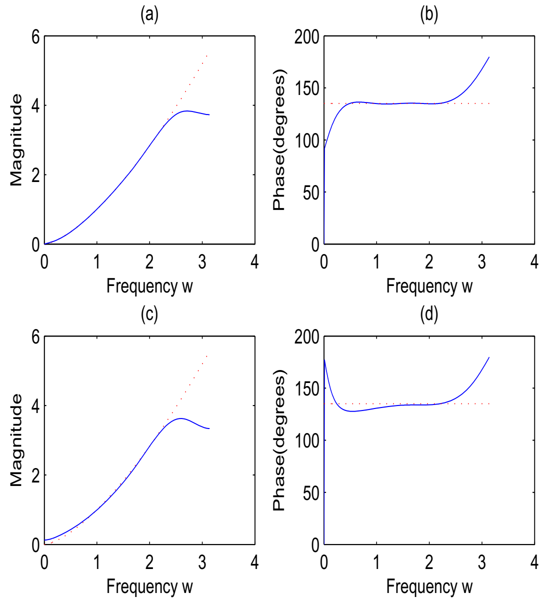

Fig. 5. The designed results (Solid line) of the fractional order FIR differentiator. (a), (b) The results of the method in [21]. (c), (d) The results of the proposed method. The dashed line is the ideal response.

parameter are chosen as $N = 60$, $I = 9$, $L = 620$, $h = 0.05$, $N_1 = 10$, $v = 0.5$ and Inverse Multiquadric RBF with $\sigma = 6.4$. Fig. 4(c),(d) show the magnitude and phase response (solid line) for the designed IIR differentiator $\bar{B}(z)$ in (37). The dashed line is the ideal response. Compared Fig. 4(a), (b) and Fig. 4(c), (d), it can be observed that the proposed RBF method has better magnitude response than conventional method. However, the phase response error of conventional approach is smaller than the proposed method. After $B(e^{jw})$ in (42) is changed to $\bar{B}(e^{jw})$, the error E with $\lambda = 0.9$ is 2.7449 for the proposed RBF design. Thus, the above result show that the proposed method has smaller error than the conventional method.

Example 4: In this example, let us compare the proposed method with the conventional method in [21] where fractional order FIR differentiator has been designed by using frequency response approximation approach.

The transfer function for the frequency response approximation is given as

$$H(e^{jw}) = \sum_{m=0}^{N} a(m)\cos mw + j\sum_{m=0}^{N} b(m)\sin mw, \qquad w \in [-\pi, \pi]$$

$$H(z) = \sum_{m=0}^{N} \frac{a(m)}{2}\,(z^m + z^{-m}) + \sum_{m=0}^{N} \frac{b(m)}{2}\,(z^m - z^{-m})$$

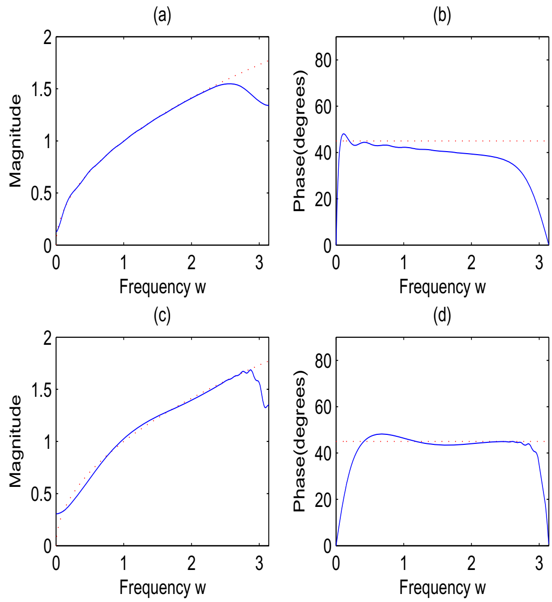

Fig. 6. The designed results (solid line) of the fractional order FIR differentiator. (a), (b) The results of the fractional delay method in [22]. (c), (d) The results of the proposed method. The dashed line is the ideal response.

$$H(z) = a(0) + \sum_{m=1}^{N} \frac{a(m)+b(m)}{2} z^{m} + \sum_{m=1}^{N} \frac{a(m)-b(m)}{2} z^{-m}$$

When the design parameters are chosen as $N = 10$, $I = 5$ and $v = 1.5$, the FIR filter coefficient $b(m)$ in (33) can be obtained from the data in column 3 of Table 1 of [21]. Fig. 5(a),(b) show the magnitude and phase response (solid line) of this conventional method. The dashed is the ideal response. For comparison under the same implementation complexity, the design parameters of the proposed method are chosen as $N = 10, I = 5, v = 1.5, L = 620, h = 0.01$ and Inverse multiquadric RBF with $\sigma = 6.4$. Fig. 5(c),(d) show the magnitude and phase responses (solid line) for the designed FIR differentiator $B(z)$ in (36). The dashed line is the ideal response. Now the error comparison is made. If $\lambda = 0.72$ is chosen, the error E of conventional method in [21] is 0.019 and the error E of proposed method is 0.0198. Thus, the results show that the error is approximately equal to the conventional method in the frequency band $[0, 0.72\pi]$. When $\lambda = 0.9$ is chosen, the error E of the conventional design method is 0.623, and the error E of the proposed RBF design method is 0.565. Thus, the above result show that the proposed method has smaller error than conventional method in the frequency band $[0, 0.9\pi]$.

Example 5: In this example, we will compare the proposed method with the conventional fractional delay method in [22].

$$G(z) = \sum_{k=0}^{N} \alpha(k) z^{-(I_k + f_k)}$$

$$G(z) = \sum_{k=0}^{N} \alpha(k) \left[\sum_{n=0}^{2I_k} \left(\prod_{m=0, m\neq n}^{2I_k} \frac{I_k + f_k - m}{n - m} \right) z^{-n} \right]$$

When the design parameters are chosen as $N = 80$, $I = 20$ and $v = 0.5$, the conventional fractional order FIR differentiator is designed by Lagrange fractional delay method used in Fig. 4 of [22]. Fig. 6(a),(b) show the designed results (solid line) of this method. The dashed line is the ideal response. It can be seen that the phase response does not fit the ideal response well. Moreover, the design parameters of the proposed method are chosen as $N = 80$, $I = 20$, $v = 0.5$, $L = 620$, $h = 0.02$ and the Inverse multiquadric RBF with $\sigma = 6.4$. Fig. 6(c),(d) show the magnitude and phase response (solid line) for the designed fractional order FIR differentiator. The dashed line is the ideal response. If $\lambda = 0.9$ is chosen, the error E of conventional fractional delay method in [22] is 0.2587, and the error E of proposed RBF method is 0.0316. Thus, the above result shows that the proposed method has a smaller error than the conventional method.

5 Conclusion

In this paper, an fractional order digital differentiator has been designed by using Inverse multiquadric radial basis function (RBF). First, the RBF interpolation approach is described. Then, the non-integer delay sample estimation is derived by using RBF approach. Next, the Grünwald-Letnikov derivative and non-integer delay sample delay are applied to obtain the transfer function of the proposed method i.e. fractional order digital differentiator. The design accuracy of the proposed method is better then the conventional methods like examples Time domain least squares method, Fractional sample delay method and Frequency response approximation method. However, only the one-dimensional case is studied here. Thus, it is interesting to extend the proposed method to design a two-dimensional fractional order digital differentiator in the future.

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