Generalized Projective Synchronization of Three-Scroll Chaotic Systems via Active Control

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Abstract. This paper investigates the generalized projective synchronization (GPS) of identical Wang 3-scroll chaotic systems (Wang, 2009) and non-identical Dadras 3-scroll chaotic system (Dadras and Momeni, 2009) and Wang 3-scroll chaotic system. The synchronization results (GPS) derived for the 3-scroll chaotic systems have been derived using active control method and established using Lyapunov stability theory. Since the Lyapunov exponents are not required for these calculations, the active control method is very effective and convenient for achieving the general projective synchronization (GPS) of the 3-scroll chaotic systems addressed in this paper. Numerical simulations are presented to demonstrate the effectiveness of the synchronization results derived in this paper.

Keywords: Active control, chaos, generalized projective synchronization, three-scroll systems, Wang system, Dadras system.

1 Introduction

Chaotic systems are nonlinear dynamical systems, which are highly sensitive to initial conditions. The sensitive nature of chaotic systems is commonly called as the *butterfly effect* [1]. Chaos is an interesting nonlinear phenomenon and has been intensively studied in the last three decades. Chaos theory is applied in many scientific disciplines, viz. Mathematics, Physics, Chemistry, Biology, Engineering, Computer Science, Robotics and Economics.

Chaos synchronization is a phenomenon that may occur when two or more chaotic oscillators are coupled or a chaotic oscillator drives another chaotic oscillator. Because of the butterfly effect which causes the exponential divergence of the trajectories of two identical chaotic systems started with nearly the same initial conditions, synchronizing two chaotic systems is seemingly a very challenging problem.

Chaos theory has wide applications in several fields like physical systems [2], chemical systems [3], ecological systems [4], secure communications ([5]-[7]) etc.

In most of the chaos synchronization approaches, the *master-slave* or *drive-response* formalism is used. If a particular chaotic system is called the *master* or *drive system*

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and another chaotic system is called the *slave* or *response system*, then the idea of synchronization is to use the output of the master system to control the slave system so that the output of the slave system tracks the output of the master system asymptotically.

The seminal work by Pecora and Carroll ([8], 1990) is followed by a variety of impressive approaches for chaos synchronization such as the sampled-data feedback synchronization method [9], OGY method [10], time-delay feedback method [11], backstepping method [12], active control method [13], adaptive control method [14], sliding control method [15], etc.

In generalized projective synchronization [16], the chaotic systems can synchronize up to a constant scaling matrix. Complete synchronization [17], anti-synchronization [18], hybrid synchronization [19], projective synchronization [20] and generalized synchronization [21] are special cases of generalized projective synchronization. The generalized projective synchronization (GPS) has applications in secure communications.

This paper is organized as follows. In Section 2, we provide a description of the 3-scroll chaotic systems studied in this paper. In Section 3, we derive results for the GPS between identical Wang 3-scroll chaotic systems (Wang, [22], 2009). In Section 4, we derive results for the GPS between non-identical Dadras 3-scroll chaotic system (Dadras and Momeni, [23], 2009) and Wang 3-scroll chaotic system (2009). In Section 5, we summarize the main results obtained in this paper.

2 Systems Description

The Wang 3-scroll chaotic system ([22], 2009) is described by the dynamics

$$\begin{aligned} \dot{x}_1 &= a(x_1 - x_2) - x_2 x_3 \\ \dot{x}_2 &= -b x_2 + x_1 x_3 \\ \dot{x}_3 &= -c x_3 + d x_1 + x_1 x_2 \end{aligned} \tag{1}$$

where x_1, x_2, x_3 are the *state* variables and a, b, c, d are constant, positive parameters of the system.

The system (1) exhibits a 3-scroll chaotic attractor when the system parameter values are chosen as $a = 0.977,\ b = 10,\ c = 4$ and $d = 0.1$.

Figure 1 depicts the state orbits of the Wang 3-scroll system (1).

The Dadras 3-scroll system ([23], 2009) is described by the dynamics

$$\begin{aligned} \dot{x}_1 &= x_2 - p x_1 + q x_2 x_3 \\ \dot{x}_2 &= r x_2 - x_1 x_3 + x_3 \\ \dot{x}_3 &= s x_1 x_2 - \epsilon x_3 \end{aligned} \tag{2}$$

where x_1, x_2, x_3 are the *state* variables and p, q, r, s, ϵ are constant, positive parameters of the system.

The system (2) exhibits a 3-scroll chaotic attractor when the system parameter values are chosen as $p = 3,\ q = 2.7,\ r = 1.7,\ s = 2$ and $\epsilon = 9$.

Figure 2 depicts the state orbits of the Dadras 3-scroll system (2).

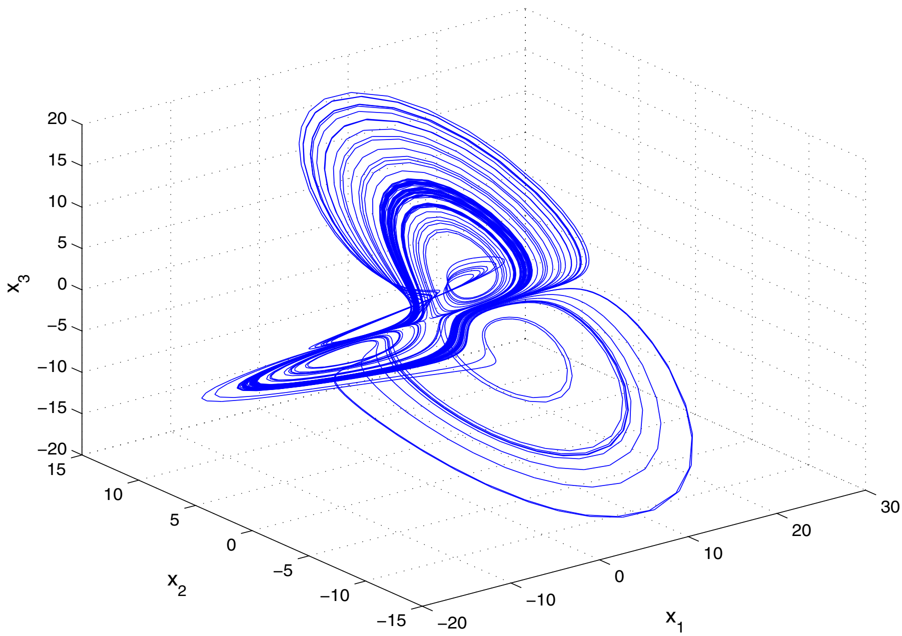

Fig. 1. State Orbits of the Wang 3-Scroll Chaotic System

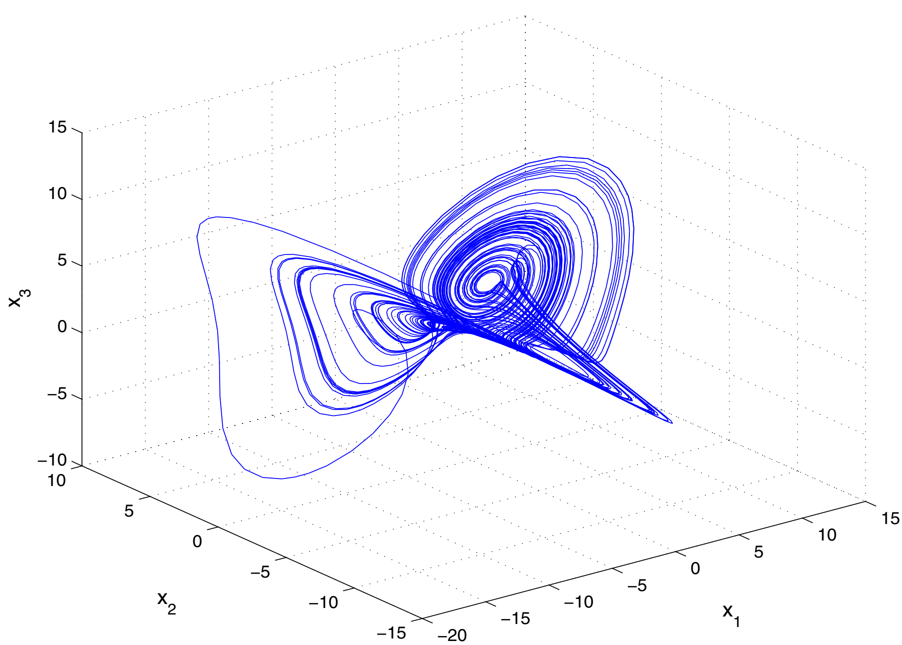

Fig. 2. State Orbits of the Dadras 3-Scroll Chaotic System

3 Generalized Projective Synchronization of Identical Wang 3-Scroll Chaotic Systems

3.1 Main Results

In this section, we discuss the design of active controller for achieving generalized projective synchronization (GPS) of identical Wang 3-scroll chaotic systems ([22], 2009).

Thus, the master system is described by the Wang dynamics

$$\begin{aligned} \dot{x}_1 &= a(x_1 - x_2) - x_2 x_3 \\ \dot{x}_2 &= -b x_2 + x_1 x_3 \\ \dot{x}_3 &= -c x_3 + d x_1 + x_1 x_2 \end{aligned} \tag{3}$$

where x_1, x_2, x_3 are the *state* variables and a, b, c, d are constant, positive parameters of the system.

Also, the slave system is described by the controlled Wang dynamics

$$\begin{aligned} \dot{y}_1 &= a(y_1 - y_2) - y_2 y_3 + u_1 \\ \dot{y}_2 &= -b y_2 + y_1 y_3 + u_2 \\ \dot{y}_3 &= -c y_3 + d y_1 + y_1 y_2 + u_3 \end{aligned} \tag{4}$$

where y_1, y_2, y_3 are the *state* variables and u_1, u_2, u_3 are the active controls.

For the GPS of (3) and (4), the synchronization errors are defined as

$$\begin{aligned} e_1 &= y_1 - \alpha_1 x_1 \\ e_2 &= y_2 - \alpha_2 x_2 \\ e_3 &= y_3 - \alpha_3 x_3 \end{aligned} \tag{5}$$

where the scales $\alpha_1, \alpha_2, \alpha_3$ are real numbers.

A simple calculation yields the error dynamics

$$\begin{aligned} \dot{e}_1 &= a(y_1 - y_2) - y_2 y_3 - \alpha_1[a(x_1 - x_2) - x_2 x_3] + u_1 \\ \dot{e}_2 &= -b y_2 + y_1 y_3 - \alpha_2[-b x_2 + x_1 x_3] + u_2 \\ \dot{e}_3 &= -c y_3 + d y_1 + y_1 y_2 - \alpha_3[-c x_3 + d x_1 + x_1 x_2] + u_3 \end{aligned} \tag{6}$$

We consider the active nonlinear controller defined by

$$\begin{aligned} u_1 &= -a(y_1 - y_2) + y_2 y_3 + \alpha_1[a(x_1 - x_2) - x_2 x_3] - k_1 e_1 \\ u_2 &= b y_2 - y_1 y_3 + \alpha_2[-b x_2 + x_1 x_3] - k_2 e_2 \\ u_3 &= c y_3 - d y_1 - y_1 y_2 + \alpha_3[-c x_3 + d x_1 + x_1 x_2] - k_3 e_3 \end{aligned} \tag{7}$$

where the gains k_1, k_2, k_3 are positive constants.

Substitution of (7) into (6) yields the closed-loop error dynamics

$$\begin{aligned} \dot{e}_1 &= -k_1 e_1 \\ \dot{e}_2 &= -k_2 e_2 \\ \dot{e}_3 &= -k_3 e_3 \end{aligned} \tag{8}$$

We consider the quadratic Lyapunov function defined by

$$V(e) = \frac{1}{2}\, e^T e = \frac{1}{2}\, \left(e_1^2 + e_2^2 + e_3^2\right) \tag{9}$$

which is positive definite on $\mathbb{R}^3$.

Differentiating (9) along the trajectories of the system (8), we get

$$\dot{V}(e) = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 \tag{10}$$

which is a negative definite function on $\mathbb{R}^3$, since k_1, k_2, k_3 are positive constants.

Thus, by Lyapunov stability theory [24], the error dynamics (8) is globally exponentially stable. Hence, we obtain the following result.

Theorem 1. *The active feedback controller (7) achieves global chaos generalized projective synchronization (GPS) between the identical Wang 3-scroll chaotic systems (3) and (4).* ■

3.2 Numerical Results

For the numerical simulations, the fourth order Runge-Kutta method is used to solve the two systems of differential equations (3) and (4) with the active controller (7).

The parameters of the identical Wang 3-scroll chaotic systems are chosen as

$$a = 0.977, \quad b = 10, \quad c = 4, \quad d = 0.1$$

The initial values for the master system (3) are taken as

$$x_1(0) = 15, \quad x_2(0) = 28, \quad x_3(0) = 7$$

The initial values for the slave system (4) are taken as

$$y_1(0) = 29, \quad y_2(0) = 10, \quad y_3(0) = 14$$

The GPS scales α_i are taken as

$$\alpha_1 = -2.5, \quad \alpha_2 = 6.3, \quad \alpha_3 = -0.7$$

We take the state feedback gains as $k_1 = 4,\;\; k_2 = 4$, and $k_3 = 4$.

Figure 3 shows the time response of the error states e_1, e_2, e_3 of the error dynamical system (6) when the active nonlinear controller (7) is deployed. From this figure, it is clear that all the error states decay to zero exponentially in 1.5 sec and thus, generalized projective synchronization is achieved between the identical Wang 3-scroll systems (3) and (4).

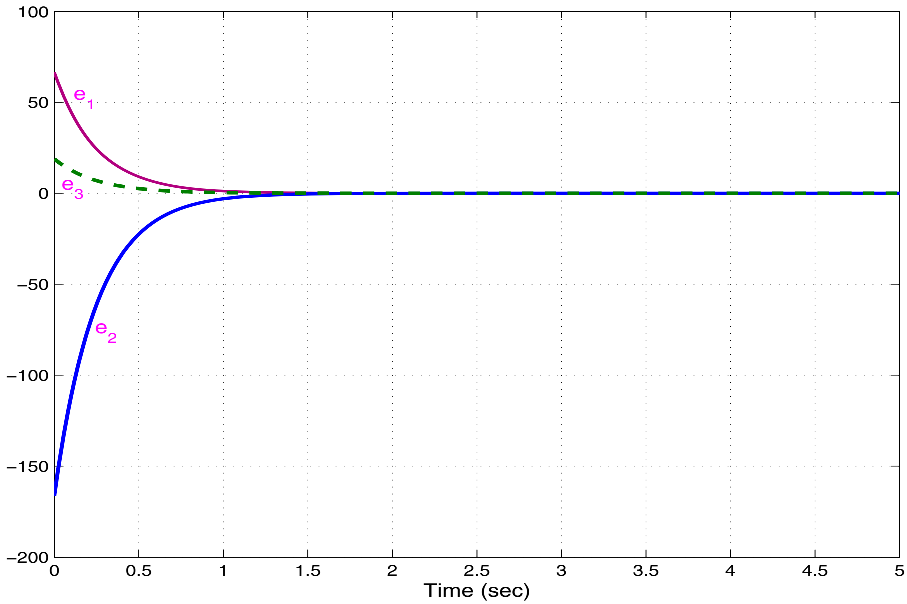

Fig. 3. Time Responses of the Error States of (6)

4 Generalized Projective Synchronization of Non-identical Dadras and Wang 3-Scroll Chaotic Systems

4.1 Main Results

In this section, we derive results for the generalized projective synchronization (GPS) of non-identical 3-scroll chaotic systems, *viz.* Dadras 3-scroll system ([23], 2009) and Wang 3-scroll system ([22], 2009).

Thus, the master system is described by the Dadras dynamics

$$\begin{aligned} \dot{x}_1 &= x_2 - px_1 + qx_2x_3 \\ \dot{x}_2 &= rx_2 - x_1x_3 + x_3 \\ \dot{x}_3 &= sx_1x_2 - \epsilon x_3 \end{aligned} \tag{11}$$

where x_1, x_2, x_3 are the *state* variables and p, q, r, s, ϵ are constant, positive parameters of the system.

Also, the slave system is described by the controlled Wang dynamics

$$\begin{aligned} \dot{y}_1 &= a(y_1 - y_2) - y_2y_3 + u_1 \\ \dot{y}_2 &= -by_2 + y_1y_3 + u_2 \\ \dot{y}_3 &= -cy_3 + dy_1 + y_1y_2 + u_3 \end{aligned} \tag{12}$$

where y_1, y_2, y_3 are the *state* variables, a, b, c, d are constant, positive parameters of the system and u_1, u_2, u_3 are the active controls.

For the GPS of (11) and (12), the synchronization errors are defined as

$$\begin{aligned} e_1 &= y_1 - \alpha_1 x_1 \\ e_2 &= y_2 - \alpha_2 x_2 \\ e_3 &= y_3 - \alpha_3 x_3 \end{aligned} \tag{13}$$

where the scales $\alpha_1, \alpha_2, \alpha_3$ are real numbers.

A simple calculation yields the error dynamics

$$\begin{aligned} \dot{e}_1 &= a(y_1 - y_2) - y_2 y_3 - \alpha_1 \left[x_2 - p x_1 + q x_2 x_3\right] + u_1 \\ \dot{e}_2 &= -b y_2 + y_1 y_3 - \alpha_2 \left[r x_2 - x_1 x_3 + x_3\right] + u_2 \\ \dot{e}_3 &= -c y_3 + d y_1 + y_1 y_2 - \alpha_3 \left[s x_1 x_2 - \epsilon x_3\right] + u_3 \end{aligned} \tag{14}$$

We consider the active nonlinear controller defined by

$$\begin{aligned} u_1 &= -a(y_1 - y_2) + y_2 y_3 + \alpha_1 \left[x_2 - p x_1 + q x_2 x_3\right] - k_1 e_1 \\ u_2 &= b y_2 - y_1 y_3 + \alpha_2 \left[r x_2 - x_1 x_3 + x_3\right] - k_2 e_2 \\ u_3 &= c y_3 - d y_1 - y_1 y_2 + \alpha_3 \left[s x_1 x_2 - \epsilon x_3\right] - k_3 e_3 \end{aligned} \tag{15}$$

where the gains k_1, k_2, k_3, k_4 are positive constants.

Substitution of (15) into (14) yields the closed-loop error dynamics

$$\begin{aligned} \dot{e}_1 &= -k_1 e_1 \\ \dot{e}_2 &= -k_2 e_2 \\ \dot{e}_3 &= -k_3 e_3 \end{aligned} \tag{16}$$

We consider the quadratic Lyapunov function defined by

$$V(e) = \frac{1}{2}\, e^T e = \frac{1}{2}\, \left(e_1^2 + e_2^2 + e_3^2\right) \tag{17}$$

which is positive definite on $\mathbb{R}^3$.

Differentiating (17) along the trajectories of the system (16), we get

$$\dot{V}(e) = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 \tag{18}$$

which is a negative definite function on $\mathbb{R}^3$, since k_1, k_2, k_3 are positive constants.

Thus, by Lyapunov stability theory [24], the error dynamics (16) is globally exponentially stable. Hence, we obtain the following result.

Theorem 2. *The active feedback controller (15) achieves global chaos generalized projective synchronization (GPS) between the non-identical Dadras 3-scroll chaotic system (11) and the Wang 3-scroll chaotic system (12).* ■

4.2 Numerical Results

For the numerical simulations, the fourth order Runge-Kutta method is used to solve the two systems of differential equations (11) and (12) with the active controller (15).

The parameters of the Dadras 3-scroll chaotic system (11) are taken as

$$p = 3, \;\; q = 2.7, \;\; r = 1.7, \;\; s = 2, \;\; \epsilon = 9$$

The parameters of the Wang 3-scroll chaotic system (12) are taken as

$$a = 0.977, \;\; b = 10, \;\; c = 4, \;\; d = 0.1$$

The initial values for the master system (11) are taken as

$$x_1(0) = 3, \;\; x_2(0) = 10, \;\; x_3(0) = 28$$

The initial values for the slave system (12) are taken as

$$y_1(0) = 24, \;\; y_2(0) = 6, \;\; y_3(0) = 15$$

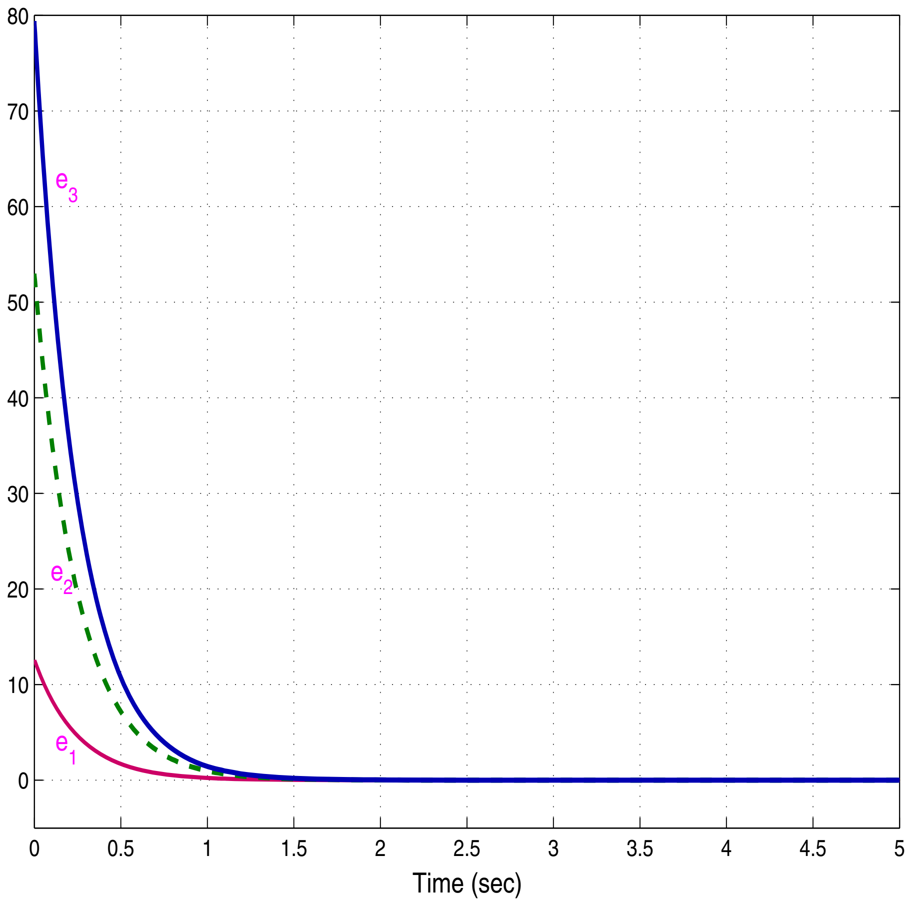

Fig. 4. Time Responses of the Error States of (14)

The GPS scales α_i are taken as

$$\alpha_1 = 3.8, \;\; \alpha_2 = -4.7, \;\; \alpha_3 = -2.3$$

We take the state feedback gains as $k_1 = 4, \; k_2 = 4$ and $k_3 = 4$.

Figure 4 shows the time response of the error states e_1, e_2, e_3 of the error dynamical system (14) when the active nonlinear controller (15) is deployed.

From this figure, it is clear that all the error states decay to zero exponentially in 1.7 sec and thus, generalized projective synchronization is achieved between the non-identical Dadras 3-scroll system (11) and Wang 3-scroll system (12).

5 Conclusions

In this paper, we derived active control laws for achieving generalized projective synchronization (GPS) of the following 3-scroll chaotic systems:

(A) Identical Wang 3-scroll systems (2009)
(B) Non-identical Dadras 3-scroll system (2009) and Wang 3-scroll system (2009).

The synchronization results (GPS) derived in this paper for the chaotic Wang and Dadras 3-scroll chaotic systems [(A) and (B)] have been proved using Lyapunov stability theory. Since Lyapunov exponents are not required for these calculations, the proposed active control method is very effective and suitable for achieving GPS of the 3-scroll chaotic systems addressed in this paper. Numerical simulations are shown to demonstrate the effectiveness of the synchronization results (GPS) derived in this paper.

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