Generalized Projective Synchronization of Hyperchaotic Lü and Hyperchaotic Cai Systems via Active Control

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Abstract. This paper investigates the problem of designing active feedback controllers for achieving generalized projective synchronization (GPS) of identical hyperchaotic Lü systems (Chen *et al.* 2006) and non-identical hyperchaotic Cai system (Wang and Cai, 2009) and hyperchaotic Lü system. The synchronization results (GPS) derived in this paper have been established using Lyapunov stability theory. Since the Lyapunov exponents are not required for these calculations, the active feedback control method is very effective and convenient for achieving the general projective synchronization (GPS) of hyperchaotic Lü and hyperchaotic Cai systems. Numerical simulations are shown to demonstrate the effectiveness of the synchronization results derived in this paper.

Keywords: Active control, hyperchaos, generalized projective synchronization, hyperchaotic Lü system, hyperchaotic Cai system.

1 Introduction

Chaotic systems are nonlinear dynamical systems, which are highly sensitive to initial conditions. The sensitive nature of chaotic systems is commonly called as the *butterfly effect* [1].

Hyperchaotic system is usually defined as a chaotic system with more than one positive Lyapunov exponent. Since hyperchaotic system has the characteristics of high capacity, high security and high efficiency, it has the potential of broad applications in nonlinear circuits, secure communications, lasers, neural networks, biological systems, and so on. Thus, the studies on hyperchaotic systems, *viz.* control, synchronization and circuit implementation are very challenging works in the chaos literature.

In most of the chaos synchronization approaches, the *master-slave* or *drive-response* formalism is used. If a particular chaotic system is called the *master* or *drive system* and another chaotic system is called the *slave* or *response system*, then the idea of synchronization is to use the output of the master system to control the slave system so that the output of the slave system tracks the output of the master system asymptotically.

The seminal work by Pecora and Carroll ([3], 1990) is followed by a variety of impressive approaches for chaos synchronization such as the sampled-data feedback

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synchronization method [4], OGY method [5], time-delay feedback method [6], backstepping method [7], active control method ([8]-[9]), adaptive control method [10], sliding control method [11], etc.

In generalized projective synchronization [12], the chaotic systems can synchronize up to a constant scaling matrix. Complete synchronization [13], anti-synchronization [14], hybrid synchronization [15], projective synchronization [16] and generalized synchronization [17] are special cases of generalized projective synchronization. The generalized projective synchronization (GPS) has applications in secure communications.

This paper deals with the problem of designing active feedback controllers for the generalized projective synchronization (GPS) of identical hyperchaotic Lü systems (Chen *et al.* [18], 2006) and non-identical hyperchaotic Cai system (Wang and Cai, [19], 2009) and hyperchaotic Lü system (2006).

This paper is organized as follows. In Section 2, we provide a description of the hyperchaotic systems studied in this paper. In Section 3, we derive results for the GPS between identical hyperchaotic Lü systems (2006). In Section 4, we derive results for the GPS between non-identical hyperchaotic Cai system (2009) and hyperchaotic Lü system (2006). In Section 5, we summarize the main results obtained in this paper.

2 Systems Description

The hyperchaotic Lü system ([18], 2006) is described by the dynamics

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) + x_4 \\ \dot{x}_2 &= -x_1 x_3 + c x_2 \\ \dot{x}_3 &= x_1 x_2 - b x_3 \\ \dot{x}_4 &= x_1 x_3 + d x_4 \end{aligned} \tag{1}$$

where x_1, x_2, x_3, x_4 are the *state* variables and a, b, c, d are constant, positive parameters of the system.

The system (1) is hyperchaotic when the system parameter values are chosen as $a = 36$, $b = 3$, $c = 20$ and $d = 1.3$.

Figure 1 depicts the state orbits of the hyperchaotic Lü system (1).

The hyperchaotic Cai system ([19], 2009) is described by the dynamics

$$\begin{aligned} \dot{x}_1 &= p(x_2 - x_1) \\ \dot{x}_2 &= q x_1 + r x_2 - x_1 x_3 + x_4 \\ \dot{x}_3 &= x_2^2 - s x_3 \\ \dot{x}_4 &= -\epsilon x_1 \end{aligned} \tag{2}$$

where x_1, x_2, x_3, x_4 are the *state* variables and p, q, r, s, ϵ are constant, positive parameters of the system.

The system (2) is hyperchaotic when the system parameter values are chosen as $p = 27.5$, $q = 3$, $r = 19.3$, $s = 2.9$ and $\epsilon = 3.3$.

Figure 2 depicts the state orbits of the hyperchaotic Cai system (2).

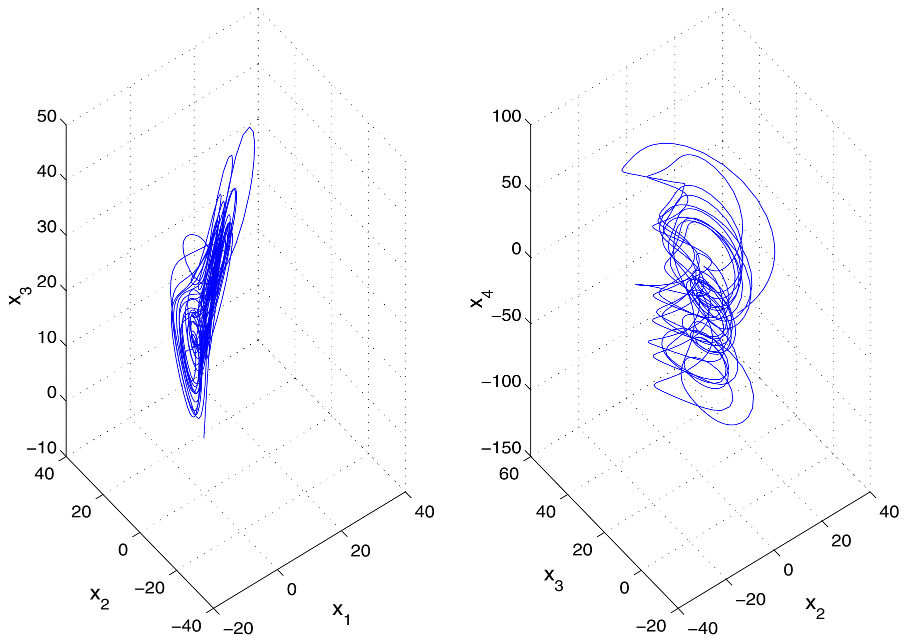

Fig. 1. State Orbits of the hyperchaotic Lü System

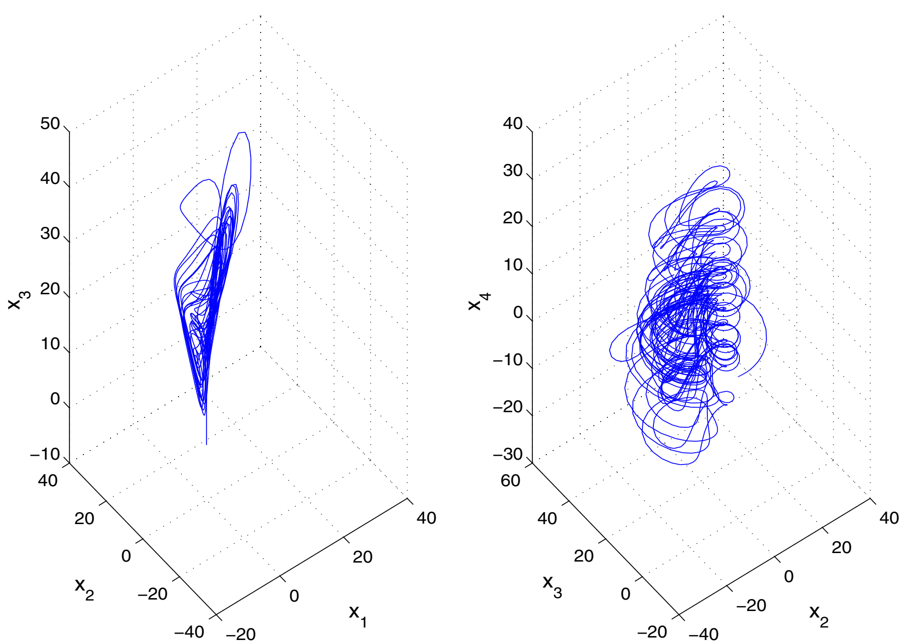

Fig. 2. State Orbits of the hyperchaotic Cai System

3 Generalized Projective Synchronization of Identical Hyperchaotic Lü Systems

3.1 Main Results

In this section, we discuss the design of active controller for achieving generalized projective synchronization (GPS) of identical hyperchaotic Lü systems ([18], 2006).

Thus, the master system is described by the hyperchaotic Lü dynamics

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) + x_4 \\ \dot{x}_2 &= -x_1 x_3 + c x_2 \\ \dot{x}_3 &= x_1 x_2 - b x_3 \\ \dot{x}_4 &= x_1 x_3 + d x_4 \end{aligned} \tag{3}$$

where x_1, x_2, x_3, x_4 are the *state* variables and a, b, c, d are constant, positive parameters of the system.

Also, the slave system is described by the controlled hyperchaotic Lü dynamics

$$\begin{aligned} \dot{y}_1 &= a(y_2 - y_1) + y_4 + u_1 \\ \dot{y}_2 &= -y_1 y_3 + c y_2 + u_2 \\ \dot{y}_3 &= y_1 y_2 - b y_3 + u_3 \\ \dot{y}_4 &= y_1 y_3 + d y_4 + u_4 \end{aligned} \tag{4}$$

where y_1, y_2, y_3, y_4 are the *state* variables and u_1, u_2, u_3, u_4 are the active controls.

For the GPS of (3) and (4), the synchronization errors are defined as

$$e_i = y_i - \alpha_i x_i, \quad (i = 1, 2, 3, 4) \tag{5}$$

where the scales $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are real numbers.

A simple calculation yields the error dynamics

$$\begin{aligned} \dot{e}_1 &= a(y_2 - y_1) + y_4 - \alpha_1[a(x_2 - x_1) + x_4] + u_1 \\ \dot{e}_2 &= -y_1 y_3 + c y_2 - \alpha_2[-x_1 x_3 + c x_2] + u_2 \\ \dot{e}_3 &= y_1 y_2 - b y_3 - \alpha_3[x_1 x_2 - b x_3] + u_3 \\ \dot{e}_4 &= y_1 y_3 + d y_4 - \alpha_4[x_1 x_3 + d x_4] + u_4 \end{aligned} \tag{6}$$

We consider the active nonlinear controller defined by

$$\begin{aligned} u_1 &= -a(y_2 - y_1) - y_4 + \alpha_1[a(x_2 - x_1) + x_4] - k_1 e_1 \\ u_2 &= y_1 y_3 - c y_2 + \alpha_2[-x_1 x_3 + c x_2] - k_2 e_2 \\ u_3 &= -y_1 y_2 + b y_3 + \alpha_3[x_1 x_2 - b x_3] - k_3 e_3 \\ u_4 &= -y_1 y_3 - d y_4 + \alpha_4[x_1 x_3 + d x_4] - k_4 e_4 \end{aligned} \tag{7}$$

where the gains k_1, k_2, k_3, k_4 are positive constants.

Substitution of (7) into (6) yields the closed-loop error dynamics

$$\begin{aligned} \dot{e}_1 &= -k_1 e_1 \\ \dot{e}_2 &= -k_2 e_2 \\ \dot{e}_3 &= -k_3 e_3 \\ \dot{e}_4 &= -k_4 e_4 \end{aligned} \tag{8}$$

We consider the quadratic Lyapunov function defined by

$$V(e) = \frac{1}{2}\, e^T e = \frac{1}{2}\, \left(e_1^2 + e_2^2 + e_3^2 + e_4^2\right) \tag{9}$$

which is positive definite on $\mathbb{R}^4$.

Differentiating (9) along the trajectories of the system (8), we get

$$\dot{V}(e) = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 \tag{10}$$

which is a negative definite function on $\mathbb{R}^4$, since k_1, k_2, k_3, k_4 are positive constants.

Thus, by Lyapunov stability theory [20], the error dynamics (8) is globally exponentially stable. Hence, we obtain the following result.

Theorem 1. *The active feedback controller (7) achieves global chaos generalized projective synchronization (GPS) between the identical hyperchaotic Lü systems (3) and (4).* ■

3.2 Numerical Results

For the numerical simulations, the fourth order Runge-Kutta method is used to solve the two systems of differential equations (3) and (4) with the active controller (7).

The parameters of the identical hyperchaotic Lü systems are chosen as

$$a = 36, \;\; b = 3, \;\; c = 20, \;\; d = 1.3$$

The initial values for the master system (3) are taken as

$$x_1(0) = 24, \;\; x_2(0) = 8, \;\; x_3(0) = 10, \;\; x_4(0) = 12$$

The initial values for the slave system (4) are taken as

$$y_1(0) = 15, \;\; y_2(0) = 12, \;\; y_3(0) = 4, \;\; y_4(0) = 20$$

The GPS scales α_i are taken as

$$\alpha_1 = 4.58, \;\; \alpha_2 = 3.49, \;\;\; \alpha_3 = -7.21, \;\; \alpha_4 = -5.34$$

We take the state feedback gains as $k_1 = 4,\;\; k_2 = 4,\;\; k_3 = 4$ and $k_4 = 4$.

Figure 3 shows the time response of the error states e_1, e_2, e_3, e_4 of the error dynamical system (6) when the active nonlinear controller (7) is deployed. From this figure, it is clear that all the error states decay to zero exponentially in 1.5 sec and thus, generalized projective synchronization is achieved between the identical hyperchaotic Lü systems (3) and (4).

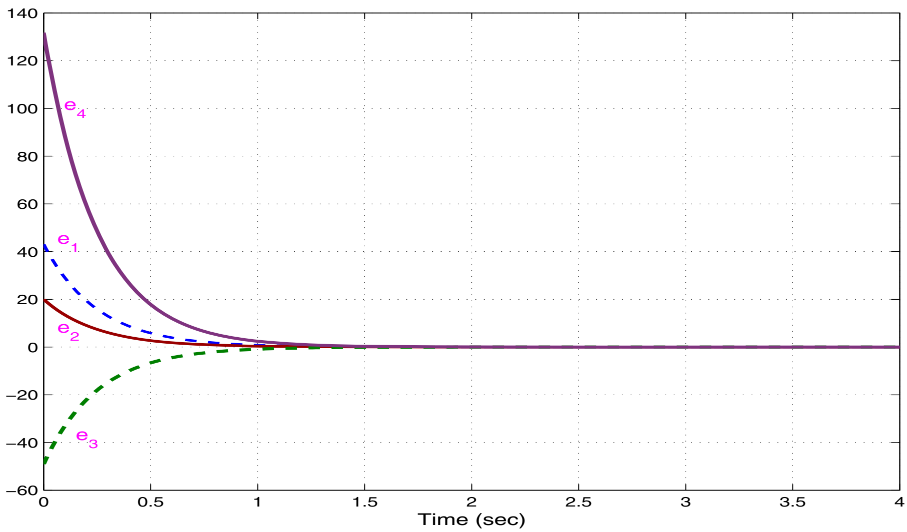

Fig. 3. Time Responses of the Error States of (6)

4 Generalized Projective Synchronization of Non-identical Hyperchaotic Lü and Hyperchaotic Cai Systems

4.1 Main Results

In this section, we derive results for the generalized projective synchronization (GPS) of non-identical hyperchaotic systems, *viz.* hyperchaotic Cai system ([19], 2009) and hyperchaotic Lü system ([18], 2006).

Thus, the master system is described by the hyperchaotic Cai dynamics

$$\begin{aligned} \dot{x}_1 &= p(x_2 - x_1) \\ \dot{x}_2 &= qx_1 + rx_2 - x_1x_3 + x_4 \\ \dot{x}_3 &= x_2^2 - sx_3 \\ \dot{x}_4 &= -\epsilon x_1 \end{aligned} \tag{11}$$

where x_1, x_2, x_3, x_4 are the *state* variables and p, q, r, s, ϵ are constant, positive parameters of the system.

Also, the slave system is described by the controlled hyperchaotic Lü dynamics

$$\begin{aligned} \dot{y}_1 &= a(y_2 - y_1) + y_4 + u_1 \\ \dot{y}_2 &= -y_1y_3 + cy_2 + u_2 \\ \dot{y}_3 &= y_1y_2 - by_3 + u_3 \\ \dot{y}_4 &= y_1y_3 + dy_4 + u_4 \end{aligned} \tag{12}$$

where y_1, y_2, y_3, y_4 are the *state* variables, a, b, c, d are constant, positive parameters of the system and u_1, u_2, u_3, u_4 are the active controls.

For the GPS of (11) and (12), the synchronization errors are defined as

$$e_i = y_i - \alpha_i x_i, \quad (i = 1, 2, 3, 4) \tag{13}$$

where the scales $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are real numbers.

A simple calculation yields the error dynamics

$$\begin{aligned} \dot{e}_1 &= a(y_2 - y_1) + y_4 - \alpha_1 \left[p(x_2 - x_1)\right] + u_1 \\ \dot{e}_2 &= -y_1 y_3 + c y_2 - \alpha_2 \left[q x_1 + r x_2 - x_1 x_3 + x_4\right] + u_2 \\ \dot{e}_3 &= y_1 y_2 - b y_3 - \alpha_3 \left[x_2^2 - s x_3\right] + u_3 \\ \dot{e}_4 &= -f y_2 - \alpha_4 \left[-\epsilon x_1\right] + u_4 \end{aligned} \tag{14}$$

We consider the active nonlinear controller defined by

$$\begin{aligned} u_1 &= -a(y_2 - y_1) - y_4 + \alpha_1 \left[p(x_2 - x_1)\right] - k_1 e_1 \\ u_2 &= y_1 y_3 - c y_2 + \alpha_2 \left[q x_1 + r x_2 - x_1 x_3 + x_4\right] - k_2 e_2 \\ u_3 &= -y_1 y_2 + b y_3 + \alpha_3 \left[x_2^2 - s x_3\right] - k_3 e_3 \\ u_4 &= f y_2 + \alpha_4 \left[-\epsilon x_1\right] - k_4 e_4 \end{aligned} \tag{15}$$

where the gains k_1, k_2, k_3, k_4 are positive constants.

Substitution of (15) into (14) yields the closed-loop error dynamics

$$\begin{aligned} \dot{e}_1 &= -k_1 e_1 \\ \dot{e}_2 &= -k_2 e_2 \\ \dot{e}_3 &= -k_3 e_3 \\ \dot{e}_4 &= -k_4 e_4 \end{aligned} \tag{16}$$

We consider the quadratic Lyapunov function defined by

$$V(e) = \frac{1}{2}\, e^T e = \frac{1}{2}\, \left(e_1^2 + e_2^2 + e_3^2 + e_4^2\right) \tag{17}$$

which is positive definite on $\mathbb{R}^4$.

Differentiating (17) along the trajectories of the system (16), we get

$$\dot{V}(e) = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 \tag{18}$$

which is a negative definite function on $\mathbb{R}^4$, since k_1, k_2, k_3, k_4 are positive constants.

Thus, by Lyapunov stability theory [20], the error dynamics (16) is globally exponentially stable. Hence, we obtain the following result.

Theorem 2. *The active feedback controller (15) achieves global chaos generalized projective synchronization (GPS) between the non-identical hyperchaotic Cai system (11) and the hyperchaotic Lü system (12).* ■

4.2 Numerical Results

For the numerical simulations, the fourth order Runge-Kutta method is used to solve the two systems of differential equations (11) and (12) with the active controller (15).

The parameters of the hyperchaotic Cai system (11) are taken as

$$p = 27.5, \quad q = 3, \quad r = 19.3, \quad s = 2.9, \quad \epsilon = 3.3$$

The parameters of the hyperchaotic Lü system (12) are taken as

$$a = 36, \quad b = 3, \quad c = 20, \quad d = 1.3$$

The initial values for the master system (11) are taken as

$$x_1(0) = 11, \; x_2(0) = 24, \; x_3(0) = 18, \; x_4(0) = 15$$

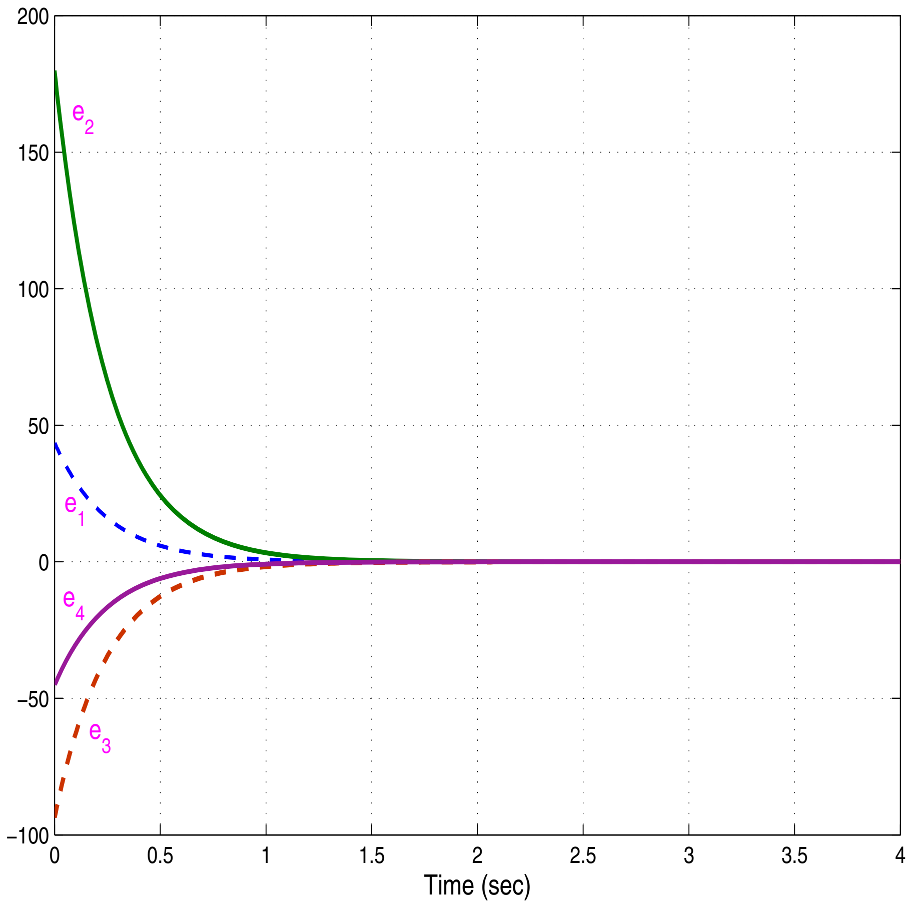

Fig. 4. Time Responses of the Error States of (14)

The initial values for the slave system (12) are taken as

$$y_1(0) = 20,\ y_2(0) = 16,\ y_3(0) = 5,\ y_4(0) = 7$$

The GPS scales α_i are taken as

$$\alpha_1 = -2.15,\ \ \alpha_2 = -6.83,\ \ \ \alpha_3 = 5.49,\ \ \alpha_4 = 3.48$$

We take the state feedback gains as $k_1 = 4,\ k_2 = 4,\ k_3 = 4$ and $k_4 = 4$.

Figure 4 shows the time response of the error states e_1, e_2, e_3, e_4 of the error dynamical system (14) when the active nonlinear controller (15) is deployed.

From this figure, it is clear that all the error states decay to zero exponentially in 1.7 sec and thus, generalized projective synchronization is achieved between the non-identical hyperchaotic Cai system (11) and hyperchaotic Lü system (12).

5 Conclusions

In this paper, we derived active control laws for achieving generalized projective synchronization (GPS) of the following hyperchaotic systems:

(A) Identical hyperchaotic Lü systems (2006)
(B) Non-identical hyperchaotic Cai system (2009) and hyperchaotic Lü system.

The synchronization results (GPS) derived in this paper for the hyperchaotic Lü and hyperchaotic Cai systems [(A) and (B)] have been proved using Lyapunov stability theory. Since Lyapunov exponents are not required for these calculations, the proposed active control method is very effective and suitable for achieving GPS of the hyperchaotic systems addressed in this paper. Numerical simulations are shown to demonstrate the effectiveness of the synchronization results (GPS) derived in this paper.

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