Multipartite Entangled Codewords for Gaussian Channels with Additive Noise and Memory

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Abstract. We study a bosonic Gaussian channel with classical additive noise and memory effects. In particular we consider correlations in the added noise which are asymmetric in the phase space quadratures and show in such a case the usefulness of entangled codewords for reliable communication. We explicitly demonstrate that optimal rates can be achieved with multiparty entangled codewords.

Keywords: Quantum Channels, Error Correction, Multipartite Entanglement.

1 Introduction

The quantum theory of information furnishes a well established framework for characterizing communication via noisy quantum channels [1,2]. To correct errors, a certain amount of redundancy is added by encoding messages using codewords of a certain length n. The transmission rate is the ratio between the length of the message and the length of the codeword. A rate is said to be reliable if, in the limit of long codewords, the errors can be corrected reaching an arbitrary high fidelity. The maximum reliable transmission rate is, by definition, the channel capacity.

There exist several concepts of channel capacities in the framework of quantum information theory, depending on whether classical (bit) or quantum (qubit) information has to be sent (and if pre-shared entanglement is available). Here we consider the case in which quantum information carriers are used to encode and distribute classical information. The approach to the problem of determining the capacity of a quantum channels is in a certain extent similar to its classical counterpart. However, qualitative differences arise once one takes in account the phenomenon of *quantum entanglement*. Indeed, the possibility of encoding information by means of *entangled codewords* is peculiar of the quantum setting and has no classical analogue. Following a conservative approach, one can encode messages into codewords which are not entangled. The maximum reliable rate achievable with unentangled codewords is called the Holevo capacity. The presence of entanglement has led to a milestone problem in quantum information, i.e. the problem of determining whether the Holevo capacity is additive or not. Additivity of the Holevo capacity would imply that entanglement cannot be useful for achieving higher transmission rates. It has been recently proven [3] that the Holevo capacity is *not* additive, hence leading to the actual possibility of achieving higher transmission rates by means of entangled codewords.

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A. Sergienko, S. Pascazio, and P. Villoresi (Eds.): QuantumCom 2009, LNICST 36, pp. 26–33, 2010.

Earlier studies on this subject have focussed on *memoryless* quantum channel, characterized by the property that the noises affecting different channel uses are identical and independent. However, in many physical settings the assumption of memoryless noise may appear rather artificial and one is naturally led to consider quantum channel with memory [4]. In a memory channel, the noises acting at different channel uses are in general nor identical neither independent. For the case of quantum channels with memory, it has been explicitly shown that the use of entangled codewords can indeed increase the transmission rate in the context of continuous Gaussian channels, like lossy bosonic ones [5]. Here we are going to consider another kind of bosonic Gaussian channel, those with classical additive noise [6], and investigate the usefulness of entangled codewords in the presence of memory effects. For such channels, the attention has been initially focussed on short codewords [7,8], and recently the channel capacity has been computed in some cases [9]. Here, after introducing a suitable model of noise correlation, we explicitly show how multipartite entangled codewords can enhance the rate of reliable communication over arbitrary number of channel uses.

2 The Model

We study a model of bosonic Gaussian quantum channel with memory. The physical system in which information is encoded is a quantum harmonic oscillator (e.g. a spatial, spectral or polarization mode of the electromagnetic field). The channel model is the so-called additive noise channel [6]. Upon n uses of the quantum channel, a set of canonical variables $\mathbf{R} := (q_1, \dots q_n, p_1, \dots p_n)$ experience the Heisenberg picture transformation

$$\mathbf{R} \mapsto \mathbf{R} + \Xi, \tag{1}$$

where $\Xi := (X_1, \dots X_2, Y_1, \dots Y_n)$ is a vector of $2n$ real-valued stochastic variables. The noise vectors are taken to be Gaussian distributed. We consider a multivariate noise distribution of the form

$$P(\Xi) \simeq \exp\left[-\frac{1}{2}\left(\Xi\, \mathbb{V}^{-1}\, \Xi^{\mathsf{T}}\right)\right]. \tag{2}$$

In [9] the capacity of such a channel has been computed for a certain choice of the covariance matrix, describing a Markovian noise, showing that the optimal rate is achieved without the use of entangled codewords. Here we slightly modify the model by introducing a certain asymmetry in the noise acting on the quadratures q_k, p_k. We expect that the lack of symmetry corresponds to entangled optimal codewords (see comments in [7] and [9]). The noise covariance matrix is chosen as follows

$$\mathbb{V} = \begin{pmatrix} \mathbb{V}^X & \mathbb{O} \\ \mathbb{O} & \mathbb{V}^Y \end{pmatrix}, \tag{3}$$

where the $n \times n$ covariance matrices $\mathbb{V}^X$, $\mathbb{V}^Y$ have components

$$\mathbb{V}^X_{hk} = \sigma \mu^{|h-k|}, \tag{4}$$
$$\mathbb{V}^Y_{hk} = \sigma \delta_{hk}, \tag{5}$$

with $\mu \in [0,1]$, $\sigma \geq 0$, and we assume $\mathbb{V}^X_{hk} = \sigma\delta_{hk}$ for $\mu = 0$.

Moving to the Schroedinger picture, we describe quantum states of n bosonic modes by means of the Wigner function. The memory channel under consideration transforms a n mode Wigner function $W^{(n)}(\mathbf{R})$ according to:

$$W^{(n)}(\mathbf{R}) \mapsto \int \prod_{k=1}^{n} dX_k dY_k \, P(\Xi) \, W^{(n)}(\mathbf{R} - \Xi). \tag{6}$$

Let us recall that a state of n bosonic modes is said to be Gaussian if its Wigner function is Gaussian in the canonical variables $\mathbf{R}$. Analogously, a quantum channel is said to be Gaussian if it transforms Gaussian states to Gaussian states. From Eq. (6), using the fact that the noise variables are distributed according to the Gaussian in Eq. (2), it follows that the channel under consideration is indeed a Gaussian channel.

Clearly, in this model the memory effects in the quantum channel come from the correlations among the noise variables X_k. The limiting case of memoryless channel is obtained for $\mu = 0$, when the multivariate distribution in Eq. (2) factorizes as the product of independent and identical Gaussian distributions, each with variance σ. We can indeed consider the parameter σ as quantifying the amount of noise present at each channel use and $\mu \in [0, 1]$ as a memory parameter. In order to avoid unphysical results, it is customary to impose suitable constraints on the maximum energy carried by the encoding codewords. In the bosonic setting, it is natural to impose a constraint in the number of field excitations in average per mode, i.e.

$$\left\langle \frac{1}{n} \sum_{k=1}^{n} \frac{q_k^2 + p_k^2}{2} \right\rangle \leq N + \frac{1}{2}, \tag{7}$$

where the average is over codewords at the channel input. For any given σ and N, we can introduce the signal-to-noise ratio $SNR = N/\sigma$.

3 Block Encoding/Decoding Schemes

The classical capacity of a quantum channel is the maximum rate of reliable communication, where the maximum is over all possible ways of encoding classical information into quantum states and all possible measurements that can be performed at the channel output to reconstructed the encoded signal. Here we consider the maximum rate that can be achieved with a given choice of the measurement to be performed at the channel output, i.e. heterodyne detection. We optimize the transmission rates over a class of block encoding and decoding schemes, and compare their performances for transmitting classical information as function of the memory parameter and of the length of the blocks.

For any n, we consider codewords of $2n$ symbols, described by the vector $\mathbf{C} := (A_1, \ldots A_n, B_1, \ldots B_n) \in \mathbb{R}^{2n}$. Codewords are taken to be distributed according to a multivariate Gaussian distribution with zero mean and covariance matric $\mathbb{V}_c$. Codewords of length $2n$ are encoded in Gaussian states of n bosonic modes, corresponding to a block of successive channel uses. For encoding the codeword $\mathbf{C}$ we use a Gaussian state described by the Wigner function

$$W_{\mathbf{C}}^{(n)}(\mathbf{R}) \simeq \exp\left[-\frac{(\mathbf{R} - \mathbf{C})\mathbb{V}_{in}^{-1}(\mathbf{R} - \mathbf{C})^{\mathsf{T}}}{2} \right]. \tag{8}$$

These states can be obtained by applying displacement operators on a fiducial Gaussian state with zero mean and covariance matrix $\mathbb{V}_{in}$. This kind of Gaussian encoding can be proven to be optimal for several Gaussian channels, due to the recently proven *minimum output entropy conjecture* [10].

Regarding the decoding part, we consider ideal heterodyne measurement to be used to decode the classical information at the output of the quantum channel. For an encoding block of length n, heterodyne measurement is performed at the corresponding n output modes. For a given encoded codeword $\mathbf{C}$, the state of the n output modes is described by the Wigner function

$$W^{(n)}_{\mathbf{C},out}(\mathbf{R}) \simeq \exp\left[-\frac{(\mathbf{R}-\mathbf{C})(\mathbb{V}_{in}+\mathbb{V})^{-1}(\mathbf{R}-\mathbf{C})^{\mathsf{T}}}{2}\right] \tag{9}$$

(this equation has been derived inserting (2), (8) into (6)), and the ensemble state, averaged over all possible input codewords, is described by the Wigner function

$$W^{(n)}_{out}(\mathbf{R}) \simeq \exp\left[-\frac{\mathbf{R}(\mathbb{V}_{in}+\mathbb{V}+\mathbb{V}_c)^{-1}\mathbf{R}^{\mathsf{T}}}{2}\right]. \tag{10}$$

For each of the n output modes, heterodyne detection provides a joint measurement of both the quadratures. From Eq. (10) it follows that the probability of obtaining the vector $\mathbf{c} := (a_1, \dots a_n, b_1, \dots b_n)$ as output of an ideal measurement is given by the Gaussian distribution

$$P(\mathbf{c}) \simeq \exp\left[-\frac{\mathbf{c}(\mathbb{V}_{in}+\mathbb{V}+\mathbb{V}_c+\mathbb{I}/2)^{-1}\mathbf{c}^{\mathsf{T}}}{2}\right], \tag{11}$$

where the term $\mathbb{I}/2$, proportional to the unit matrix, accounts for the uncertainty principle (i.e. it is the noise introduced by the heterodyne measurement). Analogously from Eq. (9) the conditional probability of measuring $\mathbf{c}$, given that the codeword $\mathbf{C}$ was sent, is the Gaussian

$$P(\mathbf{c}|\mathbf{C}) \simeq \exp\left[-\frac{(\mathbf{c}-\mathbf{C})(\mathbb{V}_{in}+\mathbb{V}+\mathbb{I}/2)^{-1}(\mathbf{c}-\mathbf{C})^{\mathsf{T}}}{2}\right]. \tag{12}$$

The Shannon entropy of the decoded codeword $\mathbf{c}$, measured in bits, is hence given by the expression

$$H(\mathbf{c}) = \frac{1}{2}\log_2\left[\det\left(\mathbb{V}_{in}+\mathbb{V}+\mathbb{V}_c+\mathbb{I}/2\right)\right]. \tag{13}$$

Analogously, the conditional entropy is

$$H(\mathbf{c}|\mathbf{C}) = \frac{1}{2}\log_2\left[\det\left(\mathbb{V}_{in}+\mathbb{V}+\mathbb{I}/2\right)\right], \tag{14}$$

independently on the value of the encoded codeword $\mathbf{C}$.

In conclusion we can write the rate of transmission, measured in bits per channel use, for a block encoding of length n and heterodyne decoding. The transmission rate is given by the mutual information per channel use

$$F_n = \frac{I(\mathbf{c};\mathbf{C})}{n} = \frac{H(\mathbf{c}) - H(\mathbf{c}|\mathbf{C})}{n} = \frac{1}{2n}\log_2\left[\frac{\det(\mathbb{V}_{in} + \mathbb{V} + \mathbb{V}_c + \mathbb{I}/2)}{\det(\mathbb{V}_{in} + \mathbb{V} + \mathbb{I}/2)}\right]. \quad (15)$$

4 Optimal Transmission Rates

The aim of this section is to maximize, for any given n, the transmission rate in Eq. (15) by optimizing over all possible covariance matrices $\mathbb{V}_c$ and $\mathbb{V}_{in}$. We will find that for $n > 1$ the optimal encoding strategy involves states which are entangled among the bosonic modes belonging to the same encoding block.

First of all, let us consider the case $n = 1$. In this case the encoding blocks are made of one bosonic mode, hence only separable states are used to encode the codewords. The noise covariance matrix is 2×2, and reads:

$$\mathbb{V} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}. \quad (16)$$

The quantum state used to encode information has covariance matrix

$$\mathbb{V}_{in} = \begin{pmatrix} i_q & i_d \\ i_d & i_p \end{pmatrix}, \quad (17)$$

where, to satisfy the uncertainty principle, the condition $i_q i_p - i_d^2 \geq 1/4$ has to imposed. Analogously, the covariance matrix of the codewords is

$$\mathbb{V}_c = \begin{pmatrix} c_q & c_d \\ c_d & c_p \end{pmatrix}. \quad (18)$$

The maximization on the encoding schemes is under the constraint

$$\frac{i_q + i_p + c_q + c_p}{2} \leq N + 1/2. \quad (19)$$

Using the Lagrange method one obtains that the maximum rate is reached in correspondence of the optimal values $i_d^{opt} = c_d^{opt} = 0$, and $i_q^{opt} = i_p^{opt} = 1/2$, $c_q^{opt} = c_p^{opt} = N$. The corresponding optimal rate is

$$R_1 = \max_{i_q, i_p, i_d, c_q, c_p, c_d} F_1 = \log_2\left(\frac{N + \sigma + 1}{\sigma + 1}\right). \quad (20)$$

Let us move to the case of encoding/decoding blocks of length $n > 1$. To optimize the transmission rate we proceed along the same line of [9]. First notice that, for any n, there exists a $n \times n$ orthogonal matrix $\mathbb{T}$ which diagonalizes the covariance matrix $\mathbb{V}^X$, i.e.

$$\sum_{h,k=1}^{n} \mathbb{T}_{jh} \mathbb{V}^X_{hk} \mathbb{T}_{lk} = \sigma_j \, \delta_{jl}. \quad (21)$$

We define the *collective* noise variables $\tilde{\Xi} = (\tilde{X}_1, \dots \tilde{X}_n, \tilde{Y}_1, \dots \tilde{Y}_n)$, where $\tilde{X}_j := \sum_k \mathbb{T}_{jk} X_k$, $\tilde{Y}_j := \sum_k \mathbb{T}_{jk} Y_k$. This new set of noise variables are hence distributed according to

$$\tilde{P}(\tilde{\Xi}) \simeq \exp\left[-\frac{1}{2}\left(\tilde{\Xi}\,\tilde{\mathbb{V}}^{-1}\,\tilde{\Xi}^{\mathsf{T}}\right)\right], \tag{22}$$

where $\tilde{\mathbb{V}} = \mathrm{diag}(\sigma_1, \dots \sigma_n, \sigma, \dots \sigma)$. We analogously define the collective field variables $\tilde{\mathbf{R}} = (\tilde{q}_1, \dots \tilde{q}_n, \tilde{p}_1, \dots \tilde{p}_n)$, where $\tilde{q}_j := \sum_k \mathbb{T}_{jk} q_k$, $\tilde{p}_j := \sum_k \mathbb{T}_{jk} p_k$. Notice that the latter is a canonical (i.e. symplectic) transformation, moreover it preserves the form of the energy constraint which reads

$$\left\langle \frac{1}{n} \sum_{j=1}^{n} \frac{\tilde{q}_j^2 + \tilde{p}_j^2}{2} \right\rangle \leq N + \frac{1}{2}. \tag{23}$$

Finally, we define the codewords $\tilde{\mathbf{C}} = (\tilde{A}_1, \dots \tilde{A}_n, \tilde{B}_1, \dots \tilde{B}_n)$ with $\tilde{A}_j := \sum_k \mathbb{T}_{jk} A_k$, $\tilde{B}_j := \sum_k \mathbb{T}_{jk} B_k$. In terms of the collective field variables the encoding Gaussian states have Wigner function of the form

$$\tilde{W}^{(n)}_{\tilde{\mathbf{C}}}(\tilde{\mathbf{R}}) \simeq \exp\left[-\frac{(\tilde{\mathbf{R}} - \tilde{\mathbf{C}})\tilde{\mathbb{V}}_{in}^{-1}(\tilde{\mathbf{R}} - \tilde{\mathbf{C}})^{\mathsf{T}}}{2}\right], \tag{24}$$

and the codewords are distributed according to

$$\tilde{P}(\tilde{\mathbf{C}}) \simeq \exp\left[-\frac{1}{2}\left(\tilde{\mathbf{C}}\,\tilde{\mathbb{V}}_c^{-1}\,\tilde{\mathbf{C}}^{\mathsf{T}}\right)\right]. \tag{25}$$

We conjecture that, in analogy with the $n = 1$ case, the optimal encoding is reached for jointly diagonal covariance matrices. We hence consider the following parametrization: $\tilde{\mathbb{V}}_{in} = \mathrm{diag}(i_{q_1}, \dots i_{q_n}, i_{p_1}, \dots i_{p_n})$, $\tilde{\mathbb{V}}_c = \mathrm{diag}(c_{q_1}, \dots c_{q_n}, c_{p_1}, \dots c_{p_n})$. Using Eq. (15) we have

$$F_n = \sum_{j=1}^{n} \frac{1}{2n} \log_2\left[\frac{(i_{q_j} + c_{q_j} + \sigma_j)(i_{p_j} + c_{p_j} + \sigma)}{(i_{q_j} + \sigma_j)(i_{p_j} + \sigma)}\right]. \tag{26}$$

The latter expression has to be maximized under the energy constraint

$$\sum_{j=1}^{n} \frac{i_{q_j} + c_{q_j} + i_{p_j} + c_{p_j}}{2n} \leq N + \frac{1}{2}. \tag{27}$$

The optimal transmission rate over an encoding block of length n is denoted $R_n = \max_{i_{q_j}, c_{q_j}, i_{p_j}, c_{p_j}} F_n$. For generic values of the parameters N, σ, μ it can be computed numerically. Figure 1 shows the ratio R_n/R_1 as function of n for several values of the memory parameter μ. The ratio expresses the gain in the transmission rate reached by encoding classical information on blocks of length n. As it is shown in the next section the optimal encoding scheme on blocks of length n involves Gaussian states which are entangled over n bosonic modes.

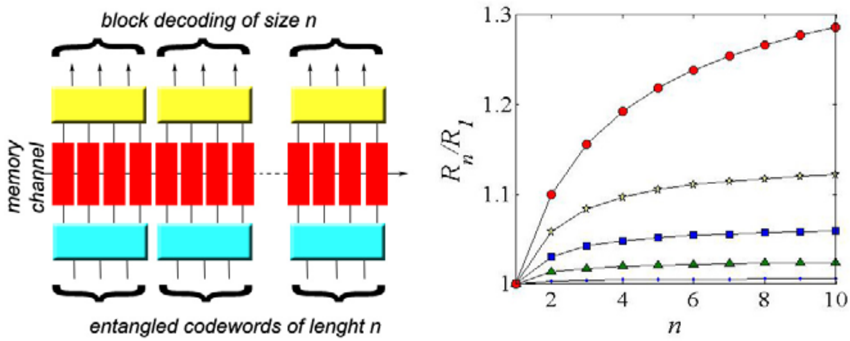

Fig. 1. On the left: block-encoding and decoding for the memory channel. On the right: the gain in the transmission rates using entangled codewords of length n, for several values of the memory parameter: dots: $\mu = 0.2$, triangles: $\mu = 0.4$, squares: $\mu = 0.6$, stars: $\mu = 0.8$, circles: $\mu = 1$. The value of the noise parameter is $\sigma = 1$, the signal-to-noise ratio is $SNR = 3$.

5 Discussion

In terms of the collective field variables $\tilde{\mathbf{R}}$ the covariance matrix of the optimal Gaussian state is diagonal of the form $\tilde{\mathbb{V}}_{in}^{opt} = \mathrm{diag}(i_{q_1}^{opt}, \ldots i_{q_n}^{opt}, i_{p_1}^{opt}, \ldots i_{p_n}^{opt})$. The optimal values $i_{q_j}^{opt}$, $i_{p_j}^{opt}$ can be in general determined numerically. It turns out that the optimal state is pure, satisfying $i_{q_j}^{opt} i_{p_j}^{opt} = 1/4$. Let us now move to the description in terms of the fields variables $\mathbf{R} = (q_1, \ldots q_n, p_1, \ldots p_n)$, where q_k, p_k are the canonical variables describing the kth channel use. This is the natural representation to study the entanglement between different channel uses. The optimal covariance matrix is

$$\mathbb{V}_{in}^{opt} = \begin{pmatrix} \mathbb{T}^{\mathsf{T}} & \mathbb{O} \\ \mathbb{O} & \mathbb{T}^{\mathsf{T}} \end{pmatrix} \tilde{\mathbb{V}}_{in}^{opt} \begin{pmatrix} \mathbb{T} & \mathbb{O} \\ \mathbb{O} & \mathbb{T} \end{pmatrix} . \qquad (28)$$

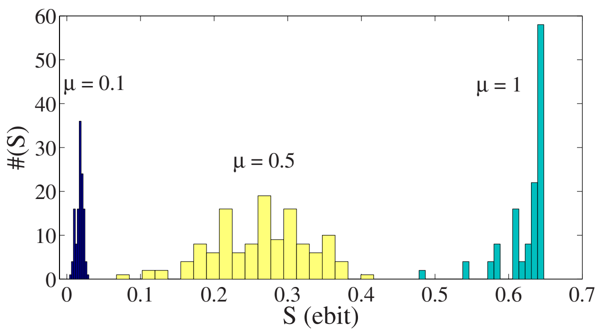

Fig. 2. The plot shows the distribution of the entanglement entropy (measured in *ebit*) over balanced bipartitions for the optimal encoding state of length $n = 10$. The histograms are for several values of the memory parameter μ and $\sigma = 1$, $SNR = 3$.

The optimal state turns out to be entangled among different channel uses. To study the entanglement in the found optimal state it is sufficient to look at the covariance matrix $\mathbb{V}_{in}^{opt}$. For a block encoding of length n, we characterize the multipartite entanglement among different channel uses by means of the distribution of bipartite entanglement among all possible bipartition of the n modes into two subsets of length d, $n-d$ [11]. We estimate the amount of bipartite entanglement using the von Neumann entropy of the reduced state of one of the two subsets (i.e. the entanglement entropy). We hence consider the distribution of the entanglement among all possible subsets of a given length d. As an illustrative example Fig. 2 shows the distribution of bipartite entanglement for $d = [n/2]$ (i.e. balanced bipartitions, $[\cdot]$ indicates the integer part) and for different values of the memory parameter.

6 Conclusion

We have studied the optimal block encoding strategy for a bosonic Gaussian memory channel with additive classical noise, where the decoding measurement is heterodyne detection. A block of length n allows the use of entangled states of n bosonic modes as codewords. We found that the optimal encoding, making use of Gaussian states, is by means of multipartite entangled states. Entanglement in the optimal multipartite encoding states has been characterized by the distribution of the entanglement entropy over all possible balanced bipartition. The optimal codeword states turns out to be separable only for $\mu = 0$ (or $n = 1$). For an encoding block of length n the mean (and other quantifiers, e.g. the mode) of the distribution of the entanglement entropy increases with the memory parameter. For a given value of the memory parameter, the rate of reliable communication increases monotonically with the length of the encoding block.

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