Entanglement Purification with Hybrid Systems

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Abstract. We propose a scheme for continuous variable entanglement purification using the interaction of two-mode squeezed beams with cold trapped atomic ensembles. In the limit of very large number of atoms, a collective atomic ensemble excitation becomes a continuous quantity. Measurement of populations in the two lower levels is of non-Gaussian nature that is necessary for the entanglement purification. We foresee the use of this scheme for implementation of a continuous variable quantum repeater.

Keywords: Entanglement manipulation and characterization, quantum description of light-matter interaction, quantum state engineering and measurements.

1 Introduction

The property of entanglement lies at the heart of quantum information processing (quantum communication, teleportation, cryptography) and quantum mechanics in general. Continuous variable (CV) entangled states are very attractive in this regard due to the relative easiness in their generation and manipulation, as compared to their discrete counterpart [1]. However, the degree of entanglement between two distant sites of a quantum network usually decreases exponentially with the length of a connecting channel, calling for implementation of some entanglement purification procedure [2]. This involves purifying mixed entangled states, such as two-mode squeezed states, after their two halves have been distributed through noisy channels. It is well-known that a procedure involving Gaussian local operations and classical communications does not lead to the enhancement of the initial entanglement [3], so non-Gaussian local operations must necessarily make part of a protocol [4].

Here, a scheme for the implementation of CV entanglement purification is proposed, using two-mode squeezed beams and two independent cold, three-level atomic ensembles in a Λ configuration (see Fig. 1). Input is provided by two-mode squeezed light with arbitrary, but less than perfect, squeezing and each mode impinges on one atomic ensemble. Squeezed beams excite one of the two Λ transitions, with the other being excited by another travelling wave field.

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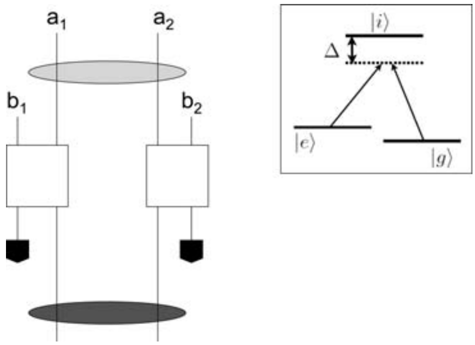

Fig. 1. Proposed setup: Two entangled light fields a_1 and a_2 are impinging on two independent but identical cold atomic ensembles describable by bosonic fields b_1 and b_2. After local interactions, the b fields are subjected to measurement while the a fields should emerge more entangled. The box shows the Λ configuration of atomic transitions. One of the two transitions is driven by the a fields and the other by classical fields.

In the limit of very large number of atoms (continuous limit), a collective ensemble excitation becomes a CV quantity (a bosonic mode) and this collective CV quantity interacts with the optical modes realizing the first part of a purification protocol. The second important part is a non-Gaussian measurement, implemented through the measurement of population of the atomic ground state. It is shown that such a protocol increases two-mode squeezing (entanglement) in the output modes with respect to that of the input ones.

2 The Model

Consider two radiation modes described by field operators a_1 and a_2 (see Fig. 1). Each of the fields drives the $|e\rangle \to |i\rangle$ transition of the ensemble of three level atoms in Λ configuration. The two atomic ensembles are considered to be independent but identical, so the couplings of each field with the atoms are taken to be equal $g_1 = g_2 = g$. Transitions $|g\rangle \to |i\rangle$ are driven by classical fields with Rabi frequencies $\Omega_{1,2}$. The coupling of the fields to the atoms is assumed to be off resonant with a large detuning Δ. The Hamiltonian of the total system can be written as ($\hbar = 1$ throughout the paper)

$$
\begin{aligned}
H = \Delta \sum_{l=1}^{2} \sum_{k=1}^{N_l} \sigma_{ii}^{k,l} + i \sum_{l=1}^{2} \sum_{k=1}^{N_l} \left(g^* \sigma_{ie}^{k,l} a_l - g \sigma_{ei}^{k,l} a_l^\dagger \right) \\
+ i \sum_{l=1}^{2} \sum_{k=1}^{N_l} \left(\Omega_l \sigma_{ig}^{k,l} - \Omega_l^* \sigma_{gi}^{k,l} \right) , \qquad (1)
\end{aligned}
$$

where index l labels the radiation mode and the atomic ensemble it impinges on and $\sigma_{nm}^{k,l}$ denotes atomic flip operator for the kth atom (out of total of N_l atoms per ensemble) in the lth ensemble.

In the case of large detuning Δ, the intermediate levels $|i\rangle$ can be adiabatically eliminated and spontaneous emission neglected. Under the collective coupling condition, the interaction shown in Fig. 1 is then described by the following Hamiltonian in the rotating frame [5]

$$H' = i \sum_{l=1}^{2} \frac{\sqrt{N_l}}{\Delta} \left(g^* \Omega_l S_{ge}^{(l)} a_l^\dagger - g \Omega_l^* S_{eg}^{(l)} a_l \right), \tag{2}$$

where $S_{nm}^{(l)} = \sum_{k=1}^{N_l} \sigma_{nm}^{k,l} / \sqrt{N_l}$ denotes the collective atomic operators for the respective ensembles. In this expression, ac-Stark shifts have been neglected as they can be compensated in practice by refining the laser frequency.

In the Holstein-Primakoff representation [6], the collective atomic operators (that are angular momentum operators) can be associated with bosonic creation and annihilation operators via the relation $S_{ge}^{(l)} = (N_l - b_l^\dagger b_l)^{1/2}\, b_l$, where b_l denotes the bosonic annihilation operator satisfying the canonical commutation relations $[b_l, b_k^\dagger] = \delta_{lk}$. In the low excitation limit, the number of atoms transferred to state $|e\rangle$ is small compared to the total number of atoms. The collective atomic operators can then be further approximated by $S_{ge}^{(l)} \approx \sqrt{N_l}\, b_l$. The Hamiltonian (2) then reduces to the following form

$$H_{BS} = i \sum_{l=1}^{2} \mu_l (a_l^\dagger b_l - a_l b_l^\dagger), \tag{3}$$

with $\mu_l = |g||\Omega_l| N_l / \Delta$ (it is possible to choose the phase of the impinging modes and of the driving lasers so that $\arg(g) = \arg(\Omega_l)$). The collective atomic ensemble excitation in the continuous limit of very large number of atoms becomes a CV quantity. The effective coupling coefficients μ_l are in general different because it is extremely difficult to prepare two large atomic ensembles with exactly the same numbers of atoms. However, one can obtain $\mu_1 = \mu_2$ by e.g. adjusting the coupling fields Ω_l accordingly. Hence, in the following it will be assumed that $\mu_1 = \mu_2 = \mu$.

The effective Hamiltonian of Eq. (3) has the familiar form reminiscent of the beam splitter interaction [7]. The coupling constant μ can be thought to be the rotation angle for the two modes a_l and b_l.

In the starting Hamiltonian (1), we could have exchanged the atomic transitions coupled to the fields. That is, we could have considered each of the quantum field driving the $|g\rangle \rightarrow |i\rangle$ transition, whilst classical fields driving transitions $|e\rangle \rightarrow |i\rangle$. The total Hamiltonian in such a case would read

$$\begin{aligned} H = {} & \Delta \sum_{l=1}^{2} \sum_{k=1}^{N_l} \sigma_{ii}^{k,l} + i \sum_{l=1}^{2} \sum_{k=1}^{N_l} \left(g^* \sigma_{ig}^{k,l} a_l - g \sigma_{gi}^{k,l} a_l^\dagger \right) \\ & + i \sum_{l=1}^{2} \sum_{k=1}^{N_l} \left(\Omega_l \sigma_{ie}^{k,l} - \Omega_l^* \sigma_{ei}^{k,l} \right), \end{aligned} \tag{4}$$

Then, by repeating the previous steps we could have arrived to the following effective Hamiltonian

$$H_{PA} = i \sum_{l=1}^{2} \nu_l (a_l^\dagger b_l^\dagger - a_l b_l), \tag{5}$$

with $\nu_l = |g||\Omega_l|N_l/\Delta$ (and choosing again phases so that $\arg(g) = \arg(\Omega_l)$). Again we could assume, without loss of generality, $\nu_1 = \nu_2 = \nu$.

The effective Hamiltonian of Eq. (5) has the familiar form reminiscent of the nondegenerate parametric amplifier [7]. The coupling constant ν can be thought to be proportional to the amplitude of the pump and second-order susceptibility.

3 The Dynamics

We are now going to study the dynamics of the system induced by the Hamiltonian H_{BS}, while that of Hamiltonian H_{PA} will be discussed later on. The Liouville equation $\dot{\rho}(t) = -i[H_{BS}, \rho(t)]$ for the state of the four modes a_1, a_2, b_1, b_2 can be transformed into a partial differential equation for the normally ordered characteristic function

$$\chi(\boldsymbol{\eta}, t) = \mathrm{Tr}\left\{ \rho(t) \prod_i e^{\eta_i \sigma_i^\dagger} \prod_i e^{\eta_i^* \sigma_i} \right\}, \tag{6}$$

with $\boldsymbol{\sigma} = (a_1, a_2, b_1, b_2)$ the vector of bosonic operators and $\boldsymbol{\eta} = (\alpha_1, \alpha_2, \beta_1, \beta_2)$ the vector of corresponding classical complex variables.

The equation of motion for the characteristic function becomes [7]

$$\begin{aligned} \frac{\partial \chi(\boldsymbol{\eta}, t)}{\partial t} = \mu \Bigg(& \alpha_1 \frac{\partial}{\partial \beta_1} + \alpha_1^* \frac{\partial}{\partial \beta_1^*} + \alpha_2 \frac{\partial}{\partial \beta_2} + \alpha_2^* \frac{\partial}{\partial \beta_2^*} \\ & -\beta_1 \frac{\partial}{\partial \alpha_1} - \beta_1^* \frac{\partial}{\partial \alpha_1^*} - \beta_2 \frac{\partial}{\partial \alpha_2} - \beta_2^* \frac{\partial}{\partial \alpha_2^*} \Bigg) \chi(\boldsymbol{\eta}, t). \end{aligned} \tag{7}$$

Let us consider as initial condition a two-mode squeezed thermal state for a_1, a_2 and vacuum state for b_1, b_2, (which means that all the atoms in the ensemble are in the ground state $|g\rangle$), that is

$$\chi(\boldsymbol{\eta}, 0) = \exp\left\{ -\left(\frac{\lambda^2}{1-\lambda^2} + n_T \right) \left(|\alpha_1|^2 + |\alpha_2|^2 \right) \right\} \exp\left\{ \frac{\lambda}{1-\lambda^2} \left(\alpha_1 \alpha_2 + \alpha_1^* \alpha_2^* \right) \right\}, \tag{8}$$

with n_T number of thermal photons and λ the two-mode squeezing parameter. Notice that for $n_T = 0$ the state for the modes a_1, a_2 is a two-mode squeezed state $\sqrt{1-\lambda^2} \sum_n \lambda^n |n\rangle_{a_1} |n\rangle_{a_2}$.

Equation (7) can be solved by assuming a Gaussian solution of the form

$$\begin{aligned} \chi(\boldsymbol{\eta}, t) = \exp\big[& -A(|\alpha_1|^2 + |\alpha_2|^2) - B(|\beta_1|^2 + |\beta_2|^2) \\ & + A_{12} \left(\alpha_1 \alpha_2 + \alpha_1^* \alpha_2^* \right) + B_{12} \left(\beta_1 \beta_2 + \beta_1^* \beta_2^* \right) \\ & + E \left(\alpha_1 \beta_1^* + \alpha_1^* \beta_1 + \alpha_2 \beta_2^* + \alpha_2^* \beta_2 \right) \\ & + C \left(\alpha_1 \beta_2 + \alpha_1^* \beta_2^* + \alpha_2 \beta_1 + \alpha_2^* \beta_1^* \right) \big], \end{aligned} \tag{9}$$

and transforming it to a set of linear differential equation for the time dependent coefficients $A, B, E, A_{12}, B_{12}, C$. Then, one can determine these coefficients as

$$A(t) = \left(\lambda^2 + n_T - \lambda^2 n_T\right) \frac{\cos^2 \mu t}{1-\lambda^2}, \tag{10a}$$

$$B(t) = \left(\lambda^2 + n_T - \lambda^2 n_T\right) \frac{\sin^2 \mu t}{1-\lambda^2}, \tag{10b}$$

$$E(t) = \frac{1}{2}\left(\lambda^2 + n_T - \lambda^2 n_T\right) \frac{\sin(2\mu t)}{1-\lambda^2}, \tag{10c}$$

$$A_{12}(t) = \lambda \frac{\cos^2 \mu t}{1-\lambda^2}, \tag{10d}$$

$$B_{12}(t) = \lambda \frac{\sin^2 \mu t}{1-\lambda^2}, \tag{10e}$$

$$C(t) = \frac{-\lambda}{2} \frac{\sin(2\mu t)}{1-\lambda^2}. \tag{10f}$$

4 Measurement

The physical scheme based on the interaction with the collective atomic ensemble is now supplemented by the simplest example of non-Gaussian measurement. This is the Fock basis measurement on modes b_1, b_2 that checks for the presence or absence of excitations in each mode. It corresponds to a measurement of populations in the two lower levels of the atomic systems (see Fig. 1). For each ensemble, laser pulse is applied on state $|e\rangle$ in order to observe quantum jumps. If the jump is not observed then the state $|e\rangle$ is not populated, i.e. all the population resides in $|g\rangle$. This situation corresponds to the projection $|0\rangle_b \langle 0|$. Otherwise, the measurement results in the projection $\mathbb{I}_b - |0\rangle_b \langle 0|$. The described procedure can be formalised by introducing the positive operator valued measure (POVM)

$$\{E_{x_1=0},\ E_{x_1=1}\} \equiv \{|0\rangle_{b_1} \langle 0|\,,\ \mathbb{I}_{b_1} - |0\rangle_{b_1} \langle 0|\}, \tag{11a}$$

$$\{E_{x_2=0},\ E_{x_2=1}\} \equiv \{|0\rangle_{b_2} \langle 0|\,,\ \mathbb{I}_{b_2} - |0\rangle_{b_2} \langle 0|\}. \tag{11b}$$

Defining $\boldsymbol{x} = (x_1,\ x_2)$, the conditioned state will be

$$\rho_{\boldsymbol{x}} = \frac{1}{p_{\boldsymbol{x}}} \{E_{\boldsymbol{x}} \rho E_{\boldsymbol{x}}^\dagger\} \tag{12}$$

with probablility

$$p_{\boldsymbol{x}} = \mathrm{Tr}\{E_{\boldsymbol{x}} \rho E_{\boldsymbol{x}}^\dagger\}, \tag{13}$$

meaning that the measurement procedure results (in average) in a mixture $\sum_{\boldsymbol{x}} p_{\boldsymbol{x}} \rho_{\boldsymbol{x}} = \sum_{\boldsymbol{x}} E_{\boldsymbol{x}} \rho E_{\boldsymbol{x}}^\dagger$.

The measurement procedure can be expressed in the language of characteristic function. First notice that the density operator for modes a_1, a_2, b_1, b_2 can be written in terms of the normally ordered characteristic function as

$$\rho = \int \frac{d^2\alpha_1}{\pi} \frac{d^2\alpha_2}{\pi} \frac{d^2\beta_1}{\pi} \frac{d^2\beta_2}{\pi} \chi(\alpha_1, \alpha_2, \beta_1, \beta_2) D^\dagger(\alpha_1) D^\dagger(\alpha_2) D^\dagger(\beta_1) D^\dagger(\beta_2) \times e^{(-|\alpha_1|^2 - |\alpha_2|^2 - |\beta_1|^2 - |\beta_2|^2)/2}, \tag{14}$$

where D denotes the displacement operator [7].

Then, let us consider the following case

$$\begin{aligned}\rho_{01} &= \frac{1}{p_{01}}\mathrm{Tr}_{b_1 b_2}\left\{\rho E_{01}^{\dagger}\right\} \\ &= \frac{1}{p_{01}}\int \frac{d^2\alpha_1}{\pi}\frac{d^2\alpha_1}{\pi}\frac{d^2\beta_1}{\pi}\frac{d^2\beta_2}{\pi}\chi(\alpha_1,\alpha_2,\beta_1,\beta_2)\mathrm{Tr}_{b_1}\left\{D^{\dagger}(\beta_1)|0\rangle_{b_1}\langle 0|\right\} \\ &\quad\times \mathrm{Tr}_{b_2}\left\{D^{\dagger}(\beta_2)\left[I_{b_2} - |0\rangle_{b_2}\langle 0|\right]\right\}D^{\dagger}(\alpha_1)D^{\dagger}(\alpha_2)e^{(-|\alpha_1|^2-|\alpha_2|^2-|\beta_1|^2-|\beta_2|^2)/2}, \\ &= \int \frac{d^2\alpha_1}{\pi}\frac{d^2\alpha_1}{\pi}\left\{\frac{1}{p_{01}}\int \frac{d^2\beta_1}{\pi}\frac{d^2\beta_2}{\pi}\chi(\alpha_1,\alpha_2,\beta_1,\beta_2)\right. \\ &\qquad\left.\times\left[\langle\beta_1|0\rangle\left(\pi\delta^2(\beta_2) - \langle\beta_2|0\rangle\right)\right]e^{(-|\beta_1|^2-|\beta_2|^2)/2}\right\} \\ &\qquad\times D^{\dagger}(\alpha_1)D^{\dagger}(\alpha_2)e^{(-|\alpha_1|^2-|\alpha_2|^2)/2}. \end{aligned} \tag{15}$$

Now the quantity inside the curly brackets can be considered as the normally ordered characteristic function corresponding to the (normalized) state ρ_{01}. It can be rewritten as

$$\begin{aligned}\chi_{01}(\alpha_1,\alpha_2) &= \frac{1}{p_{01}}\int \frac{d^2\beta_1}{\pi}\frac{d^2\beta_2}{\pi}\chi(\alpha_1,\alpha_2,\beta_1,\beta_2) \\ &\qquad\times\left[\langle\beta_1|0\rangle\left(\pi\delta^2(\beta_2) - \langle\beta_2|0\rangle\right)\right]e^{(-|\beta_1|^2-|\beta_2|^2)/2} \\ &= \frac{1}{p_{01}}\int \frac{d^2\beta_1}{\pi}\chi(\alpha_1,\alpha_2,\beta_1,\beta_2=0)e^{-|\beta_1|^2} \\ &\quad - \frac{1}{p_{01}}\int \frac{d^2\beta_1}{\pi}\frac{d^2\beta_2}{\pi}\chi(\alpha_1,\alpha_2,\beta_1,\beta_2)e^{-|\beta_1|^2-|\beta_2|^2} \end{aligned} \tag{16a}$$

The value of the probability p_{01} comes from the normalization condition

$$\chi_{01}(\alpha_1=0,\alpha_2=0) = 1. \tag{17}$$

Analogously we obtain for the other conditional states

$$\begin{aligned}\chi_{10}(\alpha_1,\alpha_2) &= \frac{1}{p_{10}}\int \frac{d^2\beta_2}{\pi}\chi(\alpha_1,\alpha_2,\beta_1=0,\beta_2)e^{-|\beta_2|^2} \\ &\quad - \frac{1}{p_{10}}\int \frac{d^2\beta_1}{\pi}\frac{d^2\beta_2}{\pi}\chi(\alpha_1,\alpha_2,\beta_1,\beta_2)e^{-|\beta_1|^2-|\beta_2|^2} \end{aligned} \tag{18a}$$

$$\begin{aligned}\chi_{11}(\alpha_1,\alpha_2) &= \frac{1}{p_{11}}\chi(\alpha_1,\alpha_2,\beta_1=0,\beta_2=0) \\ &\quad - \frac{1}{p_{11}}\int \frac{d^2\beta_1}{\pi}\chi(\alpha_1,\alpha_2,\beta_1,\beta_2=0)e^{-|\beta_1|^2} \\ &\quad - \frac{1}{p_{11}}\int \frac{d^2\beta_2}{\pi}\chi(\alpha_1,\alpha_2,\beta_1=0,\beta_2)e^{-|\beta_2|^2} \\ &\quad + \frac{1}{p_{11}}\int \frac{d^2\beta_1}{\pi}\frac{d^2\beta_2}{\pi}\chi(\alpha_1,\alpha_2,\beta_1,\beta_2)e^{-|\beta_1|^2-|\beta_2|^2} \end{aligned} \tag{18b}$$

$$\chi_{00}(\alpha_1,\alpha_2) = \frac{1}{p_{00}}\int \frac{d^2\beta_1}{\pi}\frac{d^2\beta_2}{\pi}\chi(\alpha_1,\alpha_2,\beta_1,\beta_2)e^{-|\beta_1|^2-|\beta_2|^2} \tag{18c}$$

The above expressions can be calculated from the integral

$$I(\alpha_1,\alpha_2;u,v)=\int d(u\beta_1)d(v\beta_2)\chi(\alpha_1,\alpha_2,\beta_1,\beta_2)e^{-|\beta_1|^2-|\beta_2|^2}, \tag{19}$$

where the measures are defined as follows

$$d(u\beta_1)=\begin{cases} d^2\beta_1 & u=1 \\ \pi\delta^2(\beta_1)d^2\beta_1 & u=0 \end{cases}, \tag{20a}$$

$$d(v\beta_2)=\begin{cases} d^2\beta_2 & v=1 \\ \pi\delta^2(\beta_2)d^2\beta_2 & v=0 \end{cases}. \tag{20b}$$

In summary we can write

$$\chi_{0,0}(\alpha_1,\alpha_2)=\frac{1}{p_{00}}I(\alpha_1,\alpha_2;1,1), \tag{21a}$$

$$\chi_{0,1}(\alpha_1,\alpha_2)=\frac{1}{p_{01}}\left[I(\alpha_1,\alpha_2;1,0)-I(\alpha_1,\alpha_2;1,1)\right], \tag{21b}$$

$$\chi_{1,0}(\alpha_1,\alpha_2)=\frac{1}{p_{10}}\left[I(\alpha_1,\alpha_2;0,1)-I(\alpha_1,\alpha_2;1,1)\right], \tag{21c}$$

$$\begin{aligned}\chi_{1,1}(\alpha_1,\alpha_2)=\frac{1}{p_{11}}&\left[I(\alpha_1,\alpha_2;0,0)-I(\alpha_1,\alpha_2;1,0)\right.\\ &\left.-I(\alpha_1,\alpha_2;0,1)+I(\alpha_1,\alpha_2,1,1)\right].\end{aligned} \tag{21d}$$

The value of the probability $p_{x_1x_2}$ comes from the normalization condition

$$\chi_{x_1x_2}(\alpha_1=0,\alpha_2=0)=1. \tag{22}$$

The integrals (19) can be easily evaluated and they result:

$$I(\alpha_1,\alpha_2;0,0)=\exp\left[-A\left(|\alpha_1|^2+|\alpha_2|^2\right)+A_{12}\left(\alpha_1\alpha_2+\alpha_1^*\alpha_2^*\right)\right], \tag{23a}$$

$$\begin{aligned}I(\alpha_1,\alpha_2;0,1)=\frac{1}{B+1}\exp&\left[-\left(A-\frac{C'^2}{B+1}\right)|\alpha_1|^2-\left(A-\frac{E^2}{B+1}\right)|\alpha_2|^2\right.\\ &\left.+\left(A_{12}+\frac{C'E}{B+1}\right)\left(\alpha_1\alpha_2+\alpha_1^*\alpha_2^*\right)\right],\end{aligned} \tag{23b}$$

$$\begin{aligned}I(\alpha_1,\alpha_2;1,0)=\frac{1}{B+1}\exp&\left[-\left(A-\frac{E^2}{B+1}\right)|\alpha_1|^2-\left(A-\frac{C^2}{B+1}\right)|\alpha_2|^2\right.\\ &\left.+\left(A_{12}+\frac{CE}{B+1}\right)\left(\alpha_1\alpha_2+\alpha_1^*\alpha_2^*\right)\right],\\
I(\alpha_1,\alpha_2;1,1)=\frac{1}{(B+1)^2-B_{12}^2}\exp&\left[-\left(A-\frac{(B+1)(E^2+C^2)+2B_{12}CE}{(B+1)^2-B_{12}^2}\right)\right.\\ &\times\left(|\alpha_1|^2+|\alpha_2|^2\right)\\ &+\left(A_{12}+\frac{2(B+1)CE+B_{12}(C^2+E^2)}{(B+1)^2-B_{12}^2}\right)\\ &\left.\times\left(\alpha_1\alpha_2+\alpha_1^*\alpha_2^*\right)\right],\end{aligned} \tag{23c}$$

Using Eqs.(21), (22) and (23) we also derive

$$p_{0,0} = I(0,0;1,1) = \frac{1}{(B+1)^2 - B_{12}^2}, \tag{24a}$$

$$p_{0,1} = I(0,0;1,0) - I(0,0;1,1) = \frac{1}{B+1} - \frac{1}{(B+1)^2 - B_{12}^2}, \tag{24b}$$

$$p_{1,0} = I(0,0;0,1) - I(0,0;1,1) = \frac{1}{B+1} - \frac{1}{(B+1)^2 - B_{12}^2}, \tag{24c}$$

$$\begin{aligned} p_{1,1} &= I(0,0;0,0) - I(0,0;1,0) - I(0,0;0,1) + I(0,0;1,1) \\ &= 1 - \frac{2}{B+1} + \frac{1}{(B+1)^2 - B_{12}^2}. \end{aligned} \tag{24d}$$

5 Entanglement Purification

Entanglement enhancement at the output is expected for conditioned non-Gaussian states (corresponding to non vacuum measurement outcomes). However, besides being non-Gaussian states these are also non pure state and their entanglement quantification is rather cumbersome. We adopt as an operational measure the teleportation fidelity [8]. That is, the conditional output state is assumed to be initially shared by Alice and Bob for the standard CV teleportation protocol [9]. Then, the average fidelity F for teleporting a coherent state is calculated. Actually, we get from Ref.[9], using the faltung theorem for Fourier transforms,

$$F = \int \frac{d^2\xi}{\pi} |\Phi^{in}(\xi)|^2 [\Phi^{ch}(\xi^*, \xi)]^* \tag{25}$$

where Φ stands for symmetrically ordered characteristic function. Hence, we have to intend

$$\Phi^{in}(\xi) = \exp(-|\xi|^2/2), \tag{26}$$

for the input[1] and

$$\Phi^{ch}(\xi^*, \xi) = \chi_{x_1 x_2}(\alpha_1 = \xi^*, \alpha_2 = \xi) \exp(-|\xi|^2), \tag{27}$$

for the channel. Thus Eq.(25) reads

$$F_{x_1 x_2} = \int \frac{d^2\xi}{\pi} e^{-2|\xi|^2} \chi_{x_1 x_2}(\alpha_1 = \xi^*, \alpha_2 = \xi). \tag{28}$$

[1] Since in the expression of the fidelity this appears with the modulus squared, the displacement of a coherent state does not matter.

Using (21), (23), (24) into (28) we finally arrive at

$$F_{0,0} = \frac{1}{p_{00}} \times \frac{1}{(B+1)^2 - B_{12}^2} \times \frac{1}{2+2A-2A_{12}-2\frac{(C+D)^2}{B+1-B_{12}}}, \tag{29a}$$

$$F_{0,1} = \frac{1}{p_{01}} \left[\frac{1}{B+1} \times \frac{1}{2+2A-2A_{12}-\frac{(C+D)^2}{B+1}} - \frac{1}{(B+1)^2 - B_{12}^2} \times \frac{1}{2+2A-2A_{12}-2\frac{(C+D)^2}{B+1-B_{12}}} \right], \tag{29b}$$

$$F_{1,0} = F_{0,1}, \tag{29c}$$

$$F_{1,1} = \frac{1}{p_{11}} \left[\frac{1}{2+2A-2A_{12}} - \frac{2}{B+1} \times \frac{1}{2+2A-2A_{12}-\frac{(C+D)^2}{B+1}} + \frac{1}{(B+1)^2 - B_{12}^2} \times \frac{1}{2+2A-2A_{12}-2\frac{(C+D)^2}{B+1-B_{12}}}, \right], \tag{29d}$$

The above fidelities should be compared with that coming from the initial squeezed thermal state (8).

$$f = \int \frac{d^2\xi}{\pi} e^{-2|\xi|^2} \exp\left\{ -2|\xi|^2 \left(\frac{\lambda^2+\lambda}{1-\lambda^2} + n_T \right) \right\}, \tag{30a}$$

$$= \frac{1-\lambda^2}{2\,(1-\lambda+n_T-\lambda^2 n_T)} \tag{30b}$$

By inspection, it turns out that F_{00}, F_{01} and F_{10} are always below f. On the contrary, F_{11} can be greater than f depending on both μt and λ (besides n_T). However by simply considering $F_{11} - f$ as figure of merit of the entanglement purification process could be misleading. In fact we have to also account for the probability p_{11} of having measurement outcomes 11.

Actually we can introduce an efficiency for the protocol as:

$$\mathcal{E} = \begin{cases} p_{11}\,(F_{11} - f) & if \;\; (F_{11} - f) > 0 \\ 0 & if \;\; (F_{11} - f) \leq 0 \end{cases} \tag{31}$$

This quantity is greater than zero in a wide region of parameters μt and λ for $n_T = 0$ (see Fig.2), clearly showing the possibility of entanglement concentration (we speak about entanglement concentration because for $n_T = 0$ the initial state is already pure). Fig. 2 also manifests the existence of a tradeoff between μt and λ to get the maximum of $\mathcal{E}$. The efficiency is affected by the number of thermal photons, however it remains greater than zero even for $n_T \neq 0$ (see Fig. 3) showing the possibility of entanglement purification (in this case the initial state is not pure). It is worth to remark that $\mathcal{E}$ does not decrease exponentially vs n_T, but there exist a cutoff value of n_T at wich it reaches zero [10].

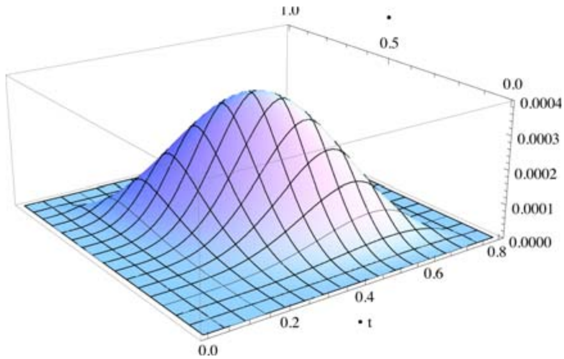

Fig. 2. Efficiency $\mathcal{E}$ vs μt and λ for $n_T = 0$

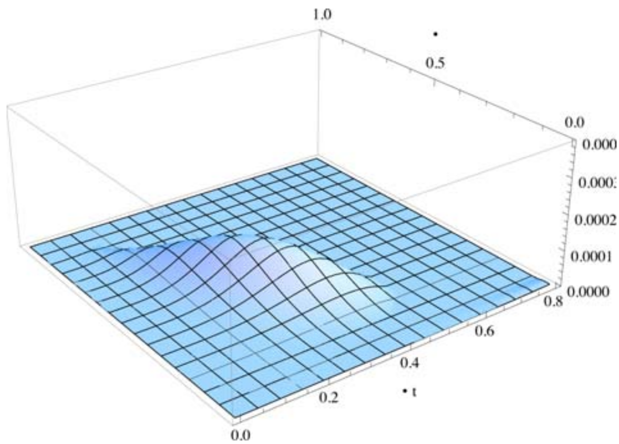

Fig. 3. Efficiency $\mathcal{E}$ vs μt and λ for $n_T = 0.05$

6 Concluding Remarks

We have proposed a realistic scheme for CV entanglement purification. It can be applied to input optical CV fields, which are then sent to two identical and independent atomic ensembles. The effective interaction between the optical and atomic systems is shown to be of the form of either beam splitter or parametric amplifier. Then supplementing the interaction with an appropriate non-Gaussian measurement on the atomic ensembles, we have shown the possibility of enhancing the two-mode squeezing in the output. It turns out that by using the effective Hamiltonian H_{PA} it is impossible to improve the teleportation fidelity [10]. This is an indication of entanglement monogamy in CV setting [11]. On the contrary, by using the effective Hamiltonian H_{BS} we have shown that it is possible to enhance the input entanglement, even if the initial weakly entangled state is not pure. From the perspective of this effective interaction, our protocol shares analogies with the generation of entangled states through photon subtraction mechanism [12].

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