Network Games with Quantum Strategies

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Abstract. Recently, Physics and Computer Science have both contributed to the field of Game Theory. The first by introducing quantum information concepts into the strategy set of the players and the second by giving methods to estimate the efficiency of equilibria and formalizing "network games". This work will aim to be a first step towards the merging of those existing ideas, by studying the behavior of players in a quantum network. It will focus on two classes of network games: formation and congestion games, showing that in some cases "classical" examples of inefficiency fail if players use quantum strategies.

Keywords: Game theory, network games, EWL protocol, quantum strategies, efficiency of equilibria, selfish routing, network formation.

1 Introduction

In game theory, a game is the study of competitive situations, formalized by a set of players (or agents) each one declaring a strategy chosen among a set of possible moves, and each one receiving a payoff based on the overall results. The outcome of the game can be predicted using the Nash Equilibrium concept, that is to say when no player can increase his utility by deviating from his initially declared strategy, while the other players' moves remain fixed. A mixed-strategy NE is similar to that, but instead of choosing a single (pure) strategy players declare a probability distribution on the strategy set.

The EWL protocol, proposed by Eisert et al. in [12], allows players to represent strategies using qubits, and sets up a classically impossible coordination creating *entanglement* between players' qubits. It was limited to 2x2 games (2 players, 2 strategies), but extensions to n-player case were discussed by Benjamin and Hayden in [14], and by many authors in following studies.

Starting from those results, this work extends studies about Network Games by Roughgarden and Tardos in [4], by using and extending EWL protocol and representing the same set of games in a quantum-network scenario. To see the different performances of the quantum and classical versions of the games, efficiency measures will be used: *Price of Stability (PoS)*, the ratio between the *best* possible equilibrium cost and the optimal cost, and *Price of Anarchy (PoA)*, which compares the *worst* possible equilibrium cost with the optimum.

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2 EWL Protocol

The EWL protocol was initially used by Jens Eisert et al. for the discussion of the "Quantum Prisoner Dilemma". It was later used for more general purposes by many others (e.g. [14] and [15]).

For a 2x2 game, the EWL scheme is as follows:

- Initially we have 2 qubits in the state $|\psi\rangle = |0\rangle \otimes |0\rangle = |00\rangle$
- A referee creates entanglement using *entangling gate*

$$\hat{J} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \end{bmatrix}, \tag{1}$$

  leaving $|\psi\rangle$ in the *maximally entangled* state

$$|\psi'\rangle = \hat{J}|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle). \tag{2}$$

  After that, qubits are sent to the respective players, which are forbidden to communicate.
- Each player i applies his own 1-qubit unitary operation s_i and successively sends the qubit back, leaving the state in

$$|\psi''\rangle = (s_1 \otimes s_2)|\psi'\rangle \tag{3}$$

- Finally the referee can determine the outcome by applying $\hat{J}^\dagger$ operation (conjugate transpose of $\hat{J}$) and measuring the qubits in their standard basis.

3 Local Formation Games

Network formation games deal with n selfish agents building a network.

In the *local* case, each player $u \in N$ is a node in a graph, and can decide to build up to $n-1$ direct arcs from himself to any other player. The strategy set for u, S_u, then contains all possible subsets of the $n-1$ incident arcs to u, and strategies can be represented by a binary vector $s_u \in S_u$. The strategy concatenation $s = (s_1, \ldots, s_n)$ forms the outcome network, $G(s)$. The goal for each agent is to have a connected $G(s)$, while taking as low as possible the sum of the distances from the other agents plus the cost α of arcs built, thus minimizing the function:

$$c_u(s) = \alpha n_u(s) + \sum_{v \in N} dist_s(u, v). \tag{4}$$

The social cost of the solution s is:

$$C(s) = \sum_{u,v \in N} dist_s(u, v) + \alpha|E| \tag{5}$$

where $|E|$ is the number of arcs[1] in $G(s)$.

[1] Counting only one α for each arc is a valid assumption, because in an equilibrium solution there is no arc (u, v) that is paid by both interested nodes.

Tardos proves in [4] that PoS can be as large as $\frac{4}{3}$ and PoA is directly dependent on the maximum diameter of the network. Using quantum strategies and entanglement in an extended EWL setting, the Price of Stability turns out to be always 1, and the maximum diameter of an equilibrium is reduced by a $\sqrt{2}$ factor.

3.1 Classical and Quantum Version

As said before, it comes natural to represent players' strategies with strings:

$$s_i = (e_1, e_2, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{n-1}, e_n) \tag{6}$$

of $n-1$ bits. To use the EWL protocol with this setting, we can create entanglement on qubits representing the same arc (u, v) in both s_u and s_v, implementing many 2-players quantum games: qubits $e_v \in s_u$ and $e_u \in s_v$ will be in an entangled state.

In the case with only 2 players, there is the following payoff matrix (C= contribute to the arc , R=refuse to contribute):

	C	**R**
C	$(\alpha+1), (\alpha+1)$	$1, (\alpha+1)$
R	$(\alpha+1), 1$	∞, ∞

The classical Nash Equilibria are (C, R) and (R, C), however with the EWL protocol these are not equilibria anymore, as it is not convenient for a player to play a pure strategy. For every unitary operation a player can chose to play, in fact, the opponent can build a perfect counter-strategy that maximizes his payoff, and vice-versa[2].

Considering Quantum mixed-strategy Nash Equilibrium of the type[3]

$$A^1 \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A^2 \sim \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \tag{7}$$

$$B^1 \sim \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad B^2 \sim \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \tag{8}$$

makes each player the unique builder of the arc with probability $\frac{1}{2}$, with a expected cost $E[c_i] = \frac{\alpha}{2} + 1$. What is changed with respect to classical equilibria is that now the agent who pays for the arc is selected randomly, and there is a decrease of the average cost of "putting an arc into game" (that was classically $\alpha + 1$). This is the key change for the PoS of the multiplayer case.

In fact, in [4], Tardos shows that the efficiency of equilibria change in function of α. The only case in which $PoS > 1$ is when $1 < \alpha < 2$, when the complete graph is optimal but the best equilibrium is a star-graph. This happens because

[2] The situation is like the Prisoner's Dilemma, see [12].

[3] The symbol "$\sim$" means that there are infinite equivalent equilibria obtained with appropriate rotations on the strategies, preserving the mutual optimality property.

no agent has incentive to pay $2 < \alpha + 1 < 3$ for building a direct arc, instead of just using the existing link and paying $dist(u, v) = 2$.

This does not hold using EWL protocol, as an agent u with strategy cost

$$c_u(s) = \alpha n_u(s) + dist_s(u, w) + \sum_{v \in N \setminus \{w\}} dist_s(u, v) \tag{9}$$

and $dist_s(u, w) \geq 2$, can "put into game" (u, w) with a new solution s', expecting

$$E[c_u(s')] = \alpha(n_u(s) + 1) + 1 + \sum_{v \in N \setminus \{w\}} dist_{s'}(u, v) \qquad \leq c_u(s). \tag{10}$$

The complete graph is a Nash Equilibrium with quantum strategies for $\alpha \leq 2$, while nothing changes for $\alpha \geq 2$. This means that with quantum strategies the PoS is always equal to 1.

About the PoA, Tardos shows in [4] that it depends on the maximum diameter[4] of a graph in an equilibrium solution, that is classically at most $2\sqrt{\alpha}$. In a quantum setting, suppose two nodes u and v are at distance $dist(u, v) \geq 2k$ for some k. Putting the arc (u, v) in competition, node u would pay $\frac{\alpha}{2}$ in expectation, while reducing the distance from nodes in the second half of the shortest-path $u - v$ of $(2k - 1) + (2k - 3) + \cdots + 1 = k^2$. The construction of the direct arc is convenient if $dist(u, v) > 2\sqrt{\frac{\alpha}{2}}$, and that's why the maximum diameter of a graph in a Nash Equilibrium with Quantum Strategies is at most $2\sqrt{\frac{\alpha}{2}} = \sqrt{2\alpha}$.

4 Global Formation Games

Does the introduction of EWL protocol define a new game with at least the same performance of the classical counterpart? It turns out that this is not true, and the following is an example.

The *global* case of formation game shows players having source-target pairs and willing to connect those by building arcs, each of them having an associated cost. If more than on player decide to build the same arc, the cost is equally shared. The *PoA* in this setting is up to $\mathcal{H}_k$, the k-th harmonic number, as shown by Tardos and Wexler in [4]. Here it is convenient to use an extended EWL protocol that, given strategy strings, creates entanglement between *all* the qubits representing the same arc. Let's show with a simple example an instance where the classical game performs better than the quantum one (figure 1). The strategy of a player is communicated to the referee using a string (e_{pb}, e_{pvt}) of 2 qubits, one for the private path and one for the public (the shared) one. The two e_{pb} qubits are in an entangled state, while the e_{pvt} ones remain independent.

Player 2 finds convenient to buy the e_{pb} link together with Player 1, who is not interested in it because the loss of $\frac{\epsilon}{2}$. Because of the entanglement, Player 1

[4] The length $max_{(u,v)} d(u, v)$ of the "longest shortest path" between any two graph vertices (u, v) of a graph, where $d(u, v)$ is a graph distance.

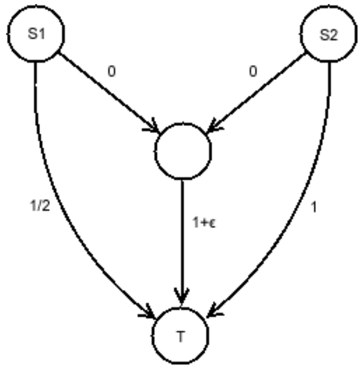

Fig. 1. A simple two-player Global Formation instance

cannot avoid to play the public link, the best he can do for his defense is to play the mixed strategies:

$$S_0 \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad\qquad S_1 \sim \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \tag{11}$$

who act as randomizers for the choice of Player 2. Player 1 will confirm (and follow) or negate the strategy of Player 2, thus ending in both or neither buying the arc with equal probability. Expected cost for Player 1 is then:

$$c_1[s] = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot (\frac{1}{2} + \frac{1+\epsilon}{2}) = \frac{3+\epsilon}{4}. \tag{12}$$

Now Player 2 has 50% probability of losing e_{pb}, so to avoid an infinite expected cost he has to build also e_{pvt}, with expected cost:

$$c_1[s] = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (1 + \frac{1+\epsilon}{2}) = \frac{5+\epsilon}{4}. \tag{13}$$

This is the only class of mixed Nash Equilibria, maximizing expectations for both players, and there are no pure NE. Let's consider the two classical cases:

- If both players would play only the e_{pvt} arcs, Player 2 could increase his payoff by playing Strategy S_1 on e_{pb} and leaving his e_{pvt} to 0. This way he could force Player 1 to follow him on the public link, paying only $\frac{1+\epsilon}{2}$ instead of 1.
- If players would use only e_{pb}, Player 1 could use the same S_1 strategy to "undo" the decision of Player 1, and use only his e_{pvt} arc, saving $\frac{\epsilon}{2}$.

PoS for this instance is then

$$PoS = \frac{\frac{3+\epsilon}{4} + \frac{5+\epsilon}{4}}{1+\epsilon} \simeq 2 \tag{14}$$

while the classical was only $1.5 + \frac{1}{\epsilon}$. Extending the game to n players, PoS becomes $\mathcal{H}_k + \frac{1+\epsilon}{2}$.

5 Congestion Games

In a network congestion game, also called *selfish routing*, players have a source node, a target node and a unit of traffic to transfer, and they have to decide the route to take. The strategy set for each agent consists of all possible paths from source to target, and the utility is the latency time he faces, considering that arcs get congested in function of the number of players using them. The unit of traffic may be splittable or unsplittable and defined as non-atomic or atomic instances. It is possible to calculate an *optimal* solution that minimizes the sum of all latencies faced by the players. However, this solution is not always an equilibrium as some players may find profitable to change route and face less latency for themselves, increasing latency for others. Introducing quantum strategies with a slightly modified EWL protocol, classical examples of inefficiency in selfish routing, present Price of Anarchy and Stability of 1. This does not mean, however, that quantum equilibria are always optimal, as we can show a network inefficient with quantum strategies.

5.1 Quantum Version of Pigou's Example

Consider the network in Figure 2. We have a set of n players, each of them wanting to route a negligible amount of unsplittable traffic from s to t. In players' strategies there are only two paths who have a different congestion function: P_1 of constant cost $c(x) = 1$, and P_2 that has load dependent cost $c(x) = x/n$.

The optimal outcome for this problem is equally splitting the agents between the paths, thus achieving a social cost of

$$cost(OPT) = \frac{n}{2} \cdot 1 + \frac{n}{2} \cdot \frac{1}{2} = \frac{3}{4}n.$$

At Nash Equilibrium all players use the P_2 path, thus

$$cost(NE) = n \cdot \frac{n}{n} = n. \tag{15}$$

We can then calculate the Price of Stability:

$$PoS = \frac{cost(NE)}{cost(OPT)} = \frac{n}{\frac{3}{4}n} = \frac{4}{3}. \tag{16}$$

To use the EWL Protocol with n players, we can create a *Entanglement by couples* (if the number is odd, the last one is free). Fixed the behaviour of other $n-2$ players, supposing that k of them (with $k < n-2$) chose the P_2 path, payoff matrix for the two remaining players is:

	P_1	P_2
P_1	(1,1)	$(1,\frac{k+1}{n})$
P_2	$(\frac{k+1}{n},1)$	$(\frac{k+2}{n},\frac{k+2}{n})$

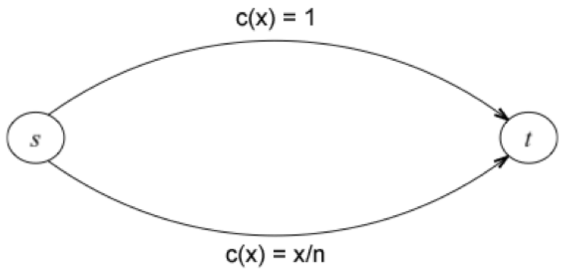

Fig. 2. Pigou's example

Note that classically the dominant strategy P_2 is still valid, and considering entanglement we reach an equilibrium with the same mixed-strategies of equations 7 and 8. The individual expectation is:

$$E[c_j] = \frac{1}{2}\left(\frac{1+\frac{k+1}{n}}{2}\right) + \frac{1}{2}\left(\frac{\frac{k+1}{n}+1}{2}\right) = \frac{1+\frac{k+1}{n}}{2}. \tag{17}$$

Because the remaining $n-2$ agents will face the same situation, this equilibrium has *exactly* one half of the agents taking the P_2 link, with an individual expected cost even lower than the classical:

$$E[c_j] = \frac{1+\frac{k+1}{n}}{2} = \frac{1+\frac{\frac{n}{2}-1+1}{n}}{2} = \frac{1+\frac{1}{2}}{2} = \frac{3}{4}. \tag{18}$$

It achieves optimal social cost and $PoS = 1$ (for this particular network):

$$cost(NE) = \sum_j c_j = n \cdot E[c_j] = \frac{3}{4}n = cost(OPT). \tag{19}$$

This quantum Pigou's example answers one important question: *are Nash Equilibria with Quantum Strategies and EWL protocol equivalent to classical correlated equilibria?* Obviously there is no correlated equilibrium for this example, because a player that is told to take the P_1 path has no incentive to obey. With quantum strategies, instead, players expect a better payoff by *taking the risk* to take the P_1 path. Finally, note that this special example does not imply that every Selfish Routing problem is optimal. In fact we can find a network where $PoS = \frac{8}{7}$, making $C(x) = \frac{2x}{n}$ in the P_2 path of Figure 2.

6 Conclusions

This work is an attempt to adapt the EWL protocol to a multiplayer game with unbounded players and to answers two natural questions about general Quantum Games, with counterexamples. It also shows how the introduction of quantum information technology in a large-scale network may lead to improvement in terms of congestion avoidance and link formation, and it is an invitation for future research and practical applications, such as quantum routers.

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