RealSurf – A Tool for the Interactive Visualization of Mathematical Models

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Abstract. For applications in fine art, architecture and engineering it is often important to visualize and to explore complex mathematical models. In former times there were static models of them collected in museums respectively in mathematical institutes. In order to check their properties for esthetical reasons it could be helpful to explore them interactively in 3D in real time. For the class of implicitly given algebraic surfaces we developed the tool RealSurf. Here we give an introduction to the program and some hints for the design of interesting surfaces.

Keywords: Implicit surface, ray tracing, models for fine art.

1 Introduction

Most of the mathematical institutes of "traditional" german universities hold a collection of mathematical models, see e.g. the Steiner surface in figure 1.

In the book [2] (edited by Gerd Fischer) the authors described several of such models, often made by gypsum, and gave mathematical explanations of them. Unfortunately this mathematical pearl is out of print today.

Because of their symmetries and their interesting esthetical behaviour this kind of objects are collected in museums. Moreover, there is a large interest in fine art for this extraordinary and sometimes beautiful mathematical constructions. See also the book by Glaeser [3] for relations of mathematical objects in fine art, architecture and engineering.

In recent times there is a strong effort in order to visualize mathematical models on a computer by the aid of methods from computer graphics, in particular (implicit) algebraic surfaces. One basic algorithm for implicitly given algebraic surfaces is the so-called ray tracing, see section 3. One of the programs for this kind of visualization is Surf, a ray tracing program for algebraic surfaces, developed by a group around Stephan Endraß, see [1].

Movies for the exploration of certain surfaces were built by a group of Herwig Hauser based on the free renderer Persistence of Vision Raytracer, see Herwig Hauser's homepage [5] for these examples. During 2008, the year of Mathematics in Germany, the Oberwolfach Research Institute of Mathematics provided

F. Huang and R.-C. Wang (Eds.): ArtsIT 2009, LNICST 30, pp. 173–180, 2010.

Fig. 1. The Steiner surface made of gypsum

an exhibition IMAGINARY combining art and algebraic geometry. The catalogue of the exhibition and many interesting programs for the visualization of mathematical problems are available at the homepage of the exhibition [4]. For the interactive visualization of algebraic surfaces there is the program SURFER, based on S. Endraß's program SURF, which creates high-resolution images of the surfaces in a background process. The renderings appear after a certain rendering time depending on the complexity of the surface.

The aim of this report is to introduce the program REALSURF [6] for an interactive visualization of algebraic surfaces in real time. The program allows the construction of mathematically and esthetically interesting surfaces, for instance for the use in fine art, and their interactive exploration. It is based on a recent technique of programming on the graphics processing unit (GPU) with shader languages. It works well for computers with recent NVIDIA graphics cards.

2 An Example

In the following consider the affine twisted cubic C curve in $\mathbb{A}^3$. That is the curve given parametrically by $x = t, y = t^2, z = t^3$. The parametric equation of the tangent surface is

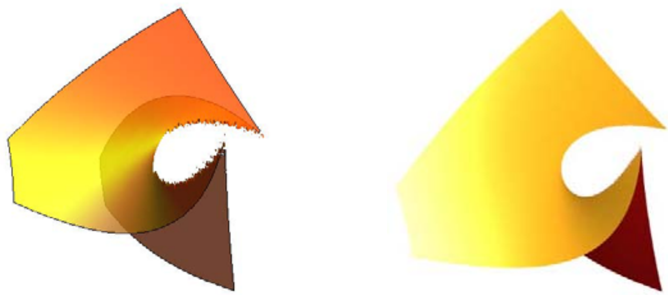
$$x = t + u,\ y = t^2 + 2tu,\ z = t^3 + 3t^2u. \tag{1}$$

An easy elimination shows that the tangent surface in its implicit form is given by

$$F = 3x^2y^2 - 4x^3z - 4y^3 + 6xyz - z^2 = 0. \tag{2}$$

The curve C is a singular curve on its tangent variety. This could be a challenge for a presentation in architecture. Figure 2a shows an image of the implicit form of the tangent variety, which was created by MATHEMATICA.

One might see the difficulty for drawing the singular locus, the original curve, correctly. In fact, this requires high numerical stability for the computation of zeros of polynomial equations.



(a) Mathematica 6 rendering (b) RealSurf rendering

Fig. 2. The tangent surface $3x^2y^2 - 4x^3z - 4y^3 + 6xyz - z^2 = 0$

3 Real Time and Singularities

A popular technique for the visualization of surfaces is based on polygonal meshes. It works well for algebraic surfaces given in parameter form, e.g. Bézier surfaces. This method does not work correctly for singularities. It could be possible that a singularity is not exhibited in a polygonal mesh, and hence does not occur for the visualization.

In Computer Graphics ray tracing is an appropriate technique in order to visualize scenes with complex details like singularities. A popular ray tracer is POV-Ray. Ray tracing techniques are also implemented in Surf. Typically ray tracing is a time consuming process, which stays in contrast to the requirement of real time visualization with interactions. An overview of the ray tracing process is given in the next subsection.

3.1 Ray Tracing

In our model of ray tracing there is a fixed world coordinate system and a scene with mathematically described objects. For each pixel of the screen a ray is sent from the eye into the scene and the nearest one among the hit points of the ray with all scene objects is computed. Afterwards the illumination at the object with the nearest hit point is calculated, which can introduce new rays in order to simulate complex visual phenomena like reflection, refraction or shadowing. To obtain fine details this iteration may require minutes respectively hours of computing time. The particular case, where only primary rays are considered, is also known as ray casting.

Most recently there is a new hardware development to allow computations and programming on the graphics processing unit (GPU) using specially designed programming languages, typically called shading languages due to their computer graphics background. Examples of shading languages are Cg, GLSL and HLSL. Prior to the execution of the program the graphics driver translates the shader code into machine instructions for the graphics card. Because of the multiprocessor concepts of recent graphics cards this procedure ensures an essential increase in computing speed as required for instance for computer games and complex visualizations.

4 RealSurf

RealSurf is a program for the interactive visualization of (implicit) algebraic surfaces. It is based on the hardware development as mentioned above and uses the OpenGL Shading Language (GLSL). Figure 3 shows a screen shot of the program for the exploration of Barth's surface of degree 10.

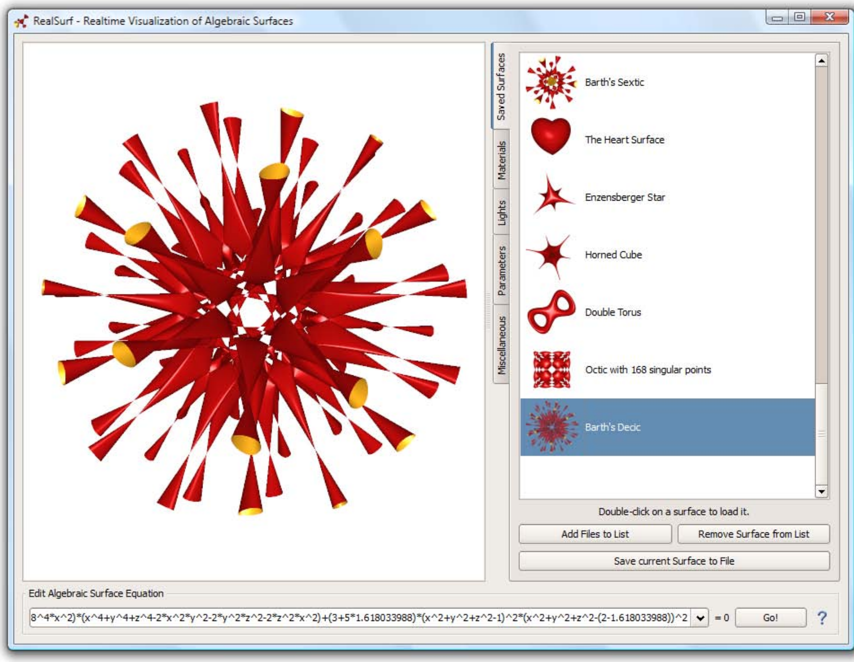


Fig. 3. The main window of RealSurf, including a gallery of predefined surfaces

Besides the shader programming techniques we had to consider several numerical aspects in order to calculate the zeros of algebraic equations in the ray-surface intersection step. Due to the numerical instabilities in the neighborhood of singularities and the low precision of calculations on graphics hardware, we first separate the roots of the polynomial equation and afterwards refine the isolating intervals to a sufficient precision for the visualization. For the details see [6].

Features of RealSurf

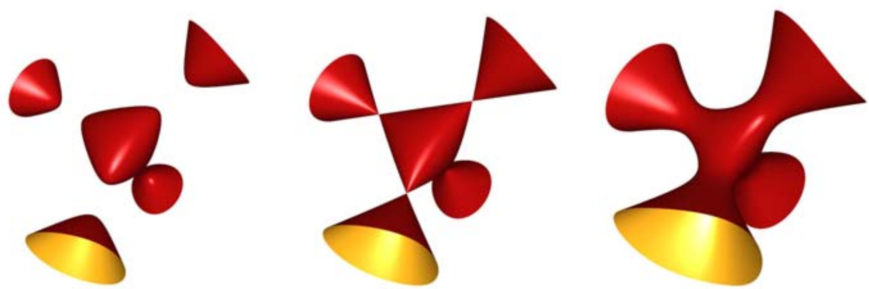
RealSurf allows to scale, rotate and translate the surface in real time by mouse actions. A collection of classical surfaces is included, own surface equations can be entered – saving and loading surfaces is supported as well.

Constants in the equation can be replaced by parameters. This procedure permits e.g. interactive deformations of surfaces. As an example we investigate the cubic surface

$$F(x, y, z) = x^2 + y^2 + z^2 - \alpha xyz - 1 = 0 \tag{3}$$

with the parameter α, see Figure 4. The program allows an interactive visualization of the surface by a continuous change of the parameter. In particular for $\alpha = 2$ it yields the well known Cayley cubic.

Furthermore several scene parameters like lighting, surface materials and objects to clip the surface against are adjustable in RealSurf. The rendered images can be saved as PNG files for use in publications and presentations.



(a) $\alpha = 1.8$ (b) $\alpha = 2.0$ (c) $\alpha = 2.8$

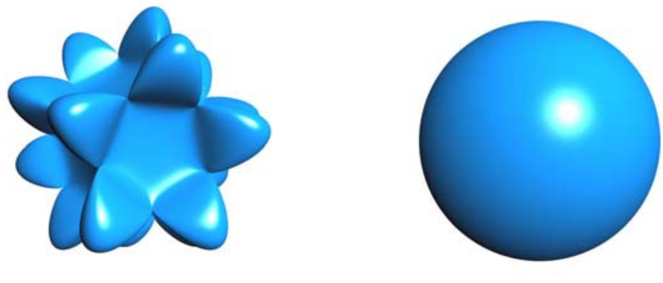
Fig. 4. The cubic for different values of α

5 Some Recipes for Creating Interesting Figures

From an artists point of view the mathematical properties of an algebraic surface may not be very interesting. Nevertheless one can start with some higher order surfaces like the Barth sextic or with variants of them like the surface in figure 5a. One possibility is to remove some terms from the complex equation, add new terms or modify coefficients and parameters.

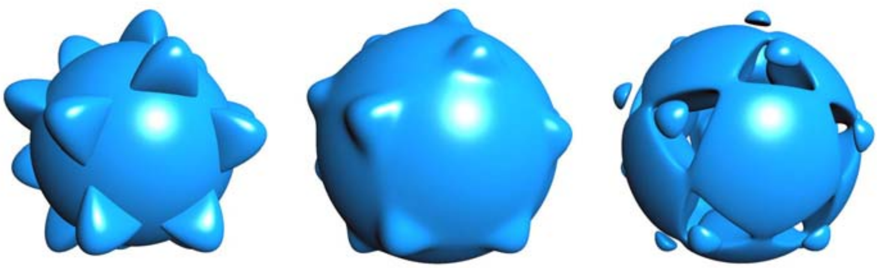
Visually more appealing surfaces can often be constructed by playing with the level sets of combinations of two or more surfaces. For example one can multiply a complex surface $F = 0$ and a simple surface $G = 0$ like in figure 5 and add a small offset to generate the level set surface $F \cdot G + \gamma = 0$, which gives a slightly disturbed union of both surfaces, thus "melting" both surfaces together. An example is shown in figure 6.

Another form of combining two surfaces is to calculate their intersection curve, which is given by $F^2 + G^2 = 0$. An example is shown in figure 7a. The infinitly thin intersection curve is thickened by the offset γ, resulting in the equation $F^2 + G^2 - \gamma = 0$, which often gives interesting surfaces like in figure 7.



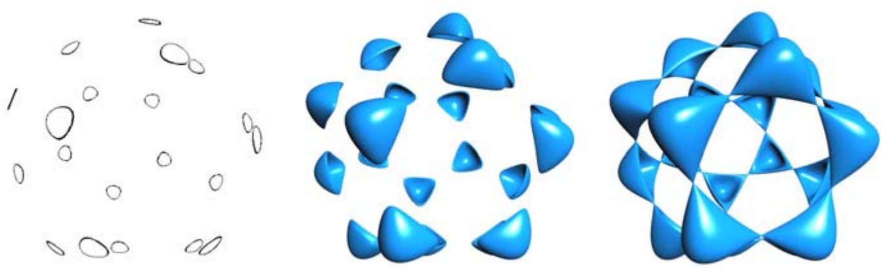
(a) The surface $F = 0$ (b) The surface $G = 0$

Fig. 5. Surfaces used in the following examples: (a) a variation of the Barth sextic with 30 cusps ($F = 4(\alpha^2x^2 - y^2)(\alpha^2y^2 - z^2)(\alpha^2z^2 - x^2) - (1 + 2\alpha)(x^2 + y^2 + z^2 - 1)^3 = 0$, $\alpha = \frac{1}{2}(1 + \sqrt{5})$), (b) a sphere ($G = x^2 + y^2 + z^2 - \beta^2 = 0$)



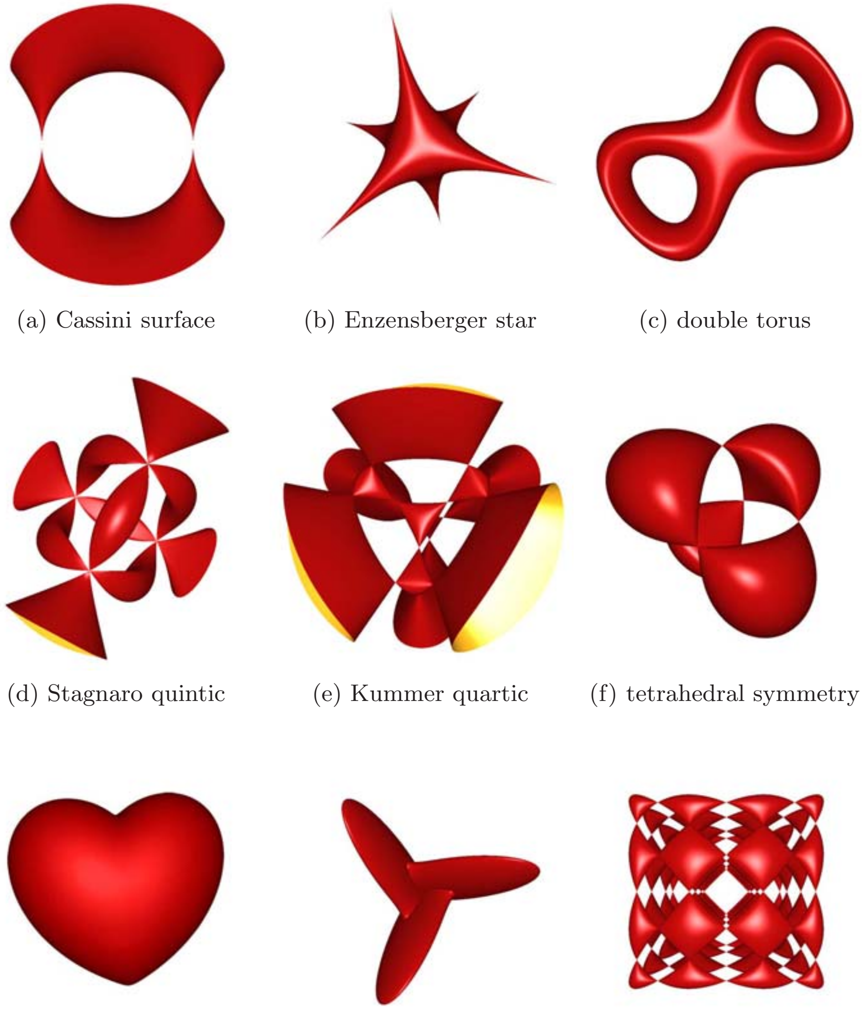
(a) $\gamma = 0$ (b) $\gamma = 0.07$ (c) $\gamma = -0.07$

Fig. 6. Level sets of the surface $F \cdot G + \gamma = 0$ for $\alpha = \frac{1}{2}(1 + \sqrt{5})$ and $\beta = \sqrt{1.3}$



(a) The intersection curve (b) $\gamma = 0.48348$ (c) $\gamma = 0.2$

Fig. 7. Starting with the intersection curve $F^2 + G^2 = 0$ of two surfaces and considering its level sets $F^2 + G^2 - \gamma = 0$ often leads to very nice surfaces. The images above were created with $\alpha = \frac{1}{2}(1 + \sqrt{5})$ and $\beta = \sqrt{2}$.



(a) Cassini surface (b) Enzensberger star (c) double torus

(d) Stagnaro quintic (e) Kummer quartic (f) tetrahedral symmetry

(g) heart surface (h) Boy surface (i) Chmutov octic

Fig. 8. For further inspiration we present some images of well known algebraic surfaces. Equations of the above surfaces can be found in [7], for example.

6 Concluding Remarks

The program RealSurf allows the interactive exploration of implicit given algebraic surfaces with singularities in real time. It is based on programming with shading languages and currently renders most surfaces up to degree 13 correctly and in real time, if the equation is not too complex.

It is available for Microsoft Windows and requires recent NVIDIA graphics hardware (GeForce 6 series and up). Upon request it is freely available via [6].

The program allows the investigation of mathematical models in an interactive way. In addition one might be able to create new esthetical objects for further use in fine art, architecture a.o. The advantage over static models is the ability to modify a surface by parameters and to obtain a 3D imagination of the model.

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