Revenue Submodularity*

Shaddin Dughmi**, Tim Roughgarden***, and Mukund Sundararajan†

Department of Computer Science, Stanford University, Stanford CA 94305, USA
{shaddin,tim,mukunds}@cs.stanford.edu

Abstract. We introduce *revenue submodularity*, the property that market expansion has diminishing returns on an auction's expected revenue. We prove that revenue submodularity is generally possible only in matroid markets, that Bayesian-optimal auctions are always revenue-submodular in such markets, and that the VCG mechanism is revenue-submodular in matroid markets with IID bidders and "sufficient competition". We also give two applications of revenue submodularity: good approximation algorithms for novel market expansion problems, and approximate revenue guarantees for the VCG mechanism with IID bidders.

1 Revenue Submodularity

Auction environments are never static. Existing bidders may leave and new bidders may join or be recruited. For this reason, it is important to study how auction revenue *changes* as a function of the bidder set.

We introduce *revenue submodularity* — essentially, the property that market expansion has diminishing returns on an auction's expected revenue. For example, in a multi-unit auction with bidders that have unit demand and IID valuations, revenue submodularity means that the auction's expected revenue is a concave function of the number of bidders. In general, an auction A is deemed revenue submodular in an environment with potential bidders U if, for every subset $S \subset U$ and bidder $i \notin S$, the increase in the auction's revenue from supplementing the bidders S by i is at most that of supplementing a set $T \subseteq S$ of bidders by the same additional bidder i.

We first identify the largest class of single-parameter domains for which general revenue-submodular results are possible: *matroid markets*, in which the feasible subsets of simultaneously winning bidders form a matroid. Fortunately, matroid markets include several interesting examples, including multi-unit auctions and unit-demand matching markets (corresponding to a transversal matroid).

---

* A working paper featuring special cases of results in this work was presented at the 3rd Workshop on Sponsored Search, May 2007, under the title "Is Efficiency Expensive?".

** Supported in part by NSF grant CCF-0448664.

*** Supported in part by NSF CAREER Award CCF-0448664, an ONR Young Investigator Award, an AFOSR MURI grant, and an Alfred P. Sloan Fellowship.

† Supported by NSF Award CCF-0448664 and a Stanford Graduate Fellowship.

S. Das et al. (Eds.): Amma 2009, LNICST 14, pp. 89–91, 2009.

We then prove a number of positive results. First is a sweeping result for (Bayesian)-optimal auctions: in every matroid market with independent (not necessarily identical) valuation distributions, the revenue-maximizing auction is revenue-submodular. The VCG mechanism, on the other hand, enjoys revenue-submodularity only under additional conditions, even when valuations are IID draws from a well-behaved distribution. We identify the key sufficient condition under which the VCG mechanism is revenue-submodular with IID bidders, which is a matroid rank condition stating that there is "sufficient competition" in the market. For example, in multi-unit auctions (uniform matroids), sufficient competition requires that the number of bidders is at least the number of items. Finally, we prove that revenue-submodularity is not a monotone property of the reserve prices used: reserve prices higher than those in an optimal mechanism preserve submodularity , but reserve prices strictly between those in the VCG mechanism (zero) and those in an optimal mechanism always have the potential to destroy revenue-submodularity, even when there is sufficient competition in the market. Formally, our main results are as follows.

Theorem 1 (Submodularity of Optimal Auctions in Matroid Markets) *Fix a matroid market M over bidders U with valuations drawn independently from arbitrary distributions $\{F_i\}_{i \in U}$. The expected revenue of the corresponding optimal auction for induced matroid markets M_S is submodular over $S \subseteq U$.*

Theorem 2 (Submodularity of the VCG Mechanism on Full-Rank Sets) *Fix a matroid market M over bidders U with valuations drawn IID from a regular distribution F. The expected revenue of VCG for induced matroid markets M_S is submodular over the full-rank sets $S \subseteq U$.*

Theorem 3 (Submodularity with Incorrect Reserve Prices)

(a) For every regular distribution F with optimal reserve price r^*, every matroid market with bidders U with valuations drawn IID from F, and every $r \geq r^*$, the expected revenue of the VCG mechanism with reserve price r is submodular on U.

(b) For every $\epsilon \in (0, 1)$, there is a regular distribution F with optimal reserve price r^* and a matroid market for which the expected revenue of the VCG mechanism with reserve price $(1 - \epsilon)r^*$ is not submodular on full-rank sets.

We obtain reasonably simple and direct proofs of these results by appropriately applying, in different ways, two elegant but powerful techniques: Myerson's characterization [2] of expected auction revenue in terms of the expected "virtual surplus" of the auction's allocation; and the submodularity that arises from optimizing a nonnegative weight vector over the independent sets of a matroid.

2 Applications

The first application of revenue submodularity is algorithmic. In the basic version of the *market expansion problem*, we are given a matroid market with a set of potential bidders, a subset of initial bidders, an auction (defined for all induced submarkets), and an expansion budget k. The goal is to recruit a set of at most k new bidders to maximize the expected revenue of the auction on the submarket induced by the original bidders together with the new recruits. This problem is easy only when the environment is completely symmetric (IID bidders in a multi-unit auction). Using the result of Nehmauser, Wolsey, and Fisher [3], we observe that "greedy market expansion" — repeatedly adding the new bidder that (myopically) increases the expected revenue of the auction as much as possible — is a constant-factor approximation algorithm provided the given auction is revenue submodular over all sets containing the initial bidders.

Our second application of revenue submodularity is to approximate revenue-maximization guarantees for the VCG mechanism. More precisely, in a matroid market with IID bidder valuations and "modest competition" — formalized using matroid connectivity — the VCG mechanism always obtains a constant fraction of the revenue of an optimal auction; moreover, the approximation guarantee tends rapidly to 1 as the degree of competition increases. As part of our proof, we generalize to arbitrary matroids the bicriteria result of Bulow and Klemperer [1]. This result suggests an explanation for the persistent use of efficient auctions in settings where the auctioneer should presumably care about maximizing revenue. In many contexts, the cost (i.e., revenue loss) of running an efficient auction is small and outweighed by the benefits (relative simplicity and optimal efficiency) even for a revenue-maximizing seller. We also extend this result to a standard model of sponsored search auctions.

References

1. Bulow, J., Klemperer, P.: Auctions versus negotiations. American Economic Review 86(1), 180–194 (1996)
2. Myerson, R.: Optimal auction design. Mathematics of Operations Research 6(1), 58–73 (1981)
3. Nemhauser, G., Wolsey, L., Fisher, M.: An analysis of the approximations for maximizing submodular set functions. Mathematical Programming 14, 265–294 (1978)