Eigenvalue Based Stability Analysis for Asymmetric Complex Dynamical Networks

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Abstract. The problem of stabilization in complex networks with asymmetric couplings forced by pinning control is studied. By using eigenvalue analysis, controllable regions for different types of coupling links are obtained. Some relevant factors on controllability such as pinning fraction and pinning strength are also investigated.

Keywords: Complex network, asymmetric network, pinning control, stability, eigenvalue analysis.

1 Introduction

We are surrounded by complex systems. Many complex systems in nature and societies can be cast into complex networks, where vertices are the elements of the systems and edges represent the interactions among them. Recently, the interplay between the complexity of the overall topology and the collective dynamics of complex networks gives rise to a host of interesting effects [1-7].

Especially, there are attempts to pinning control the dynamics of a complex network and guide it to a desired state. Previous work at this question [8-9] has typically looked at very specific coupling schemes such as symmetric coupling and uniform coupling (i.e., the nodes interact with the same coupling strength). However, in many circumstances this simplification does not match satisfactorily the peculiarities of real networks. For instance, the WWW [10-11], metabolic networks and citation networks [12-13] are all directed graphs, whose coupling matrices are asymmetric. Due to the difficulty of asymmetric coupling producing imaginary parts of eigenvalues, stabilization problem for asymmetric networks subjected to pinning control has been an immensely challenging undertaking in the current literature.

The main contribution of this paper is to using eigenvalue analysis approach to characterize both the controllable region of the stationary state and the eigenvalue distribution of the coupling and control matrices. The dependence of the controllability on both pinning fraction and pinning strength is studied through directed numerical simulations.

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2 Stabilization of Stationary State by Pinning Control

Consider a very popular complex dynamical network consisting of N identical coupled nonlinear nodes with diffusive and linear couplings

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{N} b_{ij}\Gamma x_j(t)\,,\ i = 1,2,\cdots,N\,, \tag{1}$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{im})^T \in R^m$ represents the state vector of the i-th node, and the nonlinear function $f(\cdot)$, describing the local dynamics of the nodes, is continuously differentiable and capable of producing various rich dynamical behaviors, including periodic orbits and chaotic states. $\Gamma \in R^{m\times m}$ is a 0-1 constant matrix indicating inner-coupling between the elements of the node itself, while the real matrix $B = (b_{ij})_{i,j=1}^{N}$ satisfying zero-row sum denotes the outer-coupling among the nodes of the whole network. If there is a connection from node i to j $(j \neq i)$, then $b_{ij} > 0$; otherwise $b_{ij} = 0$ $(j \neq i)$. For symmetric couplings, all the eigenvalues of the coupling matrix are real. However, asymmetric couplings produce imaginary parts of eigenvalues, which can yield additional stability and instability absent in symmetric couplings cases. Here, the whole network is supposed strong connected and the coupling matrix B is assumed to be diagonalizable although it is asymmetric. It is easily proved that 0 is an eigenvalue of the asymmetric matrix B with multiplicity 1 and the real parts of all the other eigenvalues of B are negative due to Gerschgorin's disk theorem [14].

In this paper we focus on the following problem. Supposing the isolated node accepts a chaotic solution, our central task is to stabilize network (1) onto a homogenous stationary state

$$x_1(t), x_2(t), \cdots, x_N(t) \to \bar{x}\,,\ \text{as}\ t \to \infty \tag{2}$$

where $\bar{x}$ is an equilibrium point, satisfying $f(\bar{x}) = 0$.

To achieve the goal (2), feedback pinning controllers are applied onto a portion δ of nodes in network (1). Without loss of generality, let the first l nodes be selected to be pinned, where l is the integer part of the real number δN. It should be noted that a controller will not be removed after being placed in.

Thus, the controlled network can be described as

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{N} b_{ij}\Gamma x_j(t) + u_i\,,\ i = 1,2,\cdots,N\,, \tag{3}$$

with the local state feedback controllers given by

$$u_i = -d_i\Gamma(x_i(t) - \bar{x})\,,\ i = 1,2,\cdots,N\,, \tag{4}$$

where the feedback gains $d_i = d > 0$ for $i = 1,2,\cdots,l$ and $d_i = 0$ for otherwise.

The stability of the stationary state (2) after control can be analyzed exactly by setting $e_i(t) = x_i(t) - \bar{x}$ and linearizing Eqs. (3) at state $\bar{x}$. This leads to

$$\dot{E} = E[D^T f(\bar{x})] + CE\Gamma^T\,, \tag{5}$$

where $Df(\bar{x}) \in R^{m\times m}$ is the Jacobian matrix of f evaluated at $\bar{x}$. $E^T = [e_1, e_2, \cdots, e_N] \in R^{m\times N}$ and $C = B - D$ with feedback gain matrix $D = diag(d_1, d_2, \cdots, d_N)$.

It is an easy matter to prove that C is asymmetric, and all the real parts of its eigenvalues are strictly negative with their corresponding (generalized) eigenvectors

$$\Phi = [\phi_1, \phi_2, \cdots, \phi_N] \in R^{N\times N},$$

satisfying

$$C\phi_k = \lambda_k \phi_k, \ k = 1, 2, \cdots, N.$$

By expressing each column of E on the basis $\{\phi_1, \phi_2, \cdots, \phi_N\}$, one has

$$E = \Phi\eta, \tag{6}$$

where $\eta^T = [\eta_1, \eta_2, \cdots, \eta_N] \in R^{m\times N}$.

Then, (5) can be expanded into the following equations:

$$\dot{\eta}_k = [Df(\bar{x}) + \lambda_k \Gamma]\eta_k, \ k = 1, 2, \cdots, N. \tag{7}$$

To this end, the stability problem of the $(N \times m)$-dimensional system (3) about the stationary state $\bar{x}$ is reduced to the stability problem of the much simpler N independent m-dimensional linear system (7) about the origin.

The significance of Eqs. (7) is that the stability problem of the controlled network (3) can be separated into two independent problems: one is to analyze the stable regions of Eqs. (7), which depends on the dynamics of the isolated node such as the equilibrium point $\bar{x}$, the Jacobian matrix $Df(\bar{x})$ and the inner linking structure Γ; the other is to analyze the eigenvalue distribution of the matrix C, which depends on the network topology parameters, control gain d and the number of the pinned nodes l.

According to the linear system theory, the controlled network (3) is locally exponentially stable about $\bar{x}$, if and only if all the eigenvalues of the matrix $[Df(\bar{x}) + \lambda_k \Gamma]$ have negative real parts. This criterion provides the stability boundary in the complex plane. Of course, this stability boundary depends on the dynamics of the isolated node and the inner linking matrix Γ. In the following subsection we clarify various essentially different structures of controllable regions.

2.1 Stable and Unstable Regions

Now we specify the following models as our examples. Let $x = [x_1, x_2, x_3]^T$.

(1) Lorenz model

A single Lorenz oscillator [15] is described in the dimensionless form by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} \alpha(x_2 - x_1) \\ \gamma x_1 - x_1 x_3 - x_2 \\ x_1 x_2 - \beta x_3 \end{pmatrix}, \tag{8}$$

which has a chaotic attractor when $\alpha = 10$, $\beta = 8/3$ and $\gamma = 28$. With this set of system parameters, one unstable equilibrium point is $\bar{x} = [8.4853, 8.4853, 27]^T$.

(2) Chen model

A single Chen oscillator [16] is described in the dimensionless form by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} \alpha(x_2 - x_1) \\ (\gamma - \alpha)x_1 - x_1x_3 + \gamma x_2 \\ x_1x_2 - \beta x_3 \end{pmatrix}, \tag{9}$$

which has a chaotic attractor when $\alpha = 35$, $\beta = 3$ and $\gamma = 28$. With this set of system parameters, one unstable equilibrium point is $\bar{x} = [7.9373, 7.9373, 21]^T$.

In Fig. 1 we plot the stable region of the homogenous stationary state $\bar{x}$ of Lorenz model in the complex plane for different linking matrices Γ. The curves represent the critical condition at which the largest real part of the eigenvalues of the matrix $[Df(\bar{x}) + \lambda_k\Gamma]$ is equal to zero. In the region marked by "S" (stable), the largest real part of the eigenvalues of the matrix $[Df(\bar{x}) + \lambda_k\Gamma]$ is negative, while it is positive in the region marked by "U" (unstable). In Fig. 2 we do the same as in Fig. 1 with the model replaced by the Chen model.

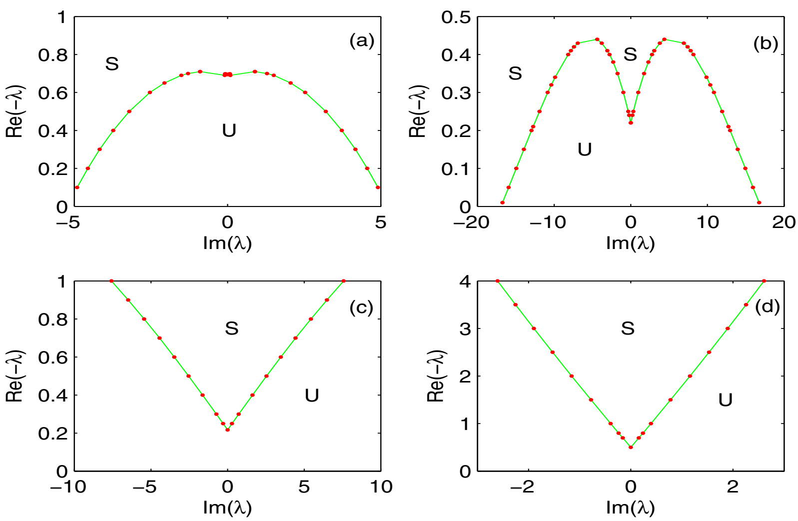

Fig. 1. Distributions of stable ("S") and unstable ("U") regions of the stationary state for the Lorenz model. The green solid lines represent the zero maximum real part of eigenvalues.

(a) $\Gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (b) $\Gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (c) $\Gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (d) $\Gamma = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

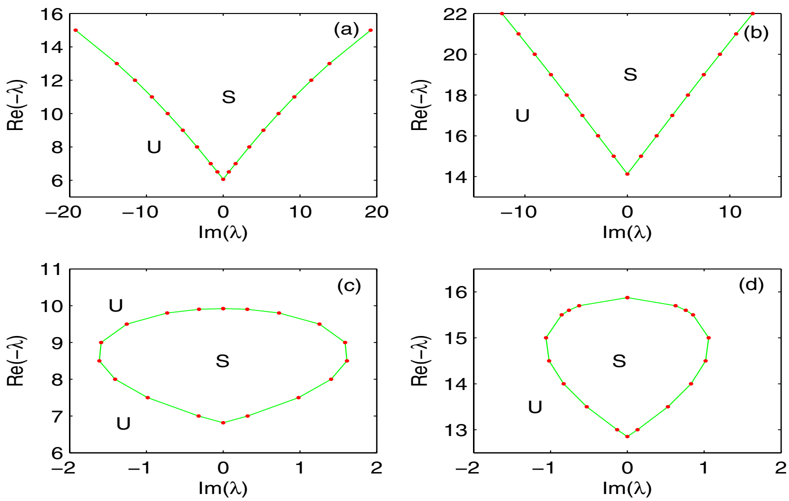

Fig. 2. The same as Fig. 1 with the Chen model considered

$$\text{(a)}\ \Gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ \text{(b)}\ \Gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\ \text{(c)}\ \Gamma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ \text{(d)}\ \Gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

It is interesting to notice that the structure of the stable regions in Figs. 1-2 can be classified into three groups. Class (i), including Figs. 1(c), 1(d), 2(a), and 2(b): the critical curves are V-shaped, then larger $\mathrm{Re}(-\lambda)$ and smaller $|\mathrm{Im}(\lambda)|$ are favorable for stabilizing the homogenous stationary states. Class (ii), including Figs. 2(c) and 2(d): the critical curves form closed circles, and then stable regions are localized in certain finite $\mathrm{Re}(-\lambda)-\mathrm{Im}(\lambda)$ regions. Too large and too small $\mathrm{Re}(-\lambda)$ and too large $|\mathrm{Im}(\lambda)|$ can definitely destroy the stability of the homogenous stationary states. Class (iii) [see Fig. 1(a)]: the critical curves are inversely U shaped, and then larger $\mathrm{Re}(-\lambda)$ and larger $|\mathrm{Im}(\lambda)|$ are favorable for stabilization of the homogenous stationary states. Fig. 1(b) is a kind of mixture of classes (i) and (iii). It is interesting to find a sharp central bottom in its stable region.

2.2 Eigenvalue Distribution

Now all the stable and unstable regions shown in Figs. 1-2 remain unchanged for the given parameters. For stabilizing the homogenous stationary state, the key point is to move all the unstable eigenvalues of B to the controllable domains by adding suitable control matrix D. In the following we consider unidirectional ring networks with 3 and 4 nodes as shown in Figs. 3(a) and 3(b), respectively.

In Fig. 4 we plot various eigenvalue distributions at $N = 3$ and $\Gamma = diag(1,0,0)$ for different d and l. It is observed from Fig. 4(a) that without control (i.e., $d = 0$),

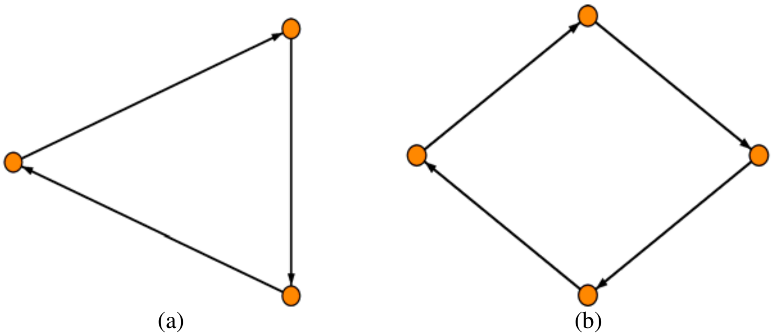

Fig. 3. (a) Unidirectional ring network: N=3. (b) Unidirectional ring network: N=4.

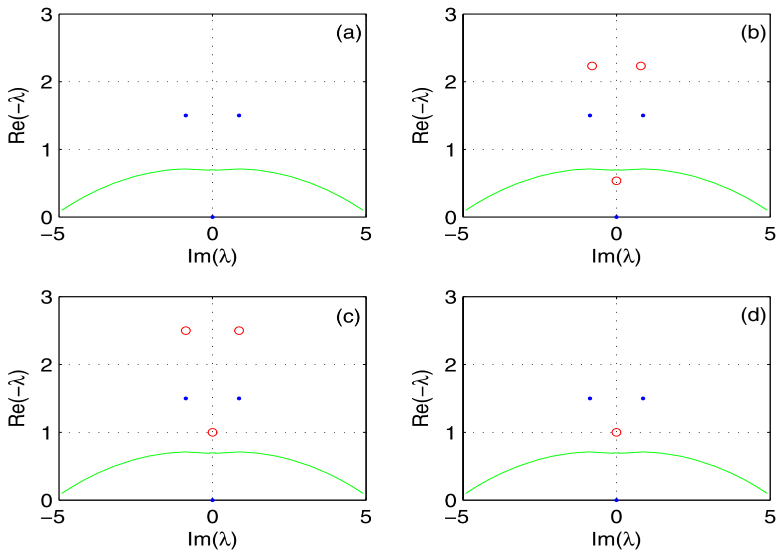

Fig. 4. Eigenvalue distribution of the asymmetric matrix C . N=3 (The network structure Fig. 3(a) is considered). (a) d=0, l=0. (b) d=1, l=2. (c) d=1, l=3. (d) d=50, l=2. The blue dots represent the eigenvalues of the matrix C with d=0 and l=0, while the red circles represent the eigenvalues of the matrix C with d≠0 and l≠0. The green lines denote the critical curves of the stable region in the case of Fig. 1(a). These notations are valid for Figs. 5-6.

there is one unstable eigenvalue, 0, locating in the unstable region. Keeping all parameters unchanged except setting $d=1$ and $l=2$, all the eigenvalues move up as shown in Fig. 4(b). However, there is still one nonzero eigenvalue sitting in the unstable region. Continuously increasing the pinning fraction until $l=3$, the unstable eigenvalue crosses the critical line and enters the stable region [see Fig. 4(c)]. In Fig. 4(d), we increase largely the feedback gain on the basis of Fig. 4(b), and find the unstable eigenvalue ultimately moves into the stable region (The other two stable eigenvalues are moved up outside of the scope of the figure). Figure 4 tells us clearly

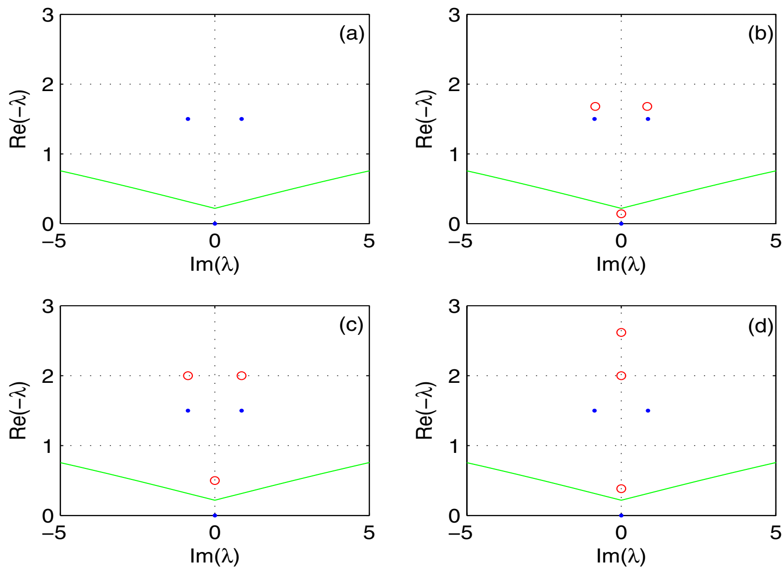

Fig. 5. Eigenvalue distribution of the asymmetric matrix C . N=3 (The network structure Fig. 3(a) is considered). (a) d=0, l=0. (b) d=0.5, l=1. (c) d=0.5, l=3. (d) d=2, l=1. The green lines denote the critical curves of the stable region in the case of Fig. 1(c).

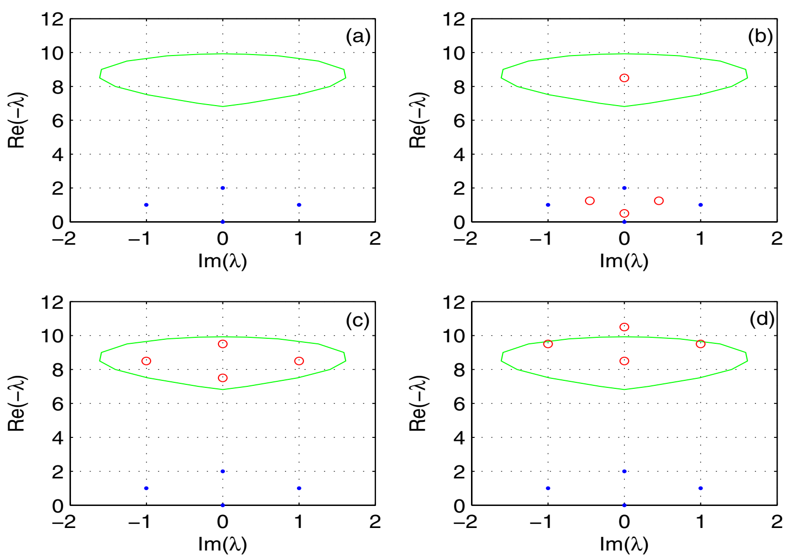

Fig. 6. Eigenvalue distribution of the asymmetric matrix C . N=4 (The network structure Fig. 3(b) is considered). (a) d=0, l=0. (b) d=7.5, l=1. (c) d=7.5, l=4. (d) d=8.5, l=4. The green lines denote the critical curves of the stable region in the case of Fig. 2(c).

the removal of the eigenvalues after control. It is concluded that increasing the pinning fraction and/or the pinning density can considerably enhance the controlling efficiency in the case of inversely U shaped region.

In Fig. 5, we do the same as Fig. 4 except the stable region distributions Fig. 1(c) is considered. All the features found in Fig. 4 are still observed in Fig. 5.

In Fig. 6 we consider the cases Fig. 2(c) and Fig. 3(b). It is observed from Fig. 6(a) that all the eigenvalues are outside the stable region without control. In Fig. 6(b) we choose only one node to be pinned. The obvious consequence is that all the eigenvalues of C move up, and then the top eigenvalue first crosses the bottom critical curve and enters the stable region. Keep all parameters unchanged from Fig. 6(b) except that the pinning fraction l is increased to $l = 4$, then all the eigenvalues move into the stable region. Continuously increasing the feedback gain d to $d = 4$ from Fig. 6(c), an interesting phenomenon takes place: the top eigenvalue first crosses the upper critical curve and enters the unstable region, which leads to destabilization. It is concluded that too large and too small l and d can definitely destroy the stabilization of the stationary state.

Figures 7 and 8 show the process of controlling the 3-node and 4-node unidirectional ring networks corresponding to Figs. 4(d) and 6(c), respectively. It is clearly that the aim state is stabilized well after control.

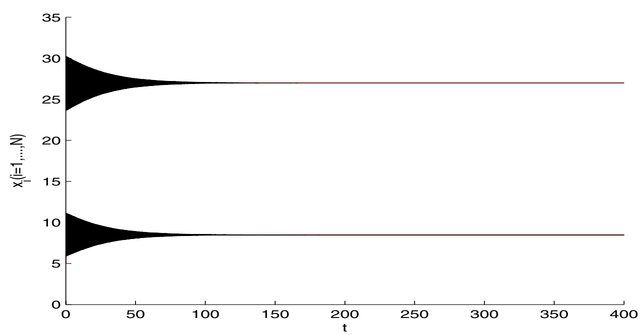

Fig. 7. Specifically pinning some nodes in a 3-node unidirectional ring network corresponding to Fig. 4(d)

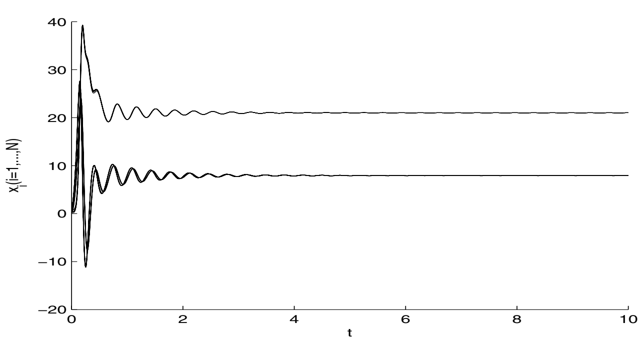

Fig. 8. Specifically pinning some nodes in a 4-node unidirectional ring network corresponding to Fig. 6(c)

3 Conclusions

In this paper, the stability problem of stationary states for asymmetric networks subjected to pinning control is discussed by eigenvalue analysis approach. The effects of pinning fraction and strength are investigated in detail. The stabilization problem of complicated high-dimensional coupled and controlled networks can be divided into two independent problems: One is the description of controllable and uncontrollable regions of the isolated node modified by an eigenvalue forcing $\lambda_k\Gamma$; the other is the eigenvalue analysis of the node coupling and controlling matrix C. The former is independent of the node interaction scheme and the control mechanism, and the latter is independent of the inner dynamics, the homogenous stationary state and the inner linking structure Γ. Both problems have been solved easily, and they, together, provide definite answers to the problems of stability of asymmetric coupled networks. For instance, one can easily reveal and classify the stability of the stationary state by examining whether and how some unstable eigenvalues enter into the stable region. Moreover, one can apply control matrix D to stabilize a stationary state by moving all the unstable eigenvalues into the stable region. The ideas in this paper can be applied to general coupled extended networks.

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