# Networks That Optimize a Trade-Off between Efficiency and Dynamical Resilience

Markus Brede[1,2,3] and Bert J.M. de Vries[3]

[1] CSIRO Marine and Atmospheric Research, Canberra ACT 2602, Australia
Markus.Brede@Csiro.au
[2] Copernicus Institute, Science, Technology and Society, Van Unnikgebouw, Heidelberglaan 2, 3584 CS Utrecht, The Netherlands
[3] Planbureau voor de Leefomgeving (PBL), P.O. Box 303, NL-3720 AH Bilthoven, The Netherlands

**Abstract.** In this paper we study networks that have been optimized to realize a trade-off between communication efficiency and dynamical resilience. While the first is related to the average shortest pathlength, we argue that the second can be measured by the largest eigenvalue of the adjacency matrix of the network. Best efficiency is realized in star-like configurations, while enhanced resilience is related to the avoidance of short loops and degree homogeneity. Thus crucially, very efficient networks are not resilient while very resilient networks lack in efficiency. Networks that realize a trade-off between both limiting cases exhibit core-periphery structures, where the average degree of core nodes decreases but core size increases as the weight is gradually shifted from a strong requirement for efficiency and limited resilience towards a smaller requirement for efficiency and a strong demand for resilience.

We argue that both, efficiency and resilience are important requirements for network design and highlight how networks can be constructed that allow for both.

**Keywords:** networks, efficiency, resilience, optimization.

## 1 Introduction

Some important engineered systems, such as for instance power systems or communication networks, are systems that consist of a large number of distributed elements. The individual elements (i.e. power generators or information processors/routers in the above two examples) are typically connected in a highly non-trivial way. At a very coarse grained level, several studies have for instance discussed scale-free networks in this context [1].

The function of these systems is determined by both the connection structure and by the dynamics of the individual elements. The efficiency of the system's function is then often related to the efficiency of communication between the individual elements, which, in turn, is determined by the average shortest pathlength of the network. For systems that realize very short communication paths

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between all individual elements one commonly observes architectures that are highly centralized and star-like [2,3].

Another requirement one would pose on an engineered system is 'robustness' to failure. The term robustness appears somewhat imprecise in this context. At a very coarse grained level, if one thinks about the complete failure of individual elements, targetted or random node removals have to be discussed. Such an analysis concentrates on the topological stability of the networks [4]. Considering random node removals robustness may be related to percolation problems on graphs, for targetted node removals the situation becomes more complex.

Some optimal network architectures, that realize a trade-off between efficiency and robustness to targetted node removals have already been discussed [5]. The results are not surprising: efficient networks are star like. If one constructs networks to be highly efficient even after removing a fraction of the most central nodes, the solution are networks consisting of a periphery made up of leaves and a highly connected core that comprises just some more nodes than are removed according to the targetted node removal procedure.

However, robustness may also be defined in a less stringent sense than above as the system's ability to recover a previous operational state after a dynamic perturbation, i.e. a shock to one of its state variables. To quantify this notion of robustness (or resilience as it is often termed in the ecology literature, cf. e.g. [6]), let us introduce the following notation. Consider a distributed system that consists of $N$ elements. The state of each element be described by a variable $X_i \in \mathcal{R}^n$ (with some $n = 1, 2, ...$ depending on the complexity of the elements). Without loss of generality the dynamics (or system function) may be given by a set of first order differential equations

$$\dot{X} = F(X). \tag{1}$$

Assume further that the desired function of the system is performed in a stationary state of Eq. (1), i.e. in some state $X^*$ for which $F(X^*) = 0$. This already allows us to define (dynamical) resilience in a more precise way: it is the speed of recovery to 'normal' system function after a perturbation to the 'normal' state $X^*$.

Mathematically, one may perform the conventional linear stability analysis, i.e. linearize the system (1) around its stationary state $X^*$, which gives

$$\frac{d(\Delta X)}{dt} = dF(X^*)\Delta X, \tag{2}$$

where $\Delta X$ describes a small (infinitessimal) perturbation to the stationary state and $dF$ is the Jacobian matrix of (1) taken at the stationary point. Equation (2) describes a linear system. Clearly, the fate of the asymptotic dynamics of initial perturbations $\Delta X_0$ are given by the largest eigenvalue of the Jacobian matrix $dF(X^*)$. If $\lambda_{\max}(dF) > 0$ the state $X^*$ is an unstable fixed point, whereas if $\lambda_{\max}(dF) < 0$ it is stable. Since one has $\max_i \Delta X_i(t) \propto \exp(\lambda_{\max} t)$, in the latter case the system will return the faster to the stationary state, the smaller the largest eigenvalue. Hence, assuming the system is stable, the system is the more resilient the smaller the largest eigenvalue of the Jacobian $dF(X^*)$.

One can also interpret the Jacobian matrix $dF(X^*)$ as a network of $N$ nodes. Nodes are connected by weighted links, the weight of which is given by the entries of $dF(X^*)$. Assuming that entries of $dF(X^*)$ are either present (in which case we set $dF_{ij} = 1$) or absent ($dF_{ij} = 0$) and that the Jacobian is symmetric, one can link the problem setting to an analysis of dominant eigenvalues of binary graphs.

In the following, for a given number of links (that is cost) we aim at constructing networks that are efficient and (dynamically) resilient in the above sense. Thus we search for connected network configurations $\Gamma$ that minimize

$$E(\alpha, \Gamma) = \alpha \lambda_{\max}(\Gamma) + (1 - \alpha) d(\Gamma), \tag{3}$$

where $\lambda_{\max}(\Gamma)$ is the largest eigenvalue of the adjacency matrix of $\Gamma$ and $d$ the average shortest pathlength, i.e.

$$d(\Gamma) = 1/(N-1) \sum_{i<j} l(i,j) \tag{4}$$

and $l(i,j)$ is the shortest pathlength between nodes $i$ and $j$.

Below, we approach this problem with a numerical optimization scheme, very similar to, e.g., [2,7]. We seed the algorithm with a random Erdös-Rényi graph [8] with a fixed number of links $L$ and then iterate the following steps:

- $k$ randomly selected links are rewired to new randomly selected link vacancies. A rewired network $\Gamma'$ is thus constructed. In the course of the optimization the search space is narrowed by gradually decreasing $k$ to one.
- The fitness $E(\Gamma')$ is calculated according to Eq. (3). If $E(\Gamma') < E(\Gamma)$ the configuration is accepted. Otherwise the original configuration is restored and another rewiring suggested.

The algorithm is terminated if no rewired configuration was accepted for $L$ iterations.

The remainder of the paper is organized as follows. We start by a brief discussion of the limiting cases $\alpha = 1$ (resilient networks) and $\alpha = 0$ (efficient networks) which are in itself of interest. Particularly the first case has recently been discussed in [9]. In the subsequent section we then analyse networks constructed for various trade-offs between resilience and efficiency. The paper is concluded by a short summary and discussion of the results.

## 2 Resilient Networks

In this section we analyse networks that have been constructed for $\alpha = 0$, i.e. only in such a way that the maximum eigenvalue of the network is minimized for a given number of links. This case has recently been discussed in [9]. Our results below, however, differ in essential points from those of Ref. [9].

Figure 1 gives an illustration of an example network with $N = 100$ nodes and $L = 250$ links that has been constructed to minimize $\lambda_{\max}$. Already from

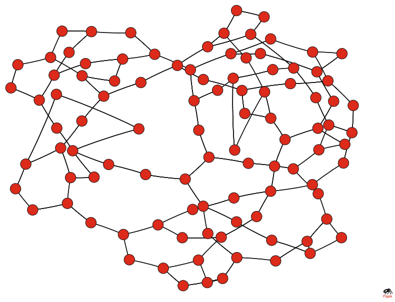

**Fig. 1.** Example of a resilient network ($N = 100$, $L = 250$ left) that minimizes $\lambda_{\max}$. Note that there are no hub nodes and that the networks are characterized by long loops.

the visualization one observes that the network has a very homogeneous degree sequence: almost every node has the same number of neighbours. Additionally, the network structure is marked by the absence of short loops and appears to be made up of many long loops that are intertwined in a low-dimensional arrangement.

To obtain a more quantitative picture, a set of 100 optimized network configurations was constructed from different randomly selected initial networks. We then measured a set of network characteristics, such as the variance $\sigma_k^2$ of the degree sequence, minimum and maximum degrees, $k_{\min}$ and $k_{\max}$, the clustering coefficient $c$ and average pathlengths $d$ and diameters. To quantify what is particular about the constructed networks we compare these quantities to those measured for the initial random networks.

For the degree variance we find $\sigma_k^2 = .25$ compared to $\sigma_k^2 = 1.7$ for random graphs with the same number of links. As also $k_{\min} = 2$ (compared to $k_{\min} = 1$ for random graphs) and $k_{\max} = 3$ (compared to $k_{\max} = 6.7$ for random graphs) this clearly demonstrates that the optimal networks become regular graphs. We also find that they are much less cliquish than random networks (clustering coefficient $c = .006$ compared to $c = .015$ for random graphs) and slightly larger (average pathlength $l = 6.5$ compared to $l = 5.4$ for random graphs) than random graphs.

The degree homogeneity would lead one to hypothesize that the $\lambda_{\max}$-minimizing networks could be similar to the 'entangled networks' that exhibit optimal stability of the synchronized state for coupled chaotic oscillators [10]. The stability of the synchronized state in the above scenario, however, can be related to a small eigenratio [11] $\lambda_{\max}/\lambda_{\min}$ (where $\lambda_{\min}$ denotes the smallest non-trivial eigenvalue in the spectrum of the adjacency matrix of an undirected graph), which can be achieved for small $\lambda_{\max}$, but has the additional requirement of large minimum eigenvalues. While being similarly uncliquish, the entangled networks were, however, found to also be small in comparison to random graphs. Following this argument, our result helps to disentangle the effects of minimizing $\lambda_{\max}$ and of maximizing $\lambda_{\min}$, which are both combined in the

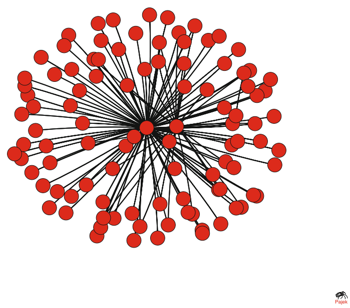

**Fig. 2.** Example of an efficient network ($N = 100$, $L = 500$). The network is characterized by the presence of one 'super-hub' node and some other highly connected hubs. Note the cliquishness.

optimization procedure performed in [10]. Hence, degree homogeneity is achieved by the minimization of the maximum eigenvalue. Small average pathlengths, however, follow from the maximization of the minimum eigenvalue.

## 3 Efficient Networks

In this section we concentrate on the construction of networks that are efficient, i.e. minimize the average pathlength. As already discussed in the literature [2,3] when links are sparse such networks are typically star-like. If more links are available than needed to construct one star, multiple, strongly interconnected hubs are constructed. The arrangement of links is thus also highly cliquish and one has average shortest pathlengths slighly smaller than 2.

More quantitatively, comparing to random networks we find an average maximum eigenvalue of $\lambda_{\max} = 10.5$ (compared to $\lambda_{\max} = 3.5$ for random networks), a degree variance $\sigma_k^2 = 99.5$ (compared to $\sigma_k^2 = 1.7$), average maximum degree $k_{\max} = 98.8$ and average minimum degree $k_{\min} = 1.2$, and a clustering coefficient $c = .26$ (compared to $c = .015$). While these results chiefly confirm the observations from the literature [2], they additionally emphasise that optimal efficiency is related to a decline in resilience, i.e. a strongly increased maximum eigenvalue. Below, we will address the problem how network topologies can be constructed that are both resilient and efficient.

## 4 The Trade-Off between Resilience and Efficiency

In this section we analyze the change in network structure as the requirements for resilience and efficiency are shifted in Eq. (3). Figure 3 illustrates some prototypical example networks that have been constructed for $\alpha = .1, .3$ and $\alpha = .7$. These example networks visualize the transition from cliquish star-like network topologies ($\alpha = .0$) towards homogeneous uncliquish networks ($\alpha = 1.$) that have been described in the previous section.

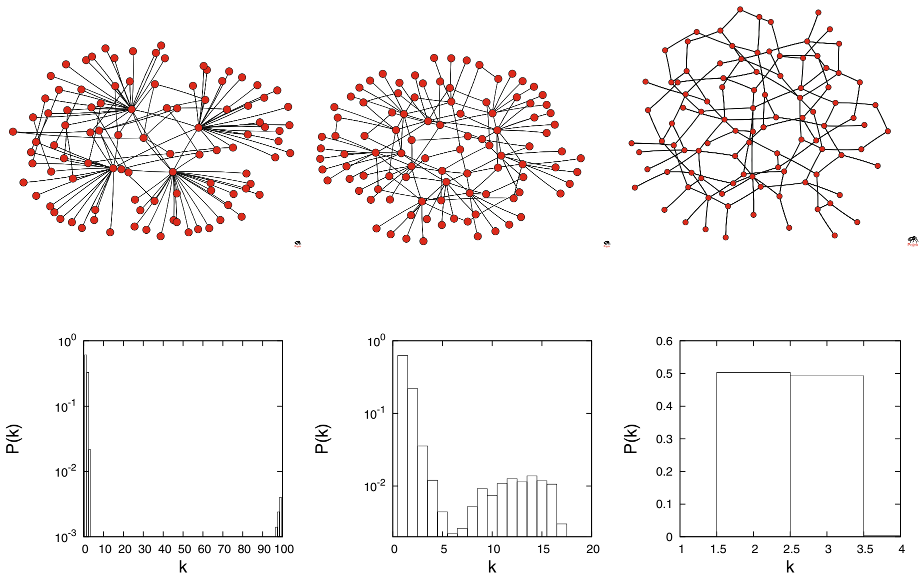


**Fig. 3.** Example networks optimized for intermediate values of $\alpha = .1, .3, .7$ (from left to right). The transition networks exhibit a core-perhiphery structure, the core consist of hub nodes, the periphery of nodes of low degree (here periphery nodes are leaves). Note, that for small $\alpha$ networks are characterised by a small core formed by hub nodes. These hub nodes are not strongly interlinked. Increasing $\alpha$ the core expands, but is formed by increasingly lower degree nodes. Finally, the core comprises almost all nodes and only a couple of nodes are left at the periphery. The second row gives the degree distributions for networks evolved for $\alpha = 0, \alpha = .3$ and $\alpha = 1$. Examples are all networks with $N = 100$ and $L = 250$.

A more thorough analysis is performed in Fig. 4 which gives the change of various network characteristics with the value of $\alpha$. The data indicate sharp transition that separate three distinct regimes, corresponding to typical network structures. The transition's behaviour is manifest in a sharp decline of the clustering coefficient at around $\alpha = .07$ which is accompanied by a sharp decrease in the maximum eigenvalue and the degree variance as well as a sudden increase in the average shortest pathlength. In the second case, for $\alpha \approx .85$ on observes a sudden increase in the minimum degree and a small increase in the clustering coefficient from a regime with $c = 0$ towards a regime with small, but non-zero clustering. Finding such sharp transitions in optimal networks is not surprising and has previously been reported in different contexts, see, e.g., [2,3].

We proceed with a more detailed analysis of the different regimes. For $\alpha < .1$ the star-like network structures discussed in section 3 are maintained. The requirement for a low level of resilience only has an insignificant effect on the structure of the optimal networks.

In the range $.1 < \alpha < .85$ the optimal network configurations exhibit a distinct core-periphery structure. As, e.g., the degree distribution plotted in the

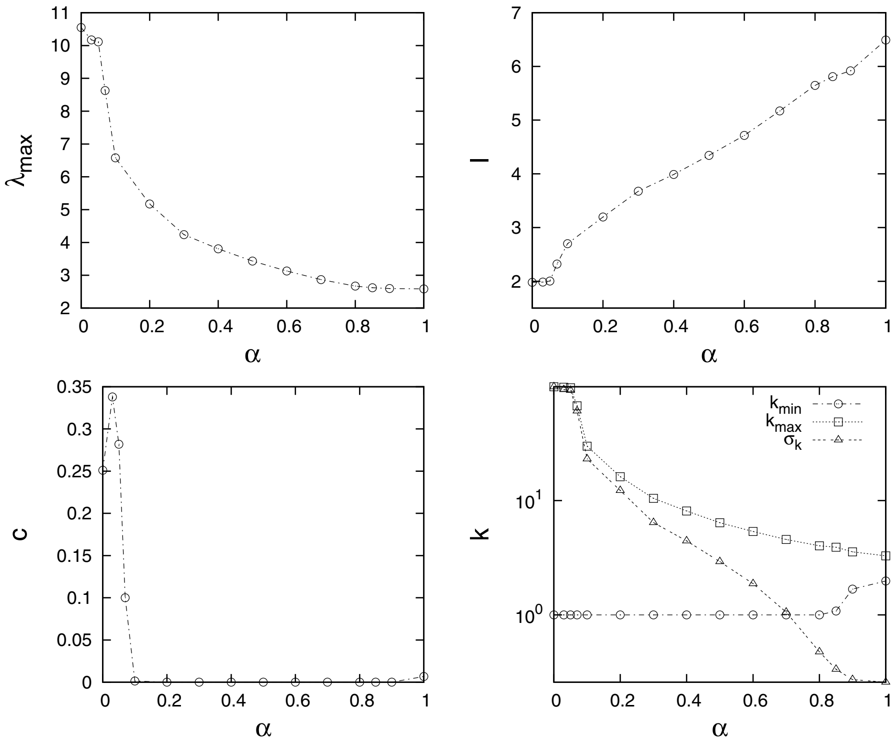


**Fig. 4.** Dependence of some (averaged) network properties on the importance trade-off between resilience and efficiency $\alpha$. (a) max. eigenvalue, (b) average shortest path-length, (c) clustering coefficient, (d) average maximum and minimum degrees and degree variance. The data are averaged over 100 optimized networks with $N = 100$ and $L = 2500$.

middle panel (second row) of Fig. 3 highlights, two groups comprised of high- and low degree nodes can be clearly separated. The high-degree nodes form a core that contains all links that are not needed to connect the periphery with the core. Periphery nodes are all leaves (cf. the bottom right hand panel of Fig. 4 which gives the average maximum and minimum degrees). Within this parameter range, when the relative weights for the requirements for efficiency and stability are shifted, the structure and size of the core change (cf. the networks illustrated in Fig. 3). Mainly, the requirement for resilience determines a cap for the maximum degree in the network. Accordingly, the network can be connected by a relatively small core. When $\alpha$ is increased this core expands, but is also made up of nodes of increasingly smaller degrees. It is also interesting to analyze the structure of the core in this parameter regime.

Notably, it is not cliquish (none of the constructed networks contained any triangle), but it is also strongly dissassortative. Properties of core nodes are very similar, such that the whole network is made up by two hierarchical levels: the well connected core and the leaves at the periphery.

For $\alpha = .85$ one finds another relatively sharp transition, that is related to the disappearance of leave nodes at the periphery. Thus, in the parameter regime $\alpha > .85$ one finds networks that are very similar to the resilience optimized networks described in section 2.

## 5 Conclusions

In this paper we have argued that both efficiency and dynamical resilience play a role in the design of engineering networks. Whereas efficiency can be related to short average pathlengths and short communication pathways between the individual elements, dynamical resilience is associated with the timescale at which a stationary state (which corresponds to the normal mode of operation of the system as we assume) can be recovered. This timescale can be related to the largest eigenvalue of the adjacency matrix of the network.

It is apparent that both requirements are conflicting demands. Efficiency is attained in star-like network structures characterized by the presence of hubs. This, however, entails a large maximum eigenvalue and a decreased reslience. Resilience is realized in degree-homogeneous networks without short loops. However, such networks are far away from allowing optimal communication.

Introducing a parameter that measures the importance of efficiency and resilience in network design we have defined various trade-offs between these two requirements. We find sharp transitions that separate three different regimes: (i) a regime where network design is essentially determined by communication efficiency, (ii) a regime in which resilience and efficiency play a balanced role in determining the network design and (iii) a regime where the structure of optimal networks is dominated by the requirement for resilience.

The networks that trade-off efficiency vs. resilience in various ways are found in the second regime. They are characterized by a clearly distinct core-periphery structure. Periphery nodes are leaves, whereas core nodes roughly have the same degrees and are strongly interlinked with each other.

As a last point, it is interesting to note that these networks that realize a trade off between efficiency and dynamical resilience are very similar in structure to those, that combine efficiency and topological robustness [5]. Interestingly, this observations suggests that all three requirements, efficiency, dynamical reslience and topological robustness can be realized in one type of network.

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