Differential Forms: A New Tool in Economics

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1 Introduction

Econophysics is the transfer of methods from natural to socio-economic sciences. This concept has first been applied to finance[1], but it is now also used in various applications of economics and social sciences [2,3]. The present paper focuses on problems in macro economics and growth. 1. Neoclassical theory [4, 5] neglects the "ex post" property of income and growth. Income Y(K, L) is assumed to be a function of capital and labor. But functions cannot model the "ex post" character of income. 2. Neoclassical theory is based on a Cobb Douglas function [6] with variable elasticity α, which may be fitted to economic data. But an undefined elasticity α leads to a descriptive rather than a predictive economic theory. The present paper introduces a new tool – differential forms and path dependent integrals – to macro economics. This is a solution to the problems above: 1. The integral of not exact differential forms is path dependent and can only be calculated "ex post" like income and economic growth. 2. Not exact differential forms can be made exact by an integrating factor, this leads to a new, well defined, unique production function F and a predictive economic theory.

2 Quesnay´s Biological Model

Economics is the science of production, distribution and consumption of commodities and services in societies. This corresponds to the supply of a living body with food and oxygen and early economists like Françoise Quesnay (1694 – 1774) have based the natural production cycle on the closed blood stream: Labor is transferred from households to agriculture and consumption goods come back from agriculture to households. Consumption goods are the rewards of labor input, fig. 1.

The modern production cycle (δP) is more complex. Labor still comes from households to industry and consumption goods come back from industry to households. But the consumption goods are no more the reward for labor input. There is a second monetary cycle (δ Y). Labor is paid by wages and consumption goods of industry are paid by consumption costs of households, fig.2.

The monetary cycle (δ Y) in fig. 2 measures the production cycle (δ P), Eq.1:

$$\oint \delta Y = -\oint \delta P \tag{1}$$

This law may be regarded as the basis of all economic calculations.

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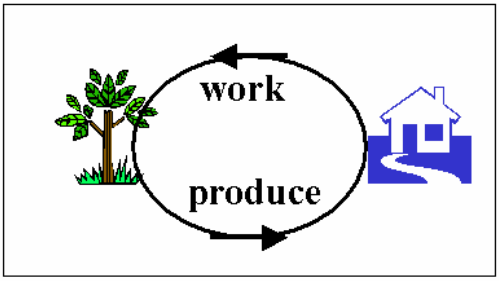

Fig. 1. Quesnay´s model of a natural production cycle of a simple economy. Work of laborers is brought from households to the agriculture. In return produce is brought as a reward from agriculture to households.

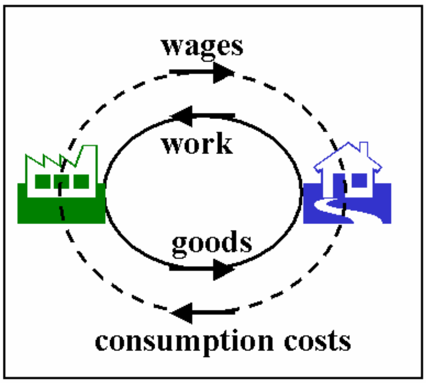

Fig. 2. The model of a modern, complex economy contains two cycles, a production cycle (δ P) (solid line), and a monetary cycle (δ Y) (dashed line)

3 The Law of Economic Survival

The monetary cycle may be split into two parts: income (Y) that is given to households by industry, and consumption costs (C) that are paid from households to industry,

$$\oint \delta Y = Y - C = S > 0 \tag{2}$$

Eq.2 is the law of survival for all economic systems: the surplus (S) must be positive in order to secure survival and growth. This is valid for households, companies, economies, societies – and all living cells. The biological model of economics, Eqs.(1) and (2), are expressed by closed integrals.

4 Differential Forms and Their Integrals

1. The closed integral of an exact differential form dY may be split into two integrals from (A) to (B) and from (B) to (A).

$$\oint dY = \int_A^B dY + \int_B^A dY = \int_A^B dY - \int_A^B dY \equiv 0 \quad (3)$$

As the integral depends only on the limits (A) and (B), both integrals cancel. The closed integral of an exact differential form dY is always zero. An exact differential in two dimensions has the form $dY = a\,dx + b\,dy$, with $\partial b / \partial x = \partial a / \partial y = \partial^2 Y / \partial x\, \partial y$. The term ∂x denotes a partial differentiation. The function Y exists “ex ante”, before integration.

Exact differential forms are used in high school calculus and are applied in economic theory. However, in calculus of more than one dimension or economics with income as a function of capital K and labor L most differential forms will be not exact! This has not been realized in neoclassical theory.

2. The closed integral of a not exact differential form δY may be split into two integrals from (A) to (B) and from (B) to (A). The integrals depend on the path of integration, and both integrals will not cancel [7 – 8]

$$\oint \delta Y = \int_A^B \delta Y + \int_B^A \delta Y = \int_A^B \delta Y - \int_A^B \delta Y \neq 0 \quad (4)$$

The closed integral of a not exact differential form δY is never zero. A not exact differential in two dimensions, $\delta Y = a\,dx + b\,dy$ looks like an exact differential form, but the relation is now $\partial b/\partial x \neq \partial a/\partial y \neq \partial^2 Y / \partial x\, \partial y$. The value of Y can only be given “ex post”, when the path x of integration, (the production process,) is known. The function Y does not exist “ex ante”, each path (x) leads to a different function Y_x. Two dimensional differential forms are generally not exact! Examples are a) heat $\delta Q(T, p)$ as a function of temperature and pressure and b) income $\delta Y(K, L)$ as a function of capital and labor.

5 The First Law of Economics

The equivalence of production and monetary cycle, Eq.1, may be interpreted as the first law of economics: The differential forms δY and δP are equal – up to a total differential form d K, as the closed integral of d K is zero,

$$\delta Y = dK - \delta P \quad (5)$$

Income is given by (δY), production by (δP), the capital of the economic system is (dK). Equation (5) may be compared to the first law of thermodynamics, $\delta Q = dE - \delta W$.

The “first law of economics” is not familiar to economists, as they do not work with differential forms. But the results are clear: 1. the monetary cycle measures the production cycle. 2: Income (δY) and production (δP) are characterized by “ex post” and must be presented by not exact differential forms.

6 The Second Law of Economics

According to the laws of calculus a not exact differential (δY) may be turned into an exact differential (dF) by an integrating factor ($1/\lambda$),

$$dF = \delta Y / \lambda \tag{6}$$

F may be called production function. The law corresponds to the second law of thermodynamics, $dS = \delta Q / T$. The production function (F) is called entropy in physics. The meaning of the parameter (λ) depends on the system: in markets it is a mean price level, in societies the living standard, in economies the GDP per capita, in thermodynamics the mean energy per particle or temperature (T). Again economists are not familiar with the “second law of economics”, but the results are familiar:

1. A production function (F) exists “ex ante” in every economic system. The production function is defined by the elements of the economic system, by the products of a market, by the wages of a company.
2. Efficient markets leads to a single price level (λ) of commodities. This price level is related to the standard of living.

7 Econophysics

The laws (5) and (6) correspond exactly to the laws of thermodynamics. Indeed, thermodynamics and statistics may be regarded as a model science for macro economics. This model character of thermodynamics is called econophysics:

Y: income, costs, profits	Q: heat
P: production	W: work
F: production function	S: entropy
K: capital	E: energy
λ: mean capital level	T: mean energy level

8 The Fundamental Law of Calculus Based Economics

Inserting Eq. (6) into Eq. (5) we obtain

$$\delta Y = dK - \lambda\, dF \tag{7}$$

Eq.(7) is the fundamental equation of calculus based economic theory, and has two parts: the exact differential form of capital dK and the not exact term ($-\lambda\, dF$) of production and growth. Dividing Eq.(7) by λ and N leads to calculus based growth,

$$\delta y / \lambda = d k / \lambda - d f \tag{8}$$

The law of relative growth, Eq.(8), contains the terms of income per capita, $\delta y = \delta Y / N$, of capital per capita $k = K / N$ and of production per capita, $f = F / N$.

Discussion: *In neoclassical theory [4 – 5] a function $Y_{neo} = f(K, N, A)$ is assumed to exist "ex ante", depending on capital K, labor force N and on technology A. For $f(K, N, A)$ the Cobb Douglas function [6] is generally used,*

$$Y_{neo} = A K^{\alpha} N^{1-\alpha} \tag{8a}$$

where α is called elasticity. By introducing income per capita, $y = Y / N$ and capital per capita, $k = K / N$, the Cobb Douglas function may be written as

$$y_{neo} = A k^{\alpha} \tag{8b}$$

The differential $d y$ is now

$$d y_{neo} = \alpha A k^{\alpha - 1} + k^{\alpha} d A \tag{8c}$$

Dividing Eq.(8c) by function y in (8b) leads to logarithmic growth of income,

$$d y_{neo} / y_{neo} = \alpha d k / k + d A / A \tag{8d}$$

In neoclassical theory logarithmic growth of income is due to logarithmic growth of capital and logarithmic growth of technology A.

But income is an "ex post" property and cannot be assumed to exist "ex ante"! The neoclassical equations (8a – d) have to be adjusted according to the laws of calculus:

a) In Eq.(8 a) the function of income Y_{neo} does not exist "ex ante". Only if the path of integration is known, e.g. at constant standard of living, λ = constant, income δY will become exact and a function Y_{λ} will exist.

b) The same holds for y_{neo} in Eq.(8 b).

c) The same holds for $d y_{neo}$ in Eq.(8 c).

d1) In Eqs.(8 d) the exact differential form $d y_{neo}$ must be replaced by the not exact differential form δy of Eq.(8).

d2) In Eq.(8 d) the per capita income function y_{neo} must be replaced by the mean income per capita or standard of living λ.

d3) The term k / α in Eq.(8 d) is to be replaced by λ.

d4) In Eq.(8 d) the logarithmic advancement of technology $d A / A$ must be replaced by reduction of individual entropy, $-d f$. Entropy is a measure of disorder of a system.

The corrections a) to d3) are only of formal importance, and obsolete in efficient markets with λ = constant. But the last point, d4) is the most important achievement of calculus based economics. The undefined term $d A / A$ of advancement of technology in Eq.(8d) is replaced by $-d f$ in Eq.(8), by individual reduction of disorder, by a better (infra)structure of the economic system. Advancement of technology means simplification of the production process and can be measured by entropy. This will be discussed in more detail in the following chapters.

9 Calculation of the Production Function

The important result of calculus based economic theory is the existence of a unique production function F. The production function F is a system function, which depends only on the parameters of the economic system. The calculation of F can only be taken from micro economics, from probability, from entropy. The importance of entropy in economics has already been stated in the literature [9]. The equivalence of production function F and entropy S leads to:

$$F = \ln \Omega \tag{9}$$

Ω is the number of possibilities to place of N objects in K (price) classes:

$$\Omega = N! / (N_1! \ldots N_k!) \tag{10}$$

For large values of N the Stirling formula $\ln N! = N \ln N - N$ may be applied,

$$F(N_k) = N \ln N - \Sigma N_k \ln N_k \tag{11}$$

The new entropy production function F depends only on the number N_k of items in K different classes of the economic system. The production function F of a market is given by the number N_k of pieces of K different commodities. The production function F of a company is given by the number N_k of people in K different jobs. The production function F of an economy is given by the number N_k of companies in K different fields of production. The properties of the new entropy production function F may be explained best by an example:

A company has N_1 permanent and N_2 temporary employees. Fig. 3 shows the new entropy production function

$$F(N_1) = (N_1 + N_2) \ln (N_1 + N_2) - N_1 \ln N_1 - N_2 \ln N_2 \tag{12}$$

and fig. 4 the Cobb Douglas function [6]

$$U(N_1) = A\, N_1^{\alpha} N_2^{1-\alpha} \tag{13}$$

plotted versus the number N_1 of permanent employees in the range from 0 to 10. The parameter in both figures is the number N_2 of temporary employees in the range from 0 to 10.

All functions in figs.(1) and (2) show marginal growth, as is observed in economics. The production function F is larger than the Cobb Douglas function U by a factor of about 1,4 for nearly all values of α. F is clearly the better production function. In addition F is independent of any arbitrary elasticity parameter α and of advancement of technology A.

The new entropy production function F is closely related to advancement of technology. Entropy measures disorder. Producing a car from many different parts means ordering and reducing the disorder of parts (- dF). Advancement of technology is again simplification of the production process, reduction of entropy of the production system.

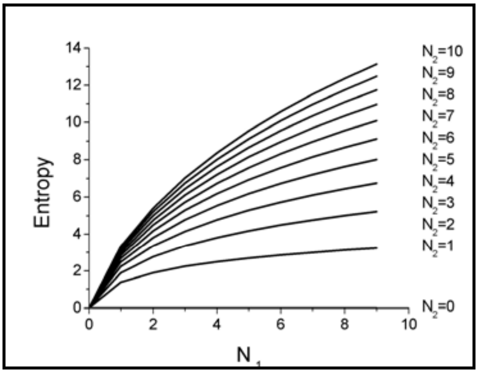

Fig. 3. shows the new production entropy function for a company with permanent and temporary employees: $F (N_2) = (N_1 + N_1) \ln (N_1 + N_1) - N_1 \ln N_1 - N_2 \ln N_2$ is plotted versus the number of permanent employees N_1 in the range from 0 to 10. The number of temporary employees N_2 is also in the range from 0 to 10.

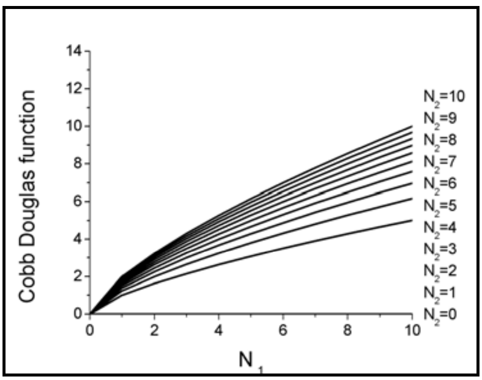

Fig. 4. shows the Cobb Douglas function $U(N_2) = N_1^{\alpha} N_2^{1-\alpha}$, which is plotted versus the number of permanent employees N_1 in the range from 0 to 10. The number of temporary employees N_2 is in the range from 0 to 10. The parameter of elasticity is chosen as $\alpha = 0{,}7$.

10 The Monetary Cycle

According to the second law of economics the integral of the monetary cycle, Eq.(2), may be carried out in the λ - F plane.

$$\oint \delta Y = \oint \lambda dF = \lambda_2 \int_A^B dF - \lambda_1 \int_A^B dF = Y - C = S > 0 \tag{14}$$

Following the idea of Carnot, the integral may be carried out along constant λ and constant F in all economic systems. In fig. 3 the integral is carried out in one direction at constant high income level (λ_2) and back at low capital level, (λ_1).

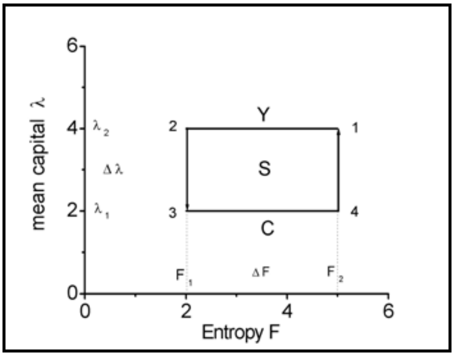

Fig. 5. Shows the integral of the monetary cycle, Eq.(2), after Carnot in the λ – F plane. The integral is carried out in one direction at constant high income level (λ_2) and back at low capital level, (λ_1) according to Eq.(14).

Income Y, costs C and surplus S correspond to

$$Y_\lambda = \lambda_2 \int_A^B dF = \lambda_2 \Delta F(N) \tag{15}$$

$$C_\lambda = -\lambda_1 \int_A^B dF = \lambda_1 \Delta F(N) \tag{16}$$

$$S_\lambda = \quad Y - C \; = \Delta\lambda \; \Delta \, F(N) \; > 0 \tag{17}$$

Income is a path dependent integral. Y_λ in Eq.(15) is the income at constant standard of living (λ_2). Costs are a path dependent integral. C_λ in Eq.(16) are the costs at constant standard of living (λ_1). With $\lambda_2 > \lambda_1$ the surplus S_λ in Eq.17 is always positive.

Fig. 3 may be interpreted by the monetary cycle of the farm selling produce. Δλ is the change in price levels bringing produce from the field to the market.

ΔF > 0 : positive change of entropy = distribution (of money) to workers in the field,

ΔF < 0 : negative change of entropy = collection (of money) from customers at markets.

1 → 2: Money collected from rich (λ_2) customers at the market, $Y_\lambda = \lambda_2 \Delta F$.
2 → 3: Money transferred from market to fields, ΔF = 0.
3 → 4: Money distributed to poor (λ_1) workers in the fields, $C_\lambda = \lambda_1 \Delta F$.
4 → 1: Money brought by workers to the market, Δ F = 0.

11 Capital Pumps

The monetary cycle is applied in all economic enterprises, in companies, in business and trade. The equivalence of economic and thermodynamic equations makes it possible to discuss the closed integral of Eqs.(2) or (14) as monetary or heat production cycle. The results in companies, business and trade may be compared to the results in generators or refrigerators:

1. a. A heat pump requires little energy to draw heat $Q_1 = T_1 \Delta S$ from a cold river and to deliver heat $Q_2 = T_2 \Delta S$ to a warm house with a high efficiency.

1. b. An import company pays low wages $C_\lambda = \lambda_1 \Delta F$ for production in a poor country and receives a high income $Y_\lambda = \lambda_2 \Delta F$ from sales in a rich country with a high efficiency. All monetary cycles of households, companies and economies or states may be called capital pumps!

2. a. A refrigerator is a Carnot machine. It starts creating cold inside and warm outside immediately, after it is plugged to the wall. But we have to close the door and separate inside and outside. The larger the difference of temperatures, the higher is the efficiency of the refrigerator.

2. b. When a company starts producing, it will automatically create a richer and a poorer side. For this capital and labor have to be separated, or the company will not work. The economic gap between capital and labor will grow with time. The larger the difference, the higher the efficiency.

Companies and business tend to optimize profits by paying wages as low as possible and selling their products as expensive as possible. In efficient markets there are limits to high prices and low wages. Competition will lead to one price level for a specific commodity and one wage level for a specific job. But unemployment immediately leads to lower wages and trusts avoid the laws of efficient market and will lead to higher prices. In a well working economy the government will try to protect the efficient market by enforcing proper antitrust rules and to keep unemployment low.

12 Economic Growth

1. The Carnot cycle of a motor requires two temperatures, inside and outside. The heat that is created in every cycle, will dissipate to the inside and outside. The distribution of heat will influence the efficiency of the motor. If the inside gets hotter and hotter at constant outside temperature (due to a cooling system), the efficiency will grow quickly. If the outside runs hot (due to a failure of the cooling system), the motor stops.

2. The monetary cycle creates two groups of people in farms, companies, business firms: farmer and laborers, owner and workers, capital and labor, first and third world, rich and poor, Y and C. Both groups together form the economic system. Accordingly, both groups will have to agree, how to divide the net output of each cycle. This is negotiated periodically by workers and employers, by unions and industry, by world trade conferences. In economic theory this is generally treated by game theory.

In a Carnot process the lower/cooler/poorer side (employees) will obtain the fraction (p) of the net output, the higher/ hotter/richer side (employer) will get the fracion (1 - p). If both groups reinvest their fraction $p (Y_2 - Y_1)$ and $(1- p) (Y_2 - Y_1)$, they will grow in time (t) with each cycle. We obtain two equations, for $C = Y_1$ (t) and Y = Y_2 (t):

$$d\,Y_1(t) = p\,(Y_2 - Y_1)\,dt \tag{18}$$

$$d\,Y_2(t) = (1-p)\,(Y_2 - Y_1)\,dt \tag{19}$$

For $p \neq ½$ the solution of this set of differential equations is given by:

$$Y_1(t) = Y_{10} + p\,[Y_{20} - Y_{10}]\,[\exp((1-2p)t) - 1]\,/(1-2p) \tag{20}$$

$$Y_2(t) = Y_{20} + (1-p)\,[Y_{20} - Y_{10}]\,[\exp(1-2p)\,t) - 1]\,/(1-2p) \tag{21}$$

For $p = ½$ the solution is given by

$$Y_1(t) = Y_{10} + ½\,[Y_{20} - Y_{10}]\,t \tag{22}$$

$$Y_2(t) = Y_{20} + ½\,[Y_{20} - Y_{10}]\,t \tag{23}$$

The equations may be applied to all interdependent systems, to workers and employers, unions and industry, or interdependent countries. The results are presented in fig. 6.

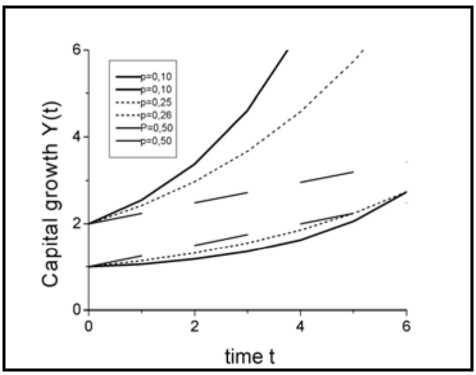

Fig. 6. Economic growth of a system of two parties (capital and labor) according to Eqs.18 and 19. One partner (Y_1) receives p percent of the surplus, the other partner (Y_2) receives (1-p) percent. For p = 0,10 we obtain high exponential growth, for p = 0,25 exponential growth is less. For p = 0,50 only linear growth is obtained.

For p < 0,50 economic growth is exponential. Surprisingly a percentage of p = 10 % of the profit is in the long run more profitable for the worker than p = 25 %.

The second surprise is the fair deal, 50:50. This distribution of profits leads only to linear growth and is the least attractive distribution between two partners.

Fig 7 shows economic growth in China and USA [9] between 1990 and 2005. The difference in GDP per capita ($C = Y_1$) and ($Y = Y_2$) is the basis for the economic motor between China and all western countries (USA). Similar exponential growth has been observed in Germany and Japan after World War II, when trade with USA and other western countries led to exponential growth in both countries.

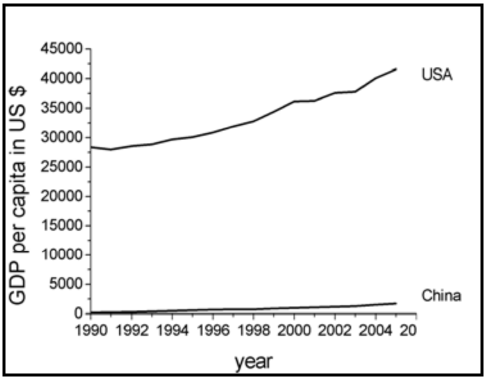

Fig. 7. Economic growth of a system of two parties (China and USA). The difference in GDP per capita [9] (Y_1) and (Y_2) is the basis of the economic motor between China and all western countries (USA).

13 Conclusion

Differential forms bring new ideas to economic theory, however, many methods of neoclassical theory and Lagrange optimization may be retained, if the Cobb Douglas function is replaced by the entropy of the economic system. Entropy makes calculation and understanding of economic processes more precise. Indeed, the concept of differential forms brings economics close to natural science.

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