Enhancement of Synchronizability of the Kuramoto Model with Assortative Degree-Frequency Mixing

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Abstract. Assortative mixing feature is an important topological property in complex networks. In this paper, we extend degree-degree mixing feature to non-identical nodes networks. We propose the degree-frequency correlation coefficient to measure the correlations between the degree and the natural frequency of oscillators. We find that the perfect assortative degree-frequency network is quite easy to synchronize. We also investigate the synchronization of complex networks with different degree-frequency coefficient.

Keywords: synchronization, Kuramoto model, correlation, assortative.

1 Introduction

Synchronization in complex systems has received increasing attention in recent years [1,2]. Recently, the development of complex networks [3,4,5,6,7,8,9,10] has provided a challenging framework for the study of synchronization of dynamical systems. People have discovered the relationships between network synchronizability and network structure topological features, such as degree distribution, shortest path length, betweenness centrality [13,12,14,15,16].

Degree-degree mixing correlation is an important topological property in complex networks. A network is said to show assortative mixing if the nodes in the network tend to be connected to other similar nodes [17,18]. Assortative/disassortative mixing is present in many real-world networks. For instance, social networks exhibit assortative mixing, whereas biological and technological networks display a disassortative mixing.It has been found that the networks synchronized better when they displayed a disassortative Degree-Degree (DD) mixing, i.e. nodes with low degree are more likely to be connected to nodes with higher degree [19,20]. Very recently, a new finding that the strong anticorrelations between natural frequencies of adjacent oscillators can enhance the synchronization behavior is presented in [21].

Our recent work discovered that the synchronizability of complex network is effected by not only the distributions of degree and natural frequency, but also the correlations of the degrees and natural frequencies of oscillators [22]. In this paper, we extend the degree-degree mixing to the degree-frequency mixing, and

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investigate the synchronization of the networks with assortative degree-frequency mixing.

The outline of the paper is as follows. In Section 2, we describe the synchronization of Kuramoto model. The degree-frequency mixing coefficient is introduced in Section 3. In Section 4, we study the synchronization of complex networks with different degree-frequency mixing patterns. Finally, some conclusions are given in Section 5.

2 Synchronization of Kuramoto Model

The Kuramoto model describes a large population of coupled limit-cycle oscillators whose natural frequencies are drawn from some prescribed distribution [23,24], whose dynamics are given by

$$\dot{\theta}_i = \omega_i + c\sum_{j=1}^{N} a_{ij}\sin(\theta_j - \theta_i), i = 1, \ldots, N \tag{1}$$

where θ_i and ω_i denote the phase and the natural frequency of the i-th oscillator, respectively. $c > 0$ is called the coupling strength. The adjacent matrix a_{ij} represents the coupling configuration of the network: if there is a connection between oscillators i and j, $a_{ij} = a_{ji} > 0$; otherwise, $a_{ij} = a_{ji} = 0$ and note $a_{ii} = 0$. The initial values of θ_i are randomly drawn from a uniform distribution in the interval $(-\pi, \pi)$.

Let the average phase $\langle\theta(t)\rangle = \frac{1}{N}\sum_{i=1}^{N}\theta_i(t)$ be the reference state. Therefore,

$$\begin{aligned}\langle\dot{\theta}(t)\rangle &= \frac{1}{N}\sum_{i=1}^{N}\dot{\theta}_i(t)\\ &= \frac{1}{N}\sum_{i=1}^{N}\omega_i + \frac{c}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\sin(\theta_j(t) - \theta_i(t))\\ &= \langle\omega\rangle\end{aligned} \tag{2}$$

which is the average frequency.

Define the error state as:

$$e_i(t) = \theta_i(t) - \langle\theta(t)\rangle, \qquad i = 1, \cdots, N \tag{3}$$

The derivative $\dot{e}_i(t)$ is:

$$\dot{e}_i(t) = \Delta\omega_i + c\sum_{j=1}^{N}\sigma_{ij}\sin(e_j(t) - e_i(t))$$

We say that $\{e_i^*, 1 \leq i \leq N\}$ are the *equilibrium points* [25,26] if

$$\dot{e}_i(t)|_{e_i^*} = 0, \qquad i = 1, \cdots, N \tag{4}$$

The dynamical network (1) is said to *synchronize* if $\dot{e}_i(t) = 0$, i.e., the phase errors $e_i(t) = \theta_i(t) - \langle\theta(t)\rangle$, $1 \leq i \leq N$ become constant asymptotically as $t \to \infty$, therefore,

$$\dot{\theta}_1(t) = \dot{\theta}_2(t) = \cdots = \dot{\theta}_N(t) = \langle\omega\rangle \tag{5}$$

When the coupling strength is small, all the oscillators act as if they are uncoupled. As the coupling becomes stronger, the frequencies of some oscillators become identical, i.e., $\dot{\theta}_i(t) = \langle\omega\rangle$ and $\dot{e}_i(t) = 0$. With the increase of the couplings, more and more oscillators achieve to synchronize. If the coupling is strong enough, finally, all $\dot{e}_i(t) = 0$.

3 Measuring Degree-Frequency Mixing

In order to reflect the degree to which the parameters are related, we introduce Degree-Frequency (DF) mixing coefficient to measure the correlation between the degrees and natural frequencies of oscillators

$$r_{k\omega} = \frac{N^{-1}\sum_{i=1}^{N} k_i\omega_i - \langle k\rangle\langle\omega\rangle}{\sqrt{N^{-1}\sum_{i=1}^{N} k_i^2 - \langle k\rangle^2}\sqrt{N^{-1}\sum_{i=1}^{N}\omega_i^2 - \langle\omega\rangle^2}} \tag{6}$$

where $\langle k\rangle$ and $\langle\omega\rangle$ are the average degree and average frequency of the network, respectively. $r_{k\omega}$ lies in the range $[-1, 1]$. When $r_{k\omega} > 0$, the correlation between the degree and natural frequency is an increasing linear relationship. We call the network is an *assortative degree-frequency mixing*. If $r_{k\omega} < 0$, we call it is a *disassortative degree-frequency mixing*. When $r_{k\omega} \approx 0$, there is no correlation mixing.

4 Synchronization of the Assortative Degree-Frequency Mixing Networks

In this section, we assume that the natural frequency of each oscillator is proportional to its degree. Then, the network is assortative degree-frequency mixing, $r_{k\omega} = 1$. We randomly exchange any two oscillators' natural frequencies to obtain a series of network models with the same network structure and the same natural frequencies but having different DF mixing. We first study the Barabsi-Albert scale-free network [7] with $N = 1000$ oscillators. The degree distribution and the natural frequency distribution are both heterogeneous. We study the effect of the assortative degree-frequency mixing pattern on the synchronization of complex network. When the network is perfect assortative mixing, i.e., $r_{k\omega} = 1$, the network is very easy to synchronize. All the oscillators could be synchronized through a very tiny coupling. With the decrease of the $r_{k\omega}$, the network becomes difficult to synchronize. When the coupling $c = 0.15$, more than 60% oscillators synchronize in the network with $r_{k\omega} = 0.823$, however, only a small fraction (less than 10%) of nodes in the network with $r_{k\omega} = 0.314$ (See Fig 1).

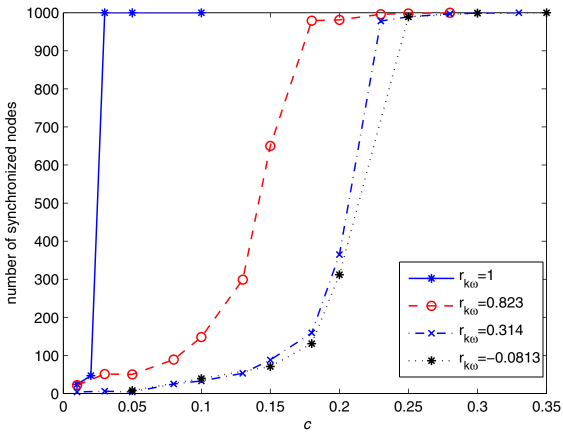

Fig. 1. The number of synchronized oscillator of the BA scale-free network with different $r_{k\omega}$. The size of networks is $N = 1000$. All quantities are averaged over 5 realizations.

We also consider another case: the network is Erdös-Rényi random network model [27] with $N = 1000$. All the nodes have the similar number of degrees. We construct a perfect DF mixing network model. Through randomly exchanging the natural frequencies of nodes, we get the networks with different DF mixing coefficients. It was found that, as the decrease of the assortativity, the network is hard to synchronize (see Fig 2), although the degree distribution and the natural frequency distribution of the oscillators are homogeneous.

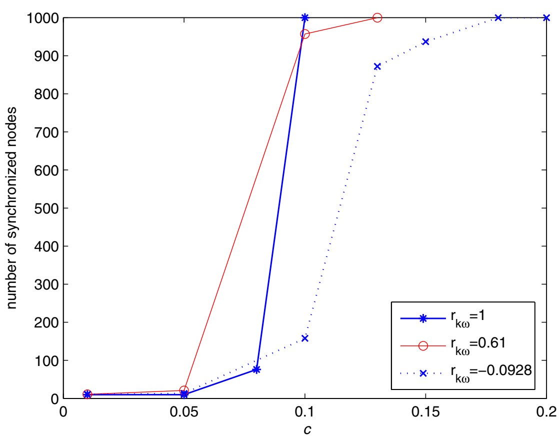

Fig. 2. The number of synchronized oscillator of the random network with different $r_{k\omega}$. The size of networks is $N = 1000$. All quantities are averaged over 5 realizations.

5 Conclusion

Synchronization of complex networks is influenced by many topological properties, such as degree distribution, shortest path length, betweenness centrality and degree-degree correlation etc. In this paper, we consider the effect of the correlation between the degrees and natural frequencies of oscillators on the synchronization. We investigated the synchronization of the assortative DF mixing networks with heterogeneous and homogeneous network topologies. We found that more oscillators can be synchronized when the network is assortative degree-frequency mixing pattern. Especially, when the networks is perfect assortative one, it can be synchronized through a very small coupling.

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