Consensus Seeking and Controlling over Directed Delayed Networks

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Abstract. In this paper, we are concerned about the consensus problem for multi-agent systems in the presence of directed information flow and arbitrary communication delays. For each agent, only delayed local information can be used to adjust its value. Information flow between connected agents can be asymmetric. It will be shown that, whatever the communication delays are in principle, consensus will eventually be achieved for strongly connected networks. Furthermore, a local controller is designed for one of the agents (but not for each agent) to control all the agents. Very weak feedback strength is proved to be effective for the control of multi-agent systems. Numerical simulations are also performed to verify our theoretical analysis.

Keywords: consensus, arbitrary communication delay, directed network.

1 Introduction

Recently, consensus problem in multi-agent systems has attracted substantial research effort from researchers in different fields [12][7][8]. The surprising focus on consensus is in part due to the emergence of such phenomenon in biological and physical system (school of fish, flocks of birds), as well as its powerful applications in sensor network, unmanned aerial vehicles and robot networks. For such multi-agent systems, certain collective behavior is generated via local communications among the agents. A large amount of literature has been devoted to the consensus behavior in nature, such as flocking, herding and schooling [12][8][6][2]. Group coordination and synchronization problems of networked coupled agents (or oscillators) have been recently addressed in [5][10][15].

Since communication delay is ubiquitous, it is of course practically necessary to take time-delay into consideration when dealing with multi-agent systems. The effect of communication delays on consensus and synchronization behaviors has been investigated in [9][14][16]. In [9], stability results were derived for multi-agent systems with identical communication delays (including agents' self-delay). In [14], consensus problem was studied for networked agents with arbitrary time-delays. However, the network topology in [14] was assumed to be connected and bidirectional or unidirectional formed in closed rings. In fact, there are a variety of practical examples for directed network, such as information flow (say between PC stations and servers) within internet; communication channels among different confidential levels in army network.

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In this paper, we will firstly study the consensus seeking problem in multi-agent systems with directed information flow and communication delays. Communication delays can be arbitrary finite constants. We will show that consensus behavior will eventually be generated if the network is strongly connected. Due to the introduction of arbitrary communication delays, the final consensus state of multi-agent system is very hard to predict. However, for many physical, social and biological systems, there is a common requirement to regulate the behavior of large ensembles of interacting agents [11][1]. Hence, external force is needed if we want to regulate the agents to certain prescribed state. In this paper, a static feedback controller is designed to control one of the agents, and the target state is propagated to other agents through local connections. Very weak feedback strength is proved to be effective for the control of multi-agent systems. Numerical example are also given to illustrate the derived results.

2 Some Preliminaries

The information exchange among agents is described by means of directed or undirected graphs. In this paper, strongly connected directed graph (including undirected graph as a special case) is considered. A directed graph is said to be strongly connected if, for each pair of nodes i and j, one can find a path from node i to node j, and meanwhile a path from node j to node i. A directed graph $\mathscr{G}$ is a pair $(\mathscr{N},\mathscr{E})$, where $\mathscr{N}=\{1,2,...,N\}$ is a finite nonempty node set and $\mathscr{E}\subseteq\mathscr{N}\times\mathscr{N}$ is an edge set of ordered pairs of nodes, called edges. The edge $(i,j)\in\mathscr{E}$ denotes that agent j can receive information from agent i. Let matrix $A=(a_{ij})\in\mathbb{R}^{N\times N}$ be the weighted adjacency matrix of the directed graph $\mathscr{G}$. We assume that the adjacency elements associated with the edges of the directed graph are positive, that is, $a_{ji}>0\Leftrightarrow(i,j)\in\mathscr{E}$. And the diagonal elements a_{ii} is assumed to be zero.

Let $\bar{A}=(\bar{a}_{ij})\in\mathbb{R}^{N\times N}$ be the Laplacian matrix of the directed graph $\mathscr{G}$ with its elements defined as follows: $\bar{a}_{ij}=a_{ij}$ for $i\neq j$, and $\bar{a}_{ii}=-\sum_{j=1,j\neq i}^{N}a_{ij}$ for $i\in\mathscr{N}$. $\bar{A}$ is irreducible due to the strong connectivity of the network topology.

3 Consensus Seeking and Controlling

By introducing communication delays, we consider the following multi-agent system with directed information flow:

$$\dot{x}_i(t)=\sum_{j=1,j\neq i}^{N}a_{ij}(x_j(t-\tau_{ij})-x_i(t)),\ \ i\in\mathscr{N} \tag{1}$$

where $x_i(t)\in\mathbb{R}^n$ denotes the state of agent i, $\tau_{ij}>0$ means the communication delay from agent j to agent i, and a_{ij} is the element of the weighted adjacency matrix. $a_{ij}>0$ implies that agent i can obtain time-delayed (with delay τ_{ij}) information $x_j(t-\tau_{ij})$ from agent j.

In model (1), communication delays τ_{ij} can be diverse, and self-delay is not considered for all agents. Consensus is said to be achieved if $x_i(t)\rightarrow c$ as $t\rightarrow\infty$ for any $i\in\mathscr{N}$, where c is a constant vector.

3.1 Consensus Seeking

In this subsection, we will show that, whatever the communication delays are, the consensus of such multi-agent system can be achieved. It means that consensus is quite robust against the communication delays.

Theorem 1. *Consider the multi-agent system* (1) *with strongly connected graph* $\mathscr{G}$*. Whatever communication delays* τ_{ij} *are, consensus is asymptotically reached under all initial conditions. That is,*

$$\lim_{t\to\infty} x_i(t) \to c, \ \ \forall \ \ i \in \mathscr{N}, \tag{2}$$

where $c \in \mathbb{R}^n$ *is a constant vector.*

Proof. According to the construction of the Laplacian matrix $\bar{A}$, one has that zero is an eigenvalue of matrix $\bar{A}$. Since the graph is strongly connected, matrix $\bar{A}$ is irreducible which implies that the real parts of the rest eigenvalues is less than zero [4]. Let $\xi = (\xi_1, \xi_2, ..., \xi_N)^T$ be the left eigenvector of Laplacian matrix $\bar{A}$ with respect to eigenvalue zero. By Perron-Frobenius theorem [4], it can be concluded that $\xi_i > 0$ for $i \in \mathscr{N}$.

Since ξ is the left eigenvalue of matrix $\bar{A}$ corresponding to eigenvalue zero, one has that $\xi\bar{A} = 0$, which implies that

$$\xi_i \bar{a}_{ii} = -\sum_{j=1, j\neq i}^{N} \xi_j \bar{a}_{ji}. \tag{3}$$

Further because $\bar{a}_{ii} = -\sum_{j=1}^{N} a_{ij}$, we can obtain that

$$\sum_{j=1}^{N} \xi_i a_{ij} = \sum_{j=1}^{N} \xi_j a_{ji}. \tag{4}$$

Consider the Lyapunov-Krasovskii functional as:

$$V(t) = V_1(t) + V_2(t), \tag{5}$$

where

$$V_1(t) = \frac{1}{2}\sum_{i=1}^{N} \xi_i x_i^T(t) x_i(t)$$

and

$$V_2(t) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{t-\tau_{ji}}^{t} \xi_j a_{ji} x_i^T(\theta) x_i(\theta) d\theta.$$

Differentiating functions $V_1(t)$ and $V_2(t)$ along the solution of system (1) gives that

$$\begin{aligned}
&\dot{V}_1(t)\\
&= \sum_{i=1}^{N} \xi_i x_i^T(t)\dot{x}_i(t)\\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} \xi_i a_{ij}[x_i^T(t)x_j(t-\tau_{ij}) - x_i^T(t)x_i(t)]\\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} \xi_i a_{ij} x_i^T(t)x_j(t-\tau_{ij}) - \sum_{i=1}^{N}\sum_{j=1}^{N} \xi_i a_{ij} x_i^T(t)x_i(t),
\end{aligned}$$

and

$$\begin{aligned}
&\dot{V}_2(t)\\
&= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \xi_j a_{ji}[x_i^T(t)x_i(t) - x_i^T(t-\tau_{ji})x_i(t-\tau_{ji})]\\
&= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \left[\xi_i a_{ij} x_i^T(t)x_i(t) - \xi_i a_{ij} x_j^T(t-\tau_{ij})x_j(t-\tau_{ij})\right].
\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}
&\dot{V}(t)\\
&= \dot{V}_1(t) + \dot{V}_2(t)\\
&= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \xi_i a_{ij} x_i^T(t)x_i(t) + \sum_{i=1}^{N}\sum_{j=1}^{N} \xi_i a_{ij} x_i^T(t)x_j(t-\tau_{ij})\\
&\quad -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \xi_i a_{ij} x_j^T(t-\tau_{ij})x_j(t-\tau_{ij})\\
&= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \xi_i a_{ij}\left[x_i^T(t)x_i(t) - 2x_i^T(t)x_j(t-\tau_{ij})\right.\\
&\quad \left. + x_j^T(t-\tau_{ij})x_j(t-\tau_{ij})\right]\\
&= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \xi_i a_{ij}\big(x_i(t) - x_j(t-\tau_{ij})\big)^T\big(x_i(t) - x_j(t-\tau_{ij})\big)\\
&\le 0. \qquad (6)
\end{aligned}$$

Hence, the limit of $V(t)$ exists and is finite. Then the boundedness of $x_i(t)$ can be obtained for $i \in \mathcal{N}$ by referring to the construction of $V(t)$. Further, system (1) implies that $\dot{x}_i(t)$ is bounded for $i \in \mathcal{N}$. Thus, we can conclude that $\ddot{V}(t)$ is also bounded. According to Barbalat's Lemma, one obtain that $\lim_{t\to\infty} a_{ij}(x_i(t) - x_j(t-\tau_{ij})) = 0$. Therefore, $\lim_{t\to\infty}(x_i(t) - x_j(t-\tau_{ij})) = 0$ for agents i and j satisfying $a_{ij} > 0$.

Since the network is strongly connected, for any two agents i and j, one can find a path from agent i to agent j and simultaneously a path from agent j to agent i. Hence, for each pair of agents $i, j \in \mathscr{N}$, one can find two constants τ_{ij}^* and τ_{ji}^*, which are certain linear combinations of all communication delays τ_{ij}, such that $x_i(t) \to x_j(t-\tau_{ij}^*)$ and $x_i(t-\tau_{ji}^*) \to x_j(t)$. Hence, $x_i(t-\tau_{ij}^*-\tau_{ji}^*) \to x_i(t)$ for each $i \in \mathscr{N}$, which implies that $x_i(t)$ tends to be periodic with constant period $\tau_{ij}^*+\tau_{ji}^*$. Noting the fact that $\dot{x}_i(t) \to 0$ as $t \to \infty$, we can conclude that $x_i(t)$ tends to a constant steady state c_j.

Since the matrix A is irreducible, it can be concluded that the largest invariant manifold of system (1) is $\mathscr{M} = \{x_1(t),x_2(t),...,x_N(t)|x_1(t) = x_2(t) = \cdots = x_N(t)\}$. This implies that there exists a constant vector c such that $c_i = c$ for each $i \in \mathscr{N}$. Hence, $x_i(t) \to c$ as $t \to \infty$. This completes the proof.

Remark 1. From Theorem 1, we can observe that consensus process is quite robust against the communication delays and the directivity of the graph. Strong connectivity of the graph can guarantee sufficient information exchange among the agents until consensus is achieved.

3.2 Consensus Controlling

It has been shown that consensus among agents will appear regardless of communication delays. Due to the introduction of arbitrary communication delays, the final consensus state c is very hard to predict. Whereas, the need of regulating the behavior of large ensembles of interacting agents is a common feature for many physical, social and biological systems [11]. For example, many regulatory mechanisms have been discovered in the context of biological, physiological and cellular processes, which play fundamental role for guaranteeing the correct function of the whole network [13]. Hence, certain controller will be designed to force all agents to certain desired constant state $x^* \in \mathbb{R}^n$.

Consider following controlled multi-agent system:

$$\dot{x}_i(t) = \sum_{j=1}^{N} a_{ij}(x_j(t-\tau_{ij}) - x_i(t)) + u_i(t), \quad i \in \mathscr{N} \tag{7}$$

with controller $u_i(t)$ designed as follows

$$u_i(t) = \begin{cases} -k(x_1(t)-x^*), & i=1; \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

where $k > 0$ is a constant. From (8), it can be seen that only the first agent is controlled.

Let $e_i(t) = x_i(t) - x^*$ be the error state between current state and the objective state, we can obtain the following error dynamical system:

$$\begin{cases} \dot{e}_1(t) = \sum_{j=2}^{N} a_{1j}(e_j(t-\tau_{1j}) - e_1(t)) - ke_1(t), \\ \dot{e}_i(t) = \sum_{j=1}^{N} a_{ij}(e_j(t-\tau_{ij}) - e_i(t)), \quad i \in \mathscr{N} \backslash \{1\}. \end{cases} \tag{9}$$

Theorem 2. *Consider the controlled multi-agent system* (9) *with strongly connected graph* $\mathscr{G}$. *Whatever communication delays* τ_{ij} *are, states of all agents will be successfully controlled to the objective state* x^*. *That is,*

$$\lim_{t\to\infty} x_i(t) \to x^*, \quad \forall \; i \in \mathscr{N}, \tag{10}$$

where $x^* \in \mathbb{R}^n$ *is the objective state.*

Proof. Consider a positive definite Lyapunov-Krasovskii functional for the controlled multi-agent system (9) as

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} \xi_i e_i^T(t) e_i(t) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{t-\tau_{ji}}^{t} \xi_j a_{ji} e_i^T(\theta) e_i(\theta) d\theta.$$

By referring to equality (4) and analysis in Theorem 1, we can obtain the derivative of this functional along the trajectories of the error dynamical system (9) as

$$\begin{aligned} &\dot{V}(t) \\ &= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \xi_i a_{ij} \big(e_i(t) - e_j(t-\tau_{ij})\big)^T \cdot \big(e_i(t) - e_j(t-\tau_{ij})\big) \\ &\quad -\xi_1 k e_1^T(t) e_1(t) \\ &\leq 0. \end{aligned} \tag{11}$$

It is obvious that $\dot{V}(t) = 0$ if and only if $e_1(t) = 0$ and $e_i(t) = e_j(t-\tau_{ij})$ for each pair of agents (i, j) satisfying $a_{ij} > 0$. Hence, the set $\mathscr{S} = \{e_i(t) = e_j(t-\tau_{ij})$ for $(i, j) \in \mathscr{E}, e_1(t) = 0\}$ is the largest invariant set contained in $\dot{V}(t) = 0$ for the controlled system. Then by using the well-known invariant principle of functional differential equations [3], starting with arbitrary initial values, the orbit of system (9) converges asymptotically to the set $\mathscr{S}$, that is, $e_i(t) \to e_j(t-\tau_{ij})$ for each pair of agents (i, j) satisfying $(i, j) \in \mathscr{E}$ and $e_1(t) \to 0$ as $t \to \infty$. Following similar analysis given in Theorem 1, we can obtain that $e_i(t) \to e_j(t)$ as $t \to \infty$. Since $e_1(t) \to 0$, one can conclude that $e_i(t) \to 0$ for each $i \in \mathscr{N}$. Therefore, $x_i(t) \to x^*$ for each $i \in \mathscr{N}$, and the proof is completed.

Remark 2. One important merit of the control scheme is that only one agent (rather than all agents) is controlled, and local connections among agents are also fully utilized during the control process. It should also be noted that the feedback strength can be very small. Hence, the designed control strategy is quite easy and cheap to implement in reality.

Remark 3. Any node in the strongly connected network can be chosen as the agent to be controlled and the objective state will also be achieved. The biggest node with the highest weighted out-degree should be a good choice to make the control process much more effective and quicker.

4 Numerical Examples

In this section, numerical examples will be given to verify the derived analytical results. Multi-agent system with 6 agents is taken into account. Topology of the communication graph is shown in Fig.1, and the constants mean the weight of the directed edges. The dimension of the agents' states are selected to be $n = 3$.

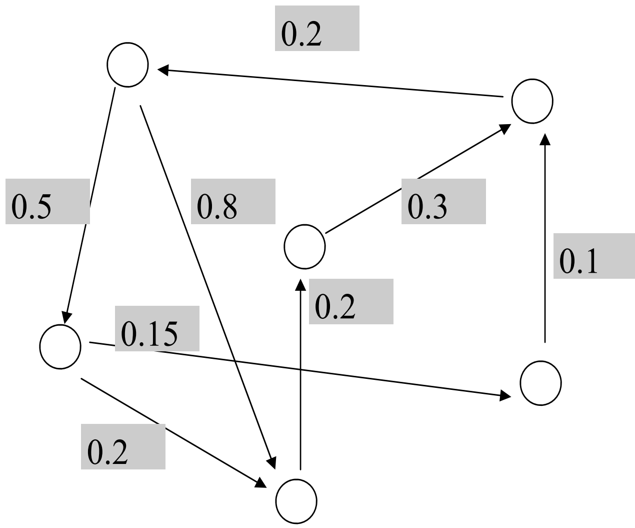

Fig. 1. Communication graph among six agents

According to the graph topology, one can obtain the corresponding weighted adjacency matrix as

$$A = \begin{bmatrix} 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.1 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2 \\ 0 & 0 & 0.15 & 0 & 0 & 0 \\ 0.8 & 0 & 0.2 & 0 & 0 & 0 \end{bmatrix}. \quad (12)$$

.

Then the Laplacian matrix $\bar{A}$ can be obtained as

$$\bar{A} = \begin{bmatrix} -0.2 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & -0.4 & 0 & 0.3 & 0.1 & 0 \\ 0.5 & 0 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.2 & 0 & 0.2 \\ 0 & 0 & 0.15 & 0 & -0.15 & 0 \\ 0.8 & 0 & 0.2 & 0 & 0 & -1 \end{bmatrix}. \quad (13)$$

$\xi = (0.1424, 0.1424, 0.0570, 0.1068, 0.0356, 0.1068)^T$ is the left eigenvector of matrix $\bar{A}$ corresponding to eigenvalue zero. By Theorem 1, we can conclude that consensus can be achieved regardless of the communication delays and initial values. With communication delays randomly chosen from $(0,3)$ and initial values randomly from $(-2,2)$, Fig.2 shows the consensus behavior of multi-agent system with adjacency matrix A given in (12). Two sub-figures in Fig.2 are plotted under the same initial values and different communication delays. It can be observed that the final consensus states in two sub-figures are different from each other due to the different communicationdelays.

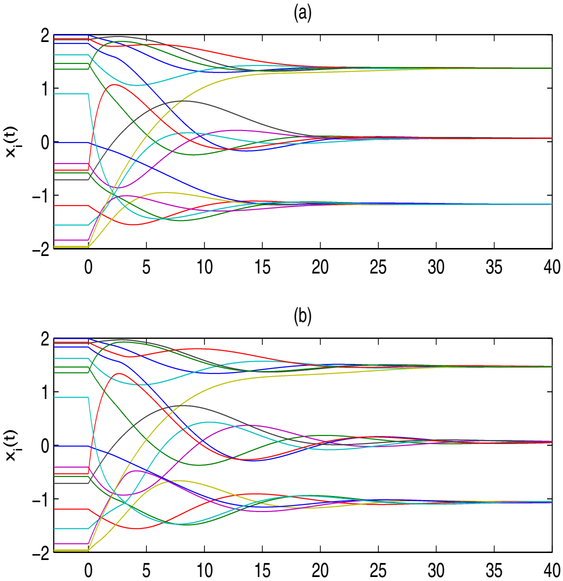

Fig. 2. Consensus seeking under different communication delays and same initial values

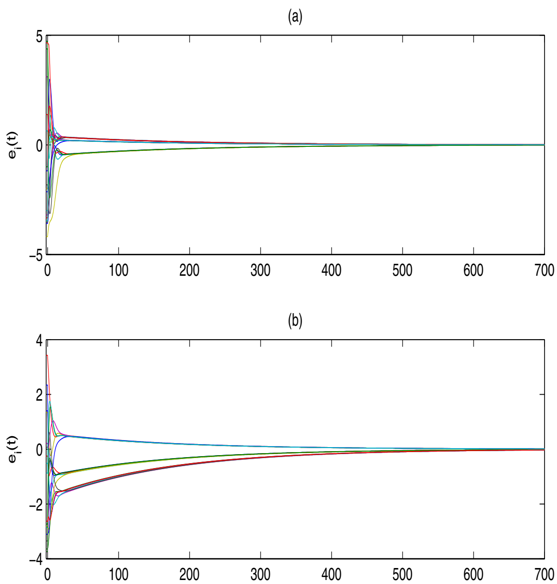

Fig. 3. Consensus controlling under different communication delays and different initial values

We can conclude that the consensus state is quite closely related with the communication delays.

In order to regulate the agents' behavior, external controller is pinned into the first agent. Let the original point $x^* = 0 \in \mathbb{R}^3$ be the objective state. Take feedback strength as $k = 0.02$. Initial values and communication noises are respectively randomly chosen from $(-5,5)$ and $(0,2)$. Fig.3 depicts the state trajectories of controlled multi-agent system with feedback strength $k = 0.02$. Both initial conditions and communications delays are different between two sub-figures in Fig.3. The effectiveness of the control strategy can be fully observed from Fig.3. Weak feedback strength 0.02 (compared with the coupling strength) is shown to be valid. Hence, it can be concluded that the designed control scheme is very cheap and easy to implement.

5 Conclusions and Future Works

In this paper, we investigate the consensus problem of multi-agent systems over directed delayed networks. Each agent adjust its state only by delayed local information, and the information flow among agents is not symmetric. It is proved that consensus will be achieved regardless of the communication delays and initial values. Control law, which is only operated on one of the agents, is successfully developed to force all agents to certain desired objective state. The control feedback strength can be very weak. Numerical simulations are given to show the validity and effectiveness of the analytical results.

Our future work involves study of consensus problem under several communication constraints, such as quantization and time-varying delay.

References

1. Battogtokh, D., Aihara, K., Tyson, J.: Synchronization of eukaryotic cells by periodic forcing. Physical Review Letters 96(14), 148102 (2006)
2. Cucker, F., Smale, S.: Emergent behavior in flocks. IEEE Transactions on Automatic Control 52(5), 852–862 (2007)
3. Hale, J., Lunel, V.: Introduction to Functional Differential Equations. Springer, New York (1993)
4. Horn, R., Johnson, C.: Matrix Analysis. Cambridge University Press, Cambridge (1990)
5. Jadbabaie, A., Lin, J., Morse, A.: Coordination of groups of mobile autonomous agents using nearest neighbor rules. IEEE Transactions on Automatic Control 48(6), 988–1001 (2003)
6. Liu, Y., Passino, K., Polycarpou, M.: Stability analysis of M-dimensional asynchronous swarms with a fixed communication topology. IEEE Transactions on Automatic Control 48(1), 76–95 (2003)
7. Moses, Y., Rajsbaum, S.: A layered analysis of consensus. SIAM Journal on Computing 31(4), 989–1021 (2002)
8. Olfati-Saber, R.: Flocking for multi-agent dynamic systems: algorithms and theory. IEEE Transactions on Automatic Control 51(3), 401–420 (2006)
9. Olfati-Saber, R., Murray, R.: Consensus problems in networks of agents with switching topology and time-delays. IEEE Transactions on Automatic Control 49(9), 1520–1533 (2004)

10. Ren, W., Beard, R., Atkins, E.: Information consensus in multivehicle cooperative control. IEEE Control Systems Magazine 27(2), 71–82 (2007)
11. Sorrentino, F., di Bernardo, M., Garofalo, F., Chen, G.: Controllability of complex networks via pinning. Physical Review E 75(4), 046103 (2007)
12. Vicsek, T., Czirók, A., Ben-Jacob, E., Cohen, I., Shochet, O.: Novel type of phase transition in a system of self-driven particles. Physical Review Letters 75(6), 1226–1229 (1995)
13. Vishnampet, S.: Regulation and Control Mechanisms in Biological Systems. Prenctice Hall (1992)
14. Wang, W., Slotine, J.: Contraction analysis of time-delayed communications and group cooperation. IEEE Transactions on Automatic Control 51(4), 712–717 (2006)
15. Wu, C.: Synchronization in Coupled Chaotic Circuits and Systems. World Scientific Publishing Company, Singapore (2002)
16. Zhu, Y., Cheng, D., Hu, X.: Synchronization of a class of networked passive systems with switching topology. In: 48th Proc. IEEE Conf. Decision Control, pp. 2271–2276 (2007)