Degree-Distribution Stability of Growing Networks

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Abstract. In this paper, we abstract a kind of stochastic processes from evolving processes of growing networks, this process is called growing network Markov chains. Thus the existence and the formulas of degree distribution are transformed to the corresponding problems of growing network Markov chains. First we investigate the growing network Markov chains, and obtain the condition in which the steady degree distribution exists and get its exact formulas. Then we apply it to various growing networks. With this method, we get a rigorous, exact and unified solution of the steady degree distribution for growing networks.

Keywords: growing network Markov chains, BA model, scale-free, degree distribution.

1 Introduction

Barabási and Albert [1] found that for many real-world networks, e.g., the World Wide Web(WWW), the fraction of vertices with degree k is proportional over a large range to a power-law tail, i.e. $P(k) \sim k^{-\gamma}$, where γ is a constant independent of the size of the network. For purpose of opening up mechanism producing scale-free property, they proposed the well known BA model and summarized the reasons: growth and preferential attachment. The proposing of BA model led to a great echo among people, with hundreds of advanced network models proposed and studied,it also gave rise to a new upsurge in studying complex networks. Some important examples are as follows:

*Example 1 (*BA model [1]*).* Barabási and Albert et al. proposed a model which starts with a small number (m_0) of vertices, at each time step add a new vertex with m $(\leq m_0)$ edges that link the new vertex to m different vertices already present in the system. To incorporate preferential attachment, they assumed the probability Π that the new vertex will be connected to a vertex i depends on the connectivity k_i of vertex i, that is $\Pi(k_i) = mk_i / \sum_j k_j$. After t steps the model leads to a random network with $t + m_0$ vertices and mt edges. Let $k_i(t)$ denotes the degree of vertex i at time t, from the mechanism of the model we know that $k_i(t)$ is a Markov chain.

*Example 2 (*Growing network with random links*).* Each vertex is choosen randomly in this case, that is, $\Pi_i = 1/t$, everything else remains the same as BA model.

*Example 3 (*LL model(i) [2]*).* The model is proposed like this: at each time step a new node with m links (edges) is added, and the probability Π_i is determined by $m\frac{(1-p)k_i+p}{\sum_j[(1-p)k_j+p]}$, where $0 \leq p \leq 1$ is a parameter characterizing the relative weights between the deterministic and random contributions to Π_i, and the summation is over the whole network at a given time. The model reduces to the BA model for $p = 0$ and it becomes a completely random network for $p = 1$. So far the evolving mechanism of the LL model has not found, so we introduce the following example in which the boundary between preferential and random are clear.

*Example 4 (*LL model(ii)*).* Everything else remains the same as the former example except that $\Pi_i = \frac{m(1-p)}{2mt+N_0}k + \frac{mp}{t+m_0}$.

*Example 5 (*Generalized collaboration networks [3]*).* Zhang P.P., et al. supposed the initial network to be several complete graphs with m_0 vertices, the sum of degree k_{i0} is k_0. At each time step add a new vertex to the network and connect it to $T - 1(T$ is a constant) different vertices already present in the system. The probability that vertex i get a link is proportional to the degree of the vertex, i.e. $\Pi_i = k_i/\sum_l k_j$. After this, link all the T vertices to form a complete graph.

*Example 6 (*ZRZ model [4]*).* The initial network $(t = 0)$ is a complete graph $(m-$complete graph) with m vertices and C_m^2 edges. At each time step a new vertex is added to the network, it will connect to all the vertices of a $m-$complete graph selected randomly, that is $m + 1$ people are collaborated in an act.

*Example 7 (*KK model [5]*).* At the beginning $(t = 1)$ we have one group with one element in it. At each time step we add a new element to the system. With probability p it will belong to one of existing groups. The probability that it joins the ith group is proportional to the size of the group (k_i/N) (the number of elements is equal to the time i.e. $N = t$). With probability $q = 1 - p$, the new element will belong to a new group.

*Example 8 (*DMS model [6]*).* At each time step a new site appears. Simultaneously, m new directed links coming out from non-specified sites are introduced. Let the connectivity q_s be the number of incoming links to a site s, i.e., to a site added at time s. The probability that a new link points to a given site s is proportional to the following characteristic of the site: $H_s = H + q_s$, thereafter called its attractiveness. All sites are born with some initial attractiveness $H \geq 0$, but afterwards it increases because of the q_s term. Note that one may allow multiple links, i.e., the connectivity of a given site may increase simultaneously by more than one.

*Example 9 (*LCD model [7]*).* Based on BA model, Bollobás et al. proposed another model which allows multiple links and loops.

Except LCD model, the existence and deduction of degree distribution of the present models (including the noted BA model) are devoid of exact mathematics basis. Recently, a mechanism for BA model is given in [8] and [9]. Moreover the existence and exact formulas of the degree distribution were also provided. In this paper we abstract a kind of Markov chains from enumerated models, we call it as growing network Markov chains. First we investigate the growing network Markov chains, and obtain the condition in which the steady degree distribution exists and get its exact formulas. Then we apply it to various growing networks. With this method, we get a rigorous, exact and unified solution of the steady degree distribution for growing networks.

2 Non-multiple Growing Network Markov Chains

For any $i = 1, 2, \cdots, k_i(t)(t = i, i+1, \cdots)$ are Markov chains non-decrease with respect to t taking values in $\{0, 1, 2, \cdots\}$. Suppose there exists a positive integer i_0, s.t, $\{k_i(t)\}(i \geq i_0)$ have the initial distribution $P\{k_i(i) = k\} = d_{k,i}$, with transition probability

$$P\{k_i(t+1) = l | k_i(t) = k\} = \begin{cases} f_t(k), & l = k+1 \\ 1 - f_t(k), & l = k \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

where $0 < f_t(k) < 1$. Let $P(k, i, t) := P\{k_i(t) = k\}(t = i, i+1, \cdots), P(k, t) := \frac{1}{t} \sum\limits_{i=1}^{t} P(k, i, t)$.

Definition 1. *The above Markov chains $\{k_i(t)\}_{t=i,i+1,\cdots}(i = 1, 2, \cdots)$ are called series of non-multiple growing network Markov chains, for short we call it non-multiple growing network Markov chains, if the limit $P(k) = \lim\limits_{t \to \infty} P(k, t)$ exists, and*

$$P(k) \geq 0, \sum_{k=0}^{\infty} P(k) = 1 \tag{2}$$

we say that the degree distribution of non-multiple growing network Markov chains exists, and $P(k)$ is the steady degree distribution of $\{k_i(t)\}$. Further, if $P(k)$ is power-law, i.e.,

$$P(k) \sim k^{-\gamma}(k \geq k_0) \tag{3}$$

$\{k_i(t)\}$ are called scale-free non-multiple growing network Markov chains.

Lemma 1. *If $\lim\limits_{i \to \infty} d_{k,i} = d_k$ exists and satisfies $\sum\limits_{k=0}^{\infty} d_k = 1$. $\lim\limits_{t \to \infty} t f_t(k) := F(k)$ also exists, and there is a non-negative integer m to satisfy $d_k = 0, k =$*

$0, 1, \cdots, m - 1, d_m > 0$, and $F(k) > 0(k = m, m + 1, \cdots)$. Then $\lim\limits_{t \to \infty} P(m, t)$ exists, moreover

$$P(m) = \frac{d_m}{1 + F(m)} \tag{4}$$

Proof. With the Markovian property, we have

$$P(m, i, t + 1) = P(m, i, t)[1 - f_t(m)], \quad (i \le t) \tag{5}$$

By the definition of $P(m, t)$ and $P(m, t + 1, t + 1) = d_{m,t+1}$, we obtain

$$P(m, t + 1) = \frac{t}{t + 1} P(m, t)[1 - f_t(m)] + \frac{1}{t + 1} d_{m,t+1} \tag{6}$$

The above difference equation has the following solution

$$P(m, t) = \frac{1}{t} \prod_{i=1}^{t-1}[1 - f_i(m)] \left\{ P(m, 1) + \sum_{l=1}^{t-1} d_{m,l+1} \prod_{j=1}^{l}[1 - f_j(m)]^{-1} \right\} \tag{7}$$

Let

$$x_t = P(m, 1) + \sum_{l=1}^{t-1} d_{m,l+1} \prod_{j=1}^{l}[1 - f_j(m)]^{-1}$$

$$y_t = t \prod_{i=1}^{t-1}[1 - f_i(m)]^{-1} > t \to \infty$$

We easily have

$$x_{t+1} - x_t = d_{m,t+1} \prod_{j=1}^{t}[1 - f_j(m)]^{-1}$$

$$y_{t+1} - y_t = [1 + tf_t(m)] \prod_{j=1}^{t}[1 - f_j(m)]^{-1} > 0$$

With $\lim\limits_{t \to \infty} tf_t(m) = F(m)$, we have

$$\frac{x_{t+1} - x_t}{y_{t+1} - y_t} = \frac{d_{m,t+1}}{1 + tf_t(m)} \to \frac{d_m}{1 + F(m)}, \quad (t \to \infty) \tag{8}$$

With the Stolz-Cesáro theorem [10], we have Eq (4) and complete the proof. □

Lemma 2. *If the conditions in Lemma 1 are all satisfied, and for $k > m$, $\lim\limits_{t \to \infty} P(k - 1, t)$ exists, then $\lim\limits_{t \to \infty} P(k, t)$ exists, moreover*

$$P(k) = \frac{F(k - 1)}{1 + F(k)} P(k - 1) + \frac{d_k}{1 + F(k)} \tag{9}$$

Proof. With the Markovian property, we have

$$P(k,i,t+1) = P(k,i,t)[1 - f_t(k)] + P(k-1,i,t)f_t(k-1) + d_{k,t+1} \quad (10)$$

The definition of $P(k,t)$ and $P(k,t+1,t+1) = d_{k,t+1}$ yield

$$P(k,t+1) = \frac{t}{t+1}P(k,t)[1 - f_t(k)] + \frac{t}{t+1}P(k-1,t)f_t(k-1) + \frac{d_{k,t+1}}{t+1} (11)$$

The above difference equation has the following solution

$$P(k,t) = \frac{1}{t} \prod_{i=1}^{t-1}[1 - f_i(k)] \times$$

$$\{P(k,1) + \sum_{l=1}^{t-1}[lP(k-1,l)f_l(k-1) + d_{k,l+1}] \prod_{j=1}^{l}[1 - f_j(k)]^{-1}\} \quad (12)$$

Similar to Lemma 1 we have Eq(9), then complete the proof.   □

Theorem 1. *If the conditions in Lemma 1 are all satisfied, the steady degree distribution of $\{k_i(t)\}$ exists, moreover*

$$P(k) = \begin{cases} \frac{d_m}{1+F(m)}, & k = m \\ \prod_{i=m}^{k-1} \frac{F(i)}{1+F(i+1)}[\frac{d_m}{1+F(m)} + \sum_{l=m}^{k-1} \frac{\frac{d_{l+1}}{1+F(l+1)}}{\prod_{j=m}^{l} \frac{F(j)}{1+F(j+1)}}], k > m \end{cases} \quad (13)$$

Proof. From Lemma 1 and Lemma 2, Eq (13) comes into existence.   □

Theorem 2. *Suppose there is a non-negative integer $M \geq m$ and satisfies $d_k = 0(k > M)$. And if there two constants A, B, satisfy $F(k) = Ak + B$, then*
(I)The degree distribution $P(k)$ satisfies

$$\sum_{k=m}^{\infty} P(k) = 1 \quad (14)$$

(II) If $A > 0$, then $\{k_i(t)\}$ are Scale-free growing network Markov chains, and

$$P(k) = \begin{cases} \frac{d_m}{1+Am+B}, & k = m \\ \prod_{i=m}^{k-1} \frac{Ai+B}{1+A(i+1)+B}[\frac{d_m}{1+Am+B} + \sum_{l=m}^{k-1} \frac{\frac{d_{l+1}}{1+A(l+1)+B}}{\prod_{j=m}^{l} \frac{Aj+B}{1+A(j+1)+B}}], & m < k \leq M \\ \frac{\Gamma(k+\frac{B}{A})}{\Gamma(k+\frac{B}{A}+1+\frac{1}{A})} \frac{\Gamma(M+\frac{B}{A}+1+\frac{1}{A})}{\Gamma(M+\frac{B}{A})} \prod_{i=m}^{M-1} \frac{Ai+B}{1+A(i+1)+B}[\frac{d_m}{1+Am+B} \\ + \sum_{l=m}^{M-1} \frac{\frac{d_{l+1}}{1+A(l+1)+B}}{\prod_{j=m}^{l} \frac{Aj+B}{1+A(j+1)+B}}] \sim k^{-(1+\frac{1}{A})}, & k > M \end{cases} \quad (15)$$

Specially, if $d_m = 1, d_k = 0 (k \neq m)$, then

$$P(k) = \begin{cases} \frac{1}{1+Am+B}, & k = m \\ \frac{\Gamma(k+\frac{B}{A})}{\Gamma(k+\frac{B}{A}+1+\frac{1}{A})} \frac{\Gamma(m+\frac{B}{A}+1+\frac{1}{A})}{\Gamma(m+\frac{B}{A})} \frac{1}{1+Am+B} \sim k^{-(1+\frac{1}{A})}, & k > m \end{cases} \tag{16}$$

(III) If $A = 0, B > 0$, $\{k_i(t)\}$ are scale-free growing network Markov chains.
(IV) The case for $A < 0$ or $A = 0, B < 0$ will never happen.

Proof. From Eq(4), (9) and the given condition, we easily obtain Eq(14). (II), (III) and (IV) are also easily proved. □

Let $k_i(t)$ denote the degree of the vertex added at time-step i evolved at time t in the former examples, and $\{k_i(t)\}$ are growing network Markov chains. Therefore we can apply Theorem 1 and Theorem 2 to the former examples.

Example 1. We have $d_m = 1$ and $f_t(k) = \frac{mk}{2t}$, so $A = \frac{m}{2}, B = 0$

$$P(k) = \begin{cases} \frac{2}{m+2}, & k = m \\ \frac{\Gamma(k)}{\Gamma(k+3)} \frac{\Gamma(m+3)}{\Gamma(m)} \frac{2}{m+2} \sim 2m^2 k^{-3}, & k > m \end{cases} \tag{17}$$

the network is scale-free with scaling exponent $\gamma = 3$. This is identical with the results in [1][6][11][8][9]. A different and exact proof of degree distribution and an evolving mechanism of this model have been provided in papers [8] and [9] .

Example 2. From the model we have $d_m = 1$ and $f_t(k) = \frac{m}{t}$, so $A = 0, B = m$. The steady degree distribution exists, and

$$P(k) = \frac{m}{1+m}P(k-1) = \left(\frac{m}{1+m}\right)^{k-m} \frac{1}{1+m} \tag{18}$$

is exponentially distributed, the network is not scale-free.

Example 3. $d_m = 1$ and $f_t(k) = m\frac{1-p}{(1-p)2mt+pt}k + m\frac{p}{(1-p)2mt+pt}$, $A = m\frac{1-p}{(1-p)2m+p}$, $B = m\frac{p}{(1-p)2m+p}$. If $p \neq 1$, the degree distribution

$$P(k) = \begin{cases} \frac{2m+(1-2m)p}{m^2+2m+(1-m^2-m)p}, & k = m \\ \frac{\Gamma(k+\frac{p}{1-p})}{\Gamma(k+\frac{p}{1-p}+3+\frac{p}{m(1-p)})} \frac{\Gamma(m+\frac{p}{1-p}+3+\frac{p}{m(1-p)})}{\Gamma(m+\frac{p}{1-p})} \frac{2m+(1-2m)p}{m^2+2m+(1-m^2-m)p} & \\ \sim k^{-(3+\frac{p}{m(1-p)})}, & k > m \end{cases} \tag{19}$$

is power-law with scaling exponent $\gamma = 3 + \frac{p}{m(1-p)}$. $P(k)$ follows exponential distribution if $p = 1$.

Example 4. We have $d_m = 1, f_t(k) = m\frac{1-p}{2mt+N_0}k + \frac{mp}{t+m_0}$. $A = \frac{1-p}{2}, B = mp$.

$$P(k) = \begin{cases} \frac{2}{2+m+mp}, & k = m \\ \frac{\Gamma(k+\frac{2mp}{1-p})}{\Gamma(k+\frac{2mp}{1-p}+1+\frac{2}{1-p})} \frac{\Gamma(m+\frac{2mp}{1-p}+1+\frac{2}{1-p})}{\Gamma(m+\frac{2mp}{1-p})} \frac{2}{2+m+mp}, & k > m \end{cases} \tag{20}$$

If $p = 1$, $P(k)$ is not power-law, and if $p \neq 1$

$$P(k) \sim k^{-(1+\frac{2}{1-p})} \tag{21}$$

is power-law with the scaling exponent $\gamma = 1 + \frac{2}{1-p}$.

Example 5. We get $d_1 = 1, f_t(k) = \frac{(T-1)k}{k_0+Tt}$, so $A = \frac{T-1}{T}, B = 0$. So the network is scale-free and the degree distribution is

$$P(k) \sim k^{-(1+\frac{T}{T-1})} \tag{22}$$

and scaling exponent is $\gamma = 1 + \frac{T}{T-1}$.

Example 6. $d_m = 1 (m > 2)$, $f_t(k) = \frac{(m-1)k}{mt+1} - \frac{m(m-2)}{mt+1}$, so $A = \frac{m-1}{m}, B = -m(m-2)$. The network is scale-free and the degree distribution is

$$P(k) \sim k^{-(1+\frac{m}{m-1})} \tag{23}$$

with scaling exponent $\gamma = 1 + \frac{m}{m-1}$.

Example 7. $d_0 = p, d_1 = 1 - p$, $f_t(k) = p\frac{k}{t}$, so $A = p$, $B = 0$, in this model $k_i(t)$ denotes the number of elements in group added at time-step i evolved at time t.

The degree distribution is

$$P(k) = \begin{cases} p, & k = 0 \\ \frac{1-p}{1+p}, & k = 1 \\ \frac{\Gamma(k)\Gamma(2+\frac{1}{p})}{\Gamma(k+1+\frac{1}{p})}\frac{1-p}{1+p}, & k > 1 \end{cases} \tag{24}$$

so it's power-law and the scaling exponent is $\gamma = 1 + \frac{1}{p}$.

3   Multiple Growing Network Markov Chains

The degree of a vertex can increase at most 1 in network of non-multiple links. However, the degree can increase more than 1 if multiple links is permitted. We have found that the probability of vertex's degree increase more than 1 is the high-level of infinitesimal. For the purpose of investigating the degree distribution of multiple linking growing network, we introduce multiple growing network Markov chains.

For any $i = 1, 2, \cdots, k_i(t)(t = i, i + 1, \cdots)$ are Markov chains non-decrease with respect to t taking values in $\{0, 1, 2, \cdots\}$, suppose there exists a positive integer i_0, s.t, $\{k_i(t)\}(i \geq i_0)$ have the initial distribution $P\{k_i(i) = k\} = d_{k,i}$, with transient probability matrix

$$\begin{pmatrix} p_{0,0} & p_{0,1} & o_{0,2}(\frac{1}{t}) & \cdots & o_{0,m'}(\frac{1}{t}) & & \\ & p_{1,1} & p_{1,2} & o_{1,3}(\frac{1}{t}) & \cdots & o_{1,m'+1}(\frac{1}{t}) & & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & p_{K,K} & p_{K,K+1} & o_{K,K+2}(\frac{1}{t}) & \cdots & o_{K,K+m'}(\frac{1}{t}) \\ & 0 & & & 1 & 0 & \cdots \\ & & & & & \ddots & \ddots \end{pmatrix}. \tag{25}$$

where $p_{k,l} = P\{k_i(t+1) = l | k_i(t) = k\}$. For $k \le K (K$ is the maximum degree of vertex i at t)

$$p_{k,l} = \begin{cases} 1 - f_t(k) - \sum_{s=k+2}^{k+m'} o_{k,s}(\frac{1}{t}), & l = k \\ f_t(k), & l = k+1 \\ o_{k,l}(\frac{1}{t}), & k+1 < l \le k+m' \\ 0, & \text{else} \end{cases} \tag{26}$$

and for $k > K$

$$p_{k,l} = \begin{cases} 1, & l = k \\ 0, & l \ne k+1 \end{cases} \tag{27}$$

Definition 2. *The above Markov chains $\{k_i(t)\}_{t=i,i+1,\cdots}(i = 1, 2, \cdots)$ are called series of multiple growing network Markov chains, for short we call it multiple growing network Markov chains , if the limit $P(k) = \lim\limits_{t \to \infty} P(k,t)$ exists, and*

$$P(k) \ge 0, \sum_{k=0}^{\infty} P(k) = 1 \tag{28}$$

we say that degree distribution of multiple growing network Markov chains exists, and $P(k)$ is the steady degree distribution of $\{k_i(t)\}$. Further, if $P(k)$ is power-law, i.e.,

$$P(k) \sim k^{-\gamma}(k \ge k_0) \tag{29}$$

$\{k_i(t)\}$ are called scale-free multiple growing network Markov chains.

Lemma 3. *If $\lim\limits_{i \to \infty} d_{k,i} = d_k$ exists and satisfies $\sum\limits_{k=0}^{\infty} d_k = 1$. $\lim\limits_{t \to \infty} t f_t(k) := F(k)$ also exist, and there is a non-negative integer m to satisfy $d_k = 0, k = 0, 1, \cdots, m-1, d_m > 0$, and $F(k) > 0(k = m, m+1, \cdots)$. Then $\lim\limits_{t \to \infty} P(m,t)$ exists, moreover*

$$P(m) = \frac{d_m}{1 + F(m)} \tag{30}$$

Proof. With the Markovian property, we have

$$P(m, i, t+1) = P(m, i, t)p_{m,m}, \quad (i \le t) \tag{31}$$

By the definition of $P(m, t)$ and $P(m, t+1, t+1) = d_{m,t+1}$, we have the following equation

$$P(m, t+1) = \frac{t}{t+1}P(m, t)[1 - f_t(m) - \sum_{s=m+2}^{m+m'} o_{m,s}(\frac{1}{t})] + \frac{1}{t+1}d_{m,t+1} \tag{32}$$

the above difference equation has the following solution

$$P(m, t) = \frac{1}{t}\prod_{i=1}^{t-1}[1 - f_i(m) - \sum_{s=m+2}^{m+m'} o_{m,s}(\frac{1}{i})] \times$$

$$\left\{ P(m, 1) + \sum_{u=1}^{t-1}d_{m,u+1}\prod_{j=1}^{u}[1 - f_j(m) - \sum_{s=m+2}^{m+m'} o_{m,s}(\frac{1}{j})]^{-1} \right\} \tag{33}$$

Similar to Lemma 1 we have Eq (30).   □

Lemma 4. *If the conditions in Lemma 3 are all satisfied, and $\lim_{t\to\infty} P(k-1, t)$ exists, then $\lim_{t\to\infty} P(k, t)$ exists, moreover*

$$P(k) = \frac{F(k-1)}{1 + F(k)}P(k-1) + \frac{d_k}{1 + F(k)} \tag{34}$$

Proof. With the property of the Markov chains, we have

$$P(k, i, t+1) = P(k, i, t)p_{k,k} + P(k-1, i, t)p_{k-1,k} + \sum_{l=2}^{m'} P(k-l, i, t)p_{k-l,k} \tag{35}$$

By the definition of $P(k, t)$ and $P(k, t+1, t+1) = d_{k,t+1}$, we have the following relation

$$P(k, t+1) = \frac{t}{t+1}P(k, t)[1 - f_t(k) - \sum_{s=k+2}^{k+m'} o_{k,s}(\frac{1}{t})] + \frac{t}{t+1}P(k-1, t)f_t(k-1)$$

$$+ \sum_{l=2}^{m} \frac{t}{t+1}P(k-l, t)o_{k-l,k}(\frac{1}{t}) + \frac{1}{t+1}d_{k,t+1} \tag{36}$$

Solve the above difference equation we have

$$P(k, t) = \frac{1}{t}\prod_{i=1}^{t-1}[1 - f_i(k) - \sum_{s=k+2}^{k+m'} o_{k,s}(\frac{1}{i})]\{P(k, 1) + \sum_{u=1}^{t-1}[uP(k-1, u)f_u(k-1)$$

$$+ \sum_{l=2}^{m'} uP(k-l, u)o_{k-l,k}(\frac{1}{u}) + d_{k,u+1}]\prod_{j=1}^{u}[1 - f_j(k) - \sum_{s=k+2}^{k+m'} o_{k,s}(\frac{1}{j})]^{-1}\} \tag{37}$$

Similar to Lemma 1 we have

$$
\begin{aligned}
P(k) &= \lim_{t\to\infty} P(k,t) \\
&= \lim_{t\to\infty} \frac{tP(k-1,t)f_t(k-1) + \sum_{l=2}^{m'} tP(k-l,t)o_{k-l,k}(\frac{1}{t}) + d_{k,t+1}}{(t+1) - t[1 - f_t(k) - \sum_{s=k+2}^{k+m'} o_{k,s}(\frac{1}{t})]} \\
&= \lim_{t\to\infty} \frac{tP(k-1,t)f_t(k-1) + d_{k,t+1}}{1 + tf_t(k) + t\sum_{s=k+2}^{k+m'} o_{k,s}(\frac{1}{t})} \\
&\quad + \lim_{t\to\infty} \frac{\sum_{l=2}^{m'} tP(k-l,t)o_{k-l,k}(\frac{1}{t})}{1 + tf_t(k) + t\sum_{s=k+2}^{k+m'} o_{k,s}(\frac{1}{t})} \\
&= \frac{F(k-1)}{1+F(k)} P(k-1) + \frac{d_k}{1+F(k)}
\end{aligned}
\tag{38}
$$

then complete the proof.   □

Theorem 3. *If the conditions in Lemma 3 are all satisfied, then the steady degree distribution of $\{k_i(t)\}$ exists. Moreover,*

$$
P(k) = \begin{cases}
\frac{d_m}{1+F(m)}, & k = m \\
\prod_{i=m}^{k-1} \frac{F(i)}{1+F(i+1)}\big[\frac{d_m}{1+F(m)} + \sum_{l=m}^{k-1} \frac{\frac{d_{l+1}}{1+F(l+1)}}{\prod_{j=m}^{l} \frac{F(j)}{1+F(j+1)}}\big], & k > m
\end{cases}
\tag{39}
$$

Proof. From Lemma 3 and Lemma 4, Eq (39) comes into existence.   □

Theorem 4. *Theorem 2 is also held when allowing multiple linking and loops.*

Theorem 5. *If $f_t(k) = a_t k + b_t + o_k(\frac{1}{t})$, we have $\lim_{t\to\infty} tf_t(k) := F(k) = Ak + B$ if and only if $\lim_{t\to\infty} ta_t = A$, $\lim_{t\to\infty} tb_t = B$.*

Example 8. From the DMS model we get: $d_0 = 1$, $f_t(k) = m\frac{k+H}{(m+H)t}$, so $A = \frac{m}{m+H}$, $B = \frac{mH}{m+H}$, with Theorem 3 and Theorem 4, we have

$$
P(k) = \begin{cases}
\frac{m+H}{m+H+mH}, & k = 0 \\
\frac{\Gamma(k-1+H)}{\Gamma(k+1+H+\frac{H}{m})}\frac{\Gamma(H+2+\frac{H}{m})}{\Gamma(H)}\frac{m+H}{m+H+mH} \sim k^{-(2+\frac{H}{m})}, & k > 0
\end{cases}
\tag{40}
$$

So the network is scale-free and degree exponent is $2 + \frac{H}{m}$.

Example 9. From the model we get $d_m = 1$, $f_t(k) = m\frac{k}{2mt+m}$, so $A = \frac{1}{2}$, $B = 0$, with Theorem 3 and Theorem 4, we have

$$
P(k) = \begin{cases}
\frac{2}{m+2}, & k = m \\
\frac{\Gamma(k)}{\Gamma(k+3)}\frac{\Gamma(m+3)}{\Gamma(m)}\frac{2}{m+2} \sim 2m^2 k^{-3}, & k > m
\end{cases}
\tag{41}
$$

it is the same as in [7].

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