Finite Time Ruin Probability in Non-standard Risk Model with Risky Investments

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Abstract. In this paper, under the assumption that the claimsize is subexponentially distributed and the insurance capital is totally invested in risky asset, some simple asymptotics of finite horizon ruin probabilities are obtained for non-homogeneous Poisson process and conditional Poisson risk models as well as renewal risk model, when the initial capital is quite large. Extremal event is described in this case because some claim can be larger than initial capital even it is large enough. The results obtained extended the corresponding results of related papers in this area.

Keywords: ruin probability, conditional Poisson process, renewal risk model, non-homogeneous Poisson process, subexponential class, regularly varying tail.

1 Models

Consider a Cramér-Lundberg model. In this model the claim sizes, X_n, $n = 1, 2, ...$, constitute a sequence of independent, identically distributed (i.i.d.) and non-negative random variables (r.v.'s) with common distribution function (d.f.) $F = 1 - \overline{F}$. The claim arrival times, $\sigma_n, n = 1, 2, ...$, form a homogenous Poisson process

$$N(t) = \sup \{n \geq 1 : \sigma_n \leq t\}, t > 0, \tag{1}$$

with a constant intensity λ, where, $\sup\phi = 0$ by convention. The total surplus of a company up to time t, with perturbed term $\sigma_0 W_0(t)$, is denoted by $U(t)$, which satisfies the following equation:

$$U(t) = u + ct - \sum_{k=1}^{N(t)} X_k + \sigma_0 W_0(s), \tag{2}$$

where, $u > 0$ is the initial capital, $c > 0$ is the constant rate of premium, $\{W_0(t), t \geq 0\}$ is a standard Brownian motion and $\sigma_0 > 0$ is the volatility coefficient of $\sigma_0 W_0(t)$. If the inter-arrival times $\sigma_1, \sigma_n - \sigma_{n-1}$ for $n = 2, 3, ...$ have a common distribution of general form, then the model above is called the renewal model.

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If in (1), parameter λ is time dependent, then $\{N(t), t \geq 0\}$ is called non-homogeneous with intensity function $\{\lambda(t), t \geq 0\}$. If for arbitrarily fixed $t, s > 0$, $N(t)$ satisfies that

$$P(N(t+s) - N(s) = k) = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} G_\Lambda(d\lambda),$$

where Λ is a r.v. with d.f. G_Λ. Then $\{N(t), t \geq 0\}$ is called conditional Poisson process.

All limit relationships in this paper, unless otherwise stated, are for $u \to \infty$. $A \sim B$ and $A \gtrsim B$ respectively mean that $\lim_{u\to\infty} \frac{A}{B} = 1$ and $\lim_{u\to\infty} \frac{A}{B} \geq 1$.

1.1 Investment in Risky Asset

If an insurer invests his capital in a risky asset, then its capital value should be specified by a geometric Brownian motion

$$dV_t = V_t(rdt + \sigma dW(t)), \tag{3}$$

where $\{W(t), t \geq 0\}$ is a standard Brownian motion and $r \geq 0, \sigma \geq 0$ are respectively called expected rate of return and volatility coefficient. It is well known that stochastic equation (3) has the following solution

$$V_t = V(0)e^{(r-\frac{1}{2}\sigma^2)t+\sigma W(t)}.$$

Therefore, at time t, the surplus with risky investment could be expressed as

$$U(t) = e^{\Delta(t)}(u + \int_0^t e^{-\Delta(s)} dU(s)), \tag{4}$$

where, $\Delta(t) = \beta t + \sigma W(t)$, $\beta = r - \sigma^2/2$.

Through out, $\{X_n, n \geq 1\}$, $\{N(t), t \geq 0\}$, $\{W(t), t \geq 0\}$ and $\{W_0(t), t \geq 0\}$ are assumed to be mutually independent.

Definition 1. *We define*

$$\psi(u, T) = P(\inf_{0\leq s\leq T} U(s) < 0 | U(0) = u),$$

the finite time ruin probability within time T. If $T = \infty$, we say that $\psi(u, \infty)$ is ultimate ruin probability.

Ruin probability reflects the possibility that the surplus process moves below zero.

In this paper, under the assumption that the risk models are non-homogenous, conditional Poisson process and renewal risk model respectively, some asymptotics of finite time ruin probabilities, with investment in risky asset, are derived.

1.2 Heavy-Tailed Claims and Some Related Results

Heavy-tailed risk has played an important role in insurance and finance because it can describe large claims; see Embrechts et al. (see [6]) for a nice review. We give here several important classes of heavy-tailed distributions for further references:

– class $\mathcal{L}$ (*Long-tailed*): a d.f. F belongs to $\mathcal{L}$ iff

$$\lim_{x\to\infty}\frac{\overline{F}(x+t)}{\overline{F}(x)}=1$$

for any t (or equivalently, for $t=1$);

– class $\mathcal{R}_{-\alpha}$: a distribution F belongs to $\mathcal{R}_{-\alpha}$ iff

$$\overline{F}(x)=x^{-\alpha}L(x),\quad x>0,$$

where $L(x)$ is a slowly varying function as $x\to\infty$ and index $-\alpha<0$. $\mathcal{R}_{-\alpha}$ is called regularly varying function class, or Pareto-like function class with index $-\alpha$.

– class $\mathcal{S}$ (*Subexponential*): a d.f. F belongs to $\mathcal{S}$ iff

$$\lim_{x\to\infty}\frac{\overline{F^{*n}}(x)}{\overline{F}(x)}=n$$

for any n (or equivalently, for $n=2$); where F^{*n} denotes the n-fold convolution of F, with convention that F^{*0} is a d.f. degenerate at 0.

These heavy-tailed classes satisfies $\mathcal{R}_{-\alpha}\subset\mathcal{S}\subset\mathcal{L}$ (see Embrechts et al. [6]). The asymptotic behavior of the ultimate ruin probability $\psi(u)$ is an important topic in risk theory. In the recent literature, the asymptotic behavior of the ruin probability with constant interest force has been extensively investigated. One of the interesting results was obtained by Klüppelberg and Stadtmuller([12]), who used a very complicated L_p transform method, proved that, in the Cramér-Lundberg risk model, if the claimsize is of regularly varying with index $-\alpha$, then

$$\psi(u)\sim\frac{\lambda}{\alpha r}\overline{F}(u),\tag{5}$$

where r is constant interest force. Asmussen ([1]) and Asmussen et al. ([2]) obtained a more general result:

$$\psi(u)\sim\frac{\lambda}{r}\int_u^\infty\frac{\overline{F}(y)}{y}\mathrm{d}y,\tag{6}$$

where the claimsize is assumed to be in $\mathcal{S}^*$, an important subclass of $\mathcal{S}$. In the case of compound Poisson model with constant interest force and without diffusion term, Tang ([16]) obtained the asymptotic formula of finite time ruin probability for sub-exponential claims. Tang ([17]) proved that, in the renewal risk model with constant interest force, if the d.f. of claimsize belongs to regularly varying class with index $-\alpha$, then ultimate ruin probability satisfies that

$$\psi(u) \sim \frac{\mathrm{E}\mathrm{e}^{-\mathrm{r}\alpha\theta_1}}{1-\mathrm{E}\mathrm{e}^{-\mathrm{r}\alpha\theta_1}}\overline{F}(u),$$

which extends (5) essentially. Jiang ([10]) extended some results to the risky case. See also Jiang ([8], [9]). Dufresne and Gerber ([4]) first researched the ruin probability for jump-diffusion Poisson process. Veraverbeke ([20]) discussed the asymptotic behavior of ruin with diffusion term.

The rest of this paper is organized as follows: In Section 2, main results of this paper are presented. In section 3, after some necessary lemmas are supplied, the proofs of the main results are completed.

2 Main Results

The following theorems are main results of this paper:

Theorem 1. *Consider non-homogenous Poisson model introduced in Section 1. If $F \in \mathcal{R}_{-\alpha}$, then it holds that*

$$\psi(u;T) \sim \overline{F}(u)\int_0^T \lambda(s)e^{-(\alpha\beta-\frac{1}{2}\alpha^2\sigma^2)s}ds. \tag{7}$$

Notes and Comments. When $F \in \mathcal{R}_{-\alpha}$ and the perturbed term disappears, the results of Tang ([10]) is consistent with this Theorem. In particular, this result is also in consistence with that of Veraverbeke ([20]), who pointed out that the diffusion term $W_0(t)$ does not influence the asymptotic behavior of the ruin probability. We should point out that the diffusion term $W(t)$ plays an essential role in influencing the interest force.

Theorem 2. *Consider conditional Poisson process introduced in Section 1. If $F \in \mathcal{R}_{-\alpha}$, then*

$$\psi(u;T) \sim \overline{F}(u)E\Lambda\int_0^T e^{-(\alpha\beta-\frac{1}{2}\alpha^2\sigma^2)s}ds. \tag{8}$$

Notes and Comments. In these two Theorems, if parameter λ is a constant and perturbed term disappears, then (7) and (8) turn to the following:

$$\psi(u;T) \sim \frac{\lambda}{\alpha r}\overline{F}(u)(1-e^{-\alpha rT}), \tag{9}$$

which is in consistence with the result of Klüppelberg and Stadtmuller ([12]).

Theorem 3. *In the renewal model with surplus (4) and $\mathcal{R}_{-\alpha}$ claims. Denote $q = \mathrm{E}e^{-(\alpha\beta-\frac{1}{2}\alpha^2\sigma^2)\theta_1}$ and $m(s)$ is the renewal function of renewal process $N(t)$. Then*

$$\psi(u;T) \sim \overline{F}(u)\int_0^T e^{-(\alpha\beta-\frac{1}{2}\alpha^2\sigma^2)s}dm(s). \tag{10}$$

Theorem 4. *In the renewal model with surplus process (4). Still denote* $q = \mathrm{E}e^{-(\alpha\beta - \frac{1}{2}\alpha^2\sigma^2)\theta_1}$. *If* $F \in \mathcal{S}$ *and* $m(s)$ *is defined in Theorem 3. Then*

$$\psi(u;T) \sim \int_0^T P(X_1 e^{-\beta s - \sigma W(s)} \geq u) dm(s). \tag{11}$$

If we denote $\psi_k(u)$ as the ruin probability when ruin happens not later than kth claim, then in the renewal case, we can obtain the following Theorem:

Theorem 5. *In the renewal model with surplus (4). Let* $q = \mathrm{E}e^{-(\alpha\beta - \frac{1}{2}\alpha^2\sigma^2)\theta_1}$. *If* $F \in \mathcal{R}_{-\alpha}$, *then*

$$\psi_k(u) \sim \overline{F}(u) \frac{q - q^{k+1}}{1-q}. \tag{12}$$

Notes and Comments. From Theorem 5, we can get the main result of Tang (2005a) easily.

3 Proofs of Main Theorems

3.1 Several Lemmas

The following lemma is well known Ross ([19]):

Lemma 1. *Let* $\{N(t)\}_{t \geq 0}$ *be a Poisson process with arrival times* $\{\sigma_k, k \geq 1\}$. *Given* $N(T) = n$ *for any fixed* $T > 0$, *the random vector* $(\sigma_1, \sigma_2, ..., \sigma_n)$ *is equal in distribution to the random vector* $\left(TU_{(1,n)}, ..., TU_{(n,n)}\right)$, *where* $\left(U_{(1,n)}, ..., U_{(n,n)}\right)$ *are the order statistics of* n *i.i.d.* $(0,1)$ *uniformly distributed random variables* $U_1, ..., U_n$.

The following lemma can be found in many standard textbooks on stochastic process, see, for example, Karatzas and Shreve ([11]).

Lemma 2. *If* $W(t)$ *is a standard Brownian Motion, then the moment of any order of* $\max_{0 \leq t \leq T} W(t)$ *exists.*

Lemma 3. *For following randomly weighted series* $W = \sum_{n=1}^{\infty} \beta_n X_n$, *where* $\{X_n, n \geq 1\}$ *is a sequence of nonnegative i.i.d.r.v.'s, with common distribution* $\overline{F} \in \mathcal{R}_{-\alpha}$, *and* $\{\beta_n, n \geq 1\}$ *is another sequence of nonnegative r.v.'s, which is independent of* $\{X_n, n \geq 1\}$. *If one of the following assumption holds:*

1. $0 < \alpha < 1$ *and for some* $\varepsilon > 0$, *it holds* $\sum_{n=1}^{\infty} \mathrm{E}({\beta_n}^{\alpha+\varepsilon} + {\beta_n}^{\alpha-\varepsilon}) < \infty$;

2. $1 \leq \alpha < \infty$ *and for some* $\varepsilon > 0$, *it holds* $\sum_{n=1}^{\infty} \mathrm{E}({\beta_n}^{\alpha+\varepsilon} + {\beta_n}^{\alpha-\varepsilon})^{\frac{1}{\alpha+\varepsilon}} < \infty$, *then*

$$P(W > u) \sim \overline{F}(u) \sum_{n=1}^{\infty} \mathrm{E}\beta_{\mathrm{n}}^{\alpha}$$

See Resnick ([18]).

Notes and Comments. The great advantage of Lemma 4, is that no information about the dependence structure of the sequence $\{\beta_n, n \geq 1\}$ is wanted. The following Lemma can be found in Tang ([15]):

Lemma 4. *Let $\{X_i, 1 \leq i \leq n\}$ be n i.i.d. subexponential r.v.s, with common distribution F. Then for any fixed $0 < a \leq b < \infty$, uniformly for all $a \leq c_i \leq b,\ \ 1 \leq i \leq n$*

$$P(\sum_{i=1}^{n} c_i X_i > u) \sim \sum_{i=1}^{n} P(c_i X_i > u).$$

3.2 Proofs of Main Results

Proof of Theorem 1.

By the definition of ruin probability, we have

$$\psi(u;T) = P\left(e^{-\Delta(t)}U(t) < 0 \ \text{ for some } \ T \geq t > 0 | U(0) = u\right). \tag{13}$$

For each $t \in (0, T]$, we have

$$\begin{aligned} & u - \sum_{i=1}^{N(t)} X_i e^{-\Delta(\sigma_i)} + \sigma_0 \int_0^t e^{-\Delta(s)} dW_0(s) \\ & \leq e^{-\Delta(t)} U(t) \\ & \leq u + c \int_0^T e^{-\Delta(s)} ds - \sum_{i=1}^{N(t)} X_i e^{-\Delta(\sigma_i)} + \sigma_0 \int_0^t e^{-\Delta(s)} dW_0(s). \end{aligned} \tag{14}$$

Without essential difficulty, one can see that $\psi(u;T)$ satisfies that

$$\psi(u;T) \geq P(\sum_{i=1}^{N(T)} X_i e^{-\Delta(\sigma_i)} \geq u + \frac{c}{\beta}\xi + \xi\eta) \tag{15}$$

and

$$\psi(u;T) \leq P(\sum_{i=1}^{N(T)} X_i e^{-\Delta(\sigma_i)} \geq u + \xi\eta), \tag{16}$$

where

$$\xi = e^{\sigma \max_{0 \leq s \leq T}(-W(s))} \quad \textit{and} \quad \eta = \sigma_0 \max_{0 \leq t \leq T} \int_0^t e^{-\beta s} dW_0(s).$$

From Ross ([19]), $N(t)$ with intensity function $\lambda(s)$ can be regarded as a random sampling of some homogenous Poisson process $\overline{N}(t)$ with constant parameter λ, where $\lambda(s) \leq \lambda$. Now we introduce the indicator function 0f event A_i, $\mathrm{I(A_i)}$. We say that A_i happens, if at time σ_i, with probability $\lambda(\sigma_i)/\lambda$, X_i is picked out. From Lemma 3

$$\begin{aligned}
P(\sum_{i=1}^{N(T)} X_i e^{-\Delta(\sigma_i)} \geq u) &= P(\sum_{i=1}^{\overline{N}(T)} X_i e^{-\Delta(\sigma_i)} \mathrm{I(A_i)} \geq \mathrm{u}) \\
&= P(\sum_{i=1}^{\infty} X_i e^{-\Delta(\sigma_i)} \mathrm{I(A_i)I(\sigma_i \leq T)} \geq \mathrm{u}) \\
&\sim \overline{F}(u) \sum_{i=1}^{\infty} \int_0^T E[e^{-\alpha\Delta(s)}(\lambda(s)/\lambda)] dF_{\sigma_i}(s) \\
&\sim \overline{F}(u) \int_0^T E[e^{-\alpha\Delta(s)}(\lambda(s)/\lambda)] dm(s) \\
&= \overline{F}(u) \int_0^T \lambda(s) e^{-(\alpha\beta - \frac{1}{2}\alpha^2\sigma^2)s} ds, \qquad (17)
\end{aligned}$$

where, we have used the fact that, the renewal function of Poisson process $m(t)$, is just λt. For any fixed $\varepsilon > 0$

$$\begin{aligned}
&P(\sum_{i=1}^{N(T)} X_i e^{-\Delta(\sigma_i)} \geq u + \frac{c}{\beta}\xi + \xi\eta) \\
&\geq P(\sum_{i=1}^{N(T)} X_i e^{-\Delta(\sigma_i)} \geq (1+\varepsilon)u) - P(\frac{c}{\beta}\xi + \xi\eta \geq \varepsilon u) \\
&\geq P(\sum_{i=1}^{N(T)} X_i e^{-\Delta(\sigma_i)} \geq (1+\varepsilon)u) - \frac{E\xi^\tau E(\eta + \frac{c}{\beta})^\tau}{(\varepsilon u)^\tau}, \qquad (18)
\end{aligned}$$

where we have used Markov inequality. Lemma 2 implies that $E\xi^\tau < \infty$. Choosing $\tau > 0$ such that

$$\frac{E\xi^\tau E(\eta + \frac{c}{\beta})^\tau}{(\varepsilon u)^\tau}$$

is the higher order infinitesimal of $P(\sum_{i=1}^{N(T)} X_i e^{-\Delta(\sigma_i)} \geq (1+\varepsilon)u)$. By (17)

$$P(\sum_{i=1}^{N(T)} X_i e^{-\Delta(\sigma_i)} \geq (1+\varepsilon)u) \sim \frac{\overline{F}(u)\int_0^T \lambda(s) e^{-(\alpha\beta - \frac{1}{2}\alpha^2\sigma^2)s} ds}{(1+\varepsilon)^\alpha}. \qquad (19)$$

By the arbitrariness of ε, we obtain that

$$P(\sum_{i=1}^{N(T)} X_i e^{-\Delta(\sigma_i)} \geq u + \frac{c}{\beta}\xi + \xi\eta) \gtrsim \overline{F}(u) \int_0^T \lambda(s) e^{-(\alpha\beta - \frac{1}{2}\alpha^2\sigma^2)s} ds. \qquad (20)$$

On the other hand

$$\begin{aligned}
P(\sum_{i=1}^{N(T)} X_i e^{-\Delta(\sigma_i)} \geq u + \xi\eta) &\leq P(\sum_{i=1}^{N(T)} X_i e^{-\Delta(\sigma_i)} \geq (1-\varepsilon)u) + P(\xi\eta \geq \varepsilon u) \\
&\leq P(\sum_{i=1}^{N(T)} X_i e^{-\Delta(\sigma_i)} \geq (1-\varepsilon)u) + \frac{E\xi^\tau E\eta^\tau}{(\varepsilon u)^\tau}. \qquad (21)
\end{aligned}$$

Similarly

$$P(\sum_{i=1}^{N(T)} X_i e^{-\Delta(\sigma_i)} \geq u + \xi\eta) \lesssim \overline{F}(u) \int_0^T \lambda(s) e^{-(\alpha\beta - \frac{1}{2}\alpha^2\sigma^2)s} ds. \tag{22}$$

□

Notes and Comments. Rewrite

$$\max_{0\leq t\leq T} \int_0^t e^{-rs} dW(s) = \max_{0\leq t\leq \frac{1-e^{-2rT}}{2r}} \int_0^{-\frac{1}{2r}\ln(1-2rt)} e^{-rs} dW(s).$$

Denote $\int_0^{-\frac{1}{2r}\ln(1-2rt)} e^{-rs} dW(s)$ by $M(t)$. We aim to prove that $M(t)$ is Brownian motion. From Fima ([7]), we only need to prove that, the quadratic variation process, $[M, M](t)$, equals to t, because $M(t)$ is a local martingale. Using the definition of the quadratic variation, we have $[M, M](t) = \int_0^{-\frac{1}{2r}\ln(1-2rt)} e^{-2rs} ds = t$. Hence, the m.g.f. of $\max_{0\leq t\leq T} \int_0^t e^{-rs} dW(s)$ exists and $E\eta^\tau$ exists. It is not difficult to check that the result of Klüppelberg and Stadtmuller ([12]) is the special case of Theorem 1 if $F \in \mathcal{R}_{-\alpha}$, λ is some constant and $T = \infty$.

Proof of Theorem 2. Similar to the proof of Theorem 1, we have

$$\begin{aligned} & P(\sum_{i=1}^{N(T)} X_i e^{-\Delta(\sigma_i)} \geq u) \\ \sim & \int_0^\infty \sum_{n=1}^\infty P\left(\sum_{i=1}^n X_i e^{-\Delta(\sigma_i)} \geq u | N(T) = n, \Lambda = \lambda\right) P(N(T) = n | \Lambda = \lambda) G(d\lambda) \\ = & \int_0^\infty \sum_{n=1}^\infty P\left(\sum_{i=1}^n X_i e^{-\Delta(TU_i)} \geq u\right) P(N(T) = n | \Lambda = \lambda) G(d\lambda) \\ \sim & \, P\left(X_1 e^{-\Delta(TU_1)} \geq u\right) \int_0^\infty \sum_{n=1}^\infty n P(N(T) = n | \Lambda = \lambda) G(d\lambda) \\ = & \, E\Lambda \overline{F}(u) \int_0^T e^{-(\alpha\beta - \frac{1}{2}\alpha^2\sigma^2)s} ds, \end{aligned} \tag{23}$$

where we have used Lemma 2. □

Proof of Theorem 3. By Lemma 3

$$\begin{aligned} P(\sum_{i=1}^{N(T)} X_i e^{-\Delta(\sigma_i)} \geq u) &= P(\sum_{i=1}^\infty X_i e^{-\Delta(\sigma_i)} \mathrm{I}(\sigma_\mathrm{i} \leq \mathrm{T}) \geq \mathrm{u}) \\ &\sim \overline{F}(u) \sum_{i=1}^\infty E e^{-\alpha\beta\sigma_i - \alpha\sigma W(\sigma_i)} \\ &= \overline{F}(u) \int_0^T e^{-(\alpha\beta - \frac{1}{2}\alpha^2\sigma^2)s} d\sum_{i=1}^\infty F(s) \end{aligned}$$

$$= \overline{F}(u)\int_0^T e^{-(\alpha\beta-\frac{1}{2}\alpha^2\sigma^2)s}dm(s). \tag{24}$$

Similarly we can end the proof. □

Proof of Theorem 4. We only consider $P(\sum_{i=1}^{N(T)} X_i e^{-\Delta(\sigma_i)} \geq u)$. Because

$$P(\sum_{i=1}^{N(T)} X_i e^{-\Delta(\sigma_i)} \geq u) = \sum_{k=1}^{\infty} P(\sum_{i=1}^{k} X_i e^{-\Delta(\sigma_i)} \geq u, N(T) = k), \tag{25}$$

while

$$\begin{aligned}
&P(\sum_{i=1}^{k} X_i e^{-\Delta(\sigma_i)} \geq u, N(T) = k)\\
&= \int_{(N(T)=k)} P(\sum_{i=1}^{k} X_i e^{-\Delta(\sigma_i)} \geq u) dF(\sigma_1, ..., \sigma_{k+1})\\
&= \int_{(N(T)=k)} E[E[P(\sum_{i=1}^{k} X_i e^{-\Delta(\sigma_i)} \geq u | W(\sigma_1), ..., W(\sigma_k))]] dF(\sigma_1, ..., \sigma_{k+1})\\
&\sim \int_{(N(T)=k)} E[E[\sum_{i=1}^{k} P(X_i e^{-\Delta(\sigma_i)} \geq u | W(\sigma_1), ..., W(\sigma_k))]] dF(\sigma_1, ..., \sigma_{k+1})\\
&= \int_{(N(T)=k)} \sum_{i=1}^{k} P(X_i e^{-\Delta(\sigma_i)} \geq u) dF(\sigma_1, ..., \sigma_{k+1})\\
&= \sum_{i=1}^{k} P(X_i e^{-\Delta(\sigma_i)} \geq u, N(T) = k),
\end{aligned} \tag{26}$$

hence

$$\begin{aligned}
P(\sum_{i=1}^{N(T)} X_i e^{-\Delta(\sigma_i)} \geq u) &= \sum_{k=1}^{\infty} P(\sum_{i=1}^{k} X_i e^{-\Delta(\sigma_i)} \geq u, N(T) = k)\\
&\sim \sum_{k=1}^{\infty} \sum_{i=1}^{k} P(X_i e^{-\Delta(\sigma_i)} \geq u, N(T) = k)\\
&= \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} P(X_i e^{-\Delta(\sigma_i)} \geq u, N(T) = k)\\
&= \sum_{i=1}^{\infty} P(X_i e^{-\Delta(\sigma_i)} \geq u, N(T) \geq i)\\
&= \sum_{i=1}^{\infty} P(X_i e^{-\Delta(\sigma_i)} \geq u, \sigma_i \leq T)\\
&= \sum_{i=1}^{\infty} \int_0^T P(X_i e^{-\Delta(s)} \geq u) dF_{\sigma_i}(s)
\end{aligned}$$

$$= \int_0^T P(X_i e^{-\Delta(s)} \geq u) dm(s). \tag{27}$$

Hence Theorem 4 is completed. □

Proof of Theorem 5. We deal with the proof by induction. For $k = 1$

$$\begin{aligned}
&\psi_1(u) \\
&= P(u + \int_0^{\theta_1} e^{-\Delta(y)} dy - X_1 e^{-\Delta(\theta_1)} < 0) \\
&= \int_0^\infty F_{\theta_1}(ds) \int_{-\infty}^\infty P(X_1 > u e^{\beta s + \sigma t} + e^{\beta s + \sigma t} \int_0^s e^{-\Delta(y)} dy) \frac{1}{\sqrt{2\pi s}} e^{-\frac{t^2}{2s}} dt \\
&\sim \overline{F}(u) \int_0^\infty e^{-\alpha\sigma s} F_{\theta_1}(ds) \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} e^{-\alpha\sigma v\sqrt{s}} dv \\
&= \overline{F}(u) q,
\end{aligned} \tag{28}$$

we have used the fact that X_1 is in $\mathcal{L}$ class. Assume that

$$\psi_n(u) \sim \overline{F}(u) \frac{q - q^{n+1}}{1 - q}, \tag{29}$$

we should prove that (29) holds for $n + 1$. Denote

$$u e^{\Delta(\theta_1))} + c \int_0^{\theta_1} e^{\Delta(\theta_1) - \Delta(y)} dy$$

by $V(\theta_1, u)$. With total probability formula

$$\begin{aligned}
&\psi_{n+1}(u) \\
&= \psi_1(u) + \overline{\psi_1}(u) \mathrm{E}[\psi_n(V - X_1, u) \mathrm{I}(V \geq X_1)] \\
&\sim \psi_1(u) + \frac{q - q^{n+1}}{1 - q} \int_0^\infty F_{\theta_1}(ds) \int_{-\infty}^\infty \Phi_{(0,s)}(dt) \int_0^\infty e^{-\alpha r s - \alpha\sigma t} F_{X_1}(dy) \overline{F}(u) \\
&\sim (q + \frac{q - q^{n+1}}{1 - q} \int_0^\infty e^{-\alpha r s} F_{\theta_1}(ds) \int_{-\infty}^\infty e^{\frac{1}{2}\alpha^2\sigma^2 s} \Phi_{(0,1)}(dt)) \overline{F}(u) \\
&= \frac{q - q^{n+2}}{1 - q} \overline{F}(u),
\end{aligned} \tag{30}$$

so by using induced assumption, Theorem 5 is finished. □

We can see that, when $n = \infty$, this Theorem turns to

$$\psi_\infty \sim \frac{q}{1 - q} \overline{F}(u),$$

which contains the result of Tang ([16]) as a special case.

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References

1. Asmussen, S.: Subexponential asymptotics for stochastic processes: extremal behavior, statonary distribution and first passage probabilities. The Annals of Appl. Prob. (8), 354–374 (1998)
2. Asmussen, S., Kalashnikov, V., Konstantinides, D., Klüppelberg, C., Tsitsiashvili, G.: A local limit theorem for random walk maxima with heavy tails. Statist. Probab. Lett. 56(4), 399–404 (2002)
3. Cai, J.: Discrete time risk models under rates of interest. Prob. Eng. Inf. Sci. (16), 309–324 (2002)
4. Dufrense, F., Gerber, H.: Risk theory for the compound Poisson process that is perturbed by diffusion. Insurance: Mathematics Economics (10), 51–59 (1991)
5. Embrechts, P., Veraverbeke, N.: Estimates for the probability of ruin with special emphasis on the possibility of large claims. Insurance: Mathematics and Economics 1, 55–72 (1982)
6. Embrechts, P., Klüppelberg, C., Mikosch, T.: Modelling Extremal Events for Insurance and Finance. Springer, Berlin (1997)
7. Fima, C.K.: Introduction to Stochastic Calculus with Applications. Imperial College Press, London (1998)
8. Jiang, T., Xu, C.: The Asymptotic Behavior of the Ruin Probability within a Random Horizon. Acta Math. Appl. Sinica 20(2), 353–356 (2004)
9. Jiang, T.: Large-deviation probabilities for maxima of sums of subexponential random variables with application to finite-time ruin probabilities. Science In China (Series A) 51(7), 1147–1338 (2008)
10. Jiang, T.: Ruin probability for risky investment of insurance capital. Journal of Systems Engineering 23(2), 148–153 (2008)
11. Karatzas, I., Shreve, S.: Brownian Motion and Stochastic Calculus. Springer, Berlin (1988)
12. Klüppelberg, C., Stadtmuller, U.: Ruin probabilities in the presence of heavy-tails and interest rates. Scand Actuar J. 1, 49–58 (1998)
13. Konstantinides, D.G., Tang, Q., Tsitsiashvili, G.S.: Estimates for the ruin probability in the classical risk model with constant interest force in the presence of heavy tails. Insurance Math. Econom. 31(3), 447–460 (2002)
14. Tang, Q.: Extremal Values of Risk Processes for Insurance and Finance: with Special Emphasis on the Possibility of Large Claims. Doctoral thesis of University of Science and Technology of China (2001)
15. Tang, Q.: The ruin probability of a discrete time risk model under constant interest rate with heavy tails. Scand. Actuar. J. (3), 229–240 (2004)
16. Tang, Q.: Asymptotic ruin probabilities of the renewal model with constant interest force and regular variation. Scand Actuar J. (1), 1–5 (2005a)
17. Tang, Q.: The finite time ruin probability of the compound Poisson model with constant interest force. J. Appl. Probab. 42(3), 608–619 (2005b)
18. Resnick, S.I., Willekens, E.: Moving averages with random coefficients and random coefficient autoregressive models. Comm. Statist. Stochastic Models 7(4), 511–525 (1991)
19. Ross, S.: Stochastic Processes. Wiley, New York (1983)
20. Veraverbeke, N.: Asymptotic estimates for the probability of ruin in a Poisson model with diffusion. Insurance Math. Econom. 13(1), 57–62 (1993)
21. Yang, H., Zhang, L.: Martingale method for ruin probability in an autoregressive model with constant interest rate. Prob. Eng. Inf. Sci. (17), 183–198 (2003)