Frequency Distributions of Sand Pile Models

Ruey-Tarng Liu

Department of Physics, National Chung-Hsing University
250, Guo-Kuang Road, Taichung, Taiwan
rtliu@phys.nchu.edu.tw

Abstract. We calculated the frequency distributions of cluster sizes in the sand pile models. Two cellular automata models differing in the rules of adding sand particles are used. For the model with local perturbation only, the distribution shows a power law behavior regardless of the spatial dimension that the sand pile is situated at. For the other model where the perturbation generated by the addition of a sand particle is not confined to one site only, the distribution is generally a power law plus an exponential cutoff. These results are consistent with what was found previously for another complex system using a model of constrained minority game. The frequency distributions in higher dimensions than two are also calculated and discussed.

Keywords: power law, frequency distribution, sand pile.

1 Introduction

Frequency distributions have been found to follow a power law in many complex systems[1]. One famous example is the Gutenberg-Richter power law for earthquakes[2]: the number of occurrences of earthquakes in a given region has a power-law relation to the magnitudes of the earthquakes. Bak[3] proposed the concept of self-organized criticality (SOC) to explain how such a complex system works. They demonstrated SOC by means of a sandpile model in which one builds a sand pile by dropping sand particles one by one from the top. The sand pile will grow until it reaches its critical state with a maximal slope, which, however, is unstable. Adding particles further will cause an avalanche after which the sand pile will gradually rebuild itself back to the critical state. Thus the system constantly self-organizes itself to approach the critical state which is then followed by an avalanche of a certain scale. They used a cellular automaton model to simulate the dynamics of a sand pile[4] and claimed that frequency distribution of the sizes of the avalanches indeed followed a power law. Some variants of their models have since been proposed and solved or simulated[5,6,7]. One general conclusion of these theoretical works is that SOC is a good description for the dynamics of a sand pile, but the frequency distributions of the cluster sizes do not necessarily follow a power law. In many cases, a power law with an exponential cutoff is observed. One is reminiscent of the results of a different complex system called the constrained minority game[8], the frequency distributions of which also shows either a power law or a power law with an exponential cutoff, depending on the relative magnitude of two parameters used in that system.

J. Zhou (Ed.): Complex 2009, Part II, LNICST 5, pp. 1743 – 1749, 2009.

In this article we investigate the original sand pile model of Bak et al[4] in various spatial dimensions and one variant of it using a different rule for adding sand particles. Our aim is to understand what parameters of a sand pile model are the dominant factors in determining the behavior its frequency distribution.

2 Sand Pile Models and Simulation Results

We first construct cellular automaton models for a sand pile. Because there is azimuthal symmetry for a conic sand pile, for simplicity we consider only a quarter of a sand pile bounded by two perpendicular walls. The height of sands at position (x, y), $x > 0$, $y > 0$, is denoted by $h(x, y)$. The slope of the sand pile at (x, y) is defined as sum of height differences between the position (x, y) and its two neighboring sites down along the hill: $s(x, y) = [h(x, y) - h(x+1, y)] + [h(x, y) - h(x, y+1)]$. The dynamics of the model consists of the processes of adding and falling of the sand particles. We add a sand particle at a random position every time. On each adding, we check the slopes at all positions. When $s(x, y)$ is larger than a predefined critical value s_c, one sand particles at (x, y) will drop, either to position $(x+1, y)$ or $(x, y+1)$. Thus for our model, the dynamics of the sand pile follow the following rules:

Adding rule: when a sand particle is added at a random position (x, y), do

$$h(x, y) \rightarrow h(x, y) + 1. \tag{1}$$

Falling rule: when $s(x, y) > s_c$, do

$$\begin{aligned} & h(x, y) \rightarrow h(x, y) - 1 \\ & h(x, y+1) \rightarrow h(x, y+1) + 1 \quad or \quad h(x+1, y) \rightarrow h(x+1, y) + 1 \end{aligned} \tag{2}$$

Alternatively, we use the slope $s(x, y)$ as the dynamic variable and let on average a 0.5 sand particle falls to each of the two positions $(x+1, y)$ and $(x, y+1)$ when $s(x, y) > s_c$. Accordingly, the rules for simulation are:

Adding rule: when a sand particle is added at a random position (x, y), do

$$\begin{aligned} s(x, y) &\rightarrow s(x, y) + 2 \\ s(x, y-1) &\rightarrow s(x, y-1) - 1 \\ s(x-1, y) &\rightarrow s(x-1, y) - 1 \end{aligned} \tag{3}$$

Falling rule: when $s(x, y) > s_c$, do

$$\begin{aligned} s(x, y) &\rightarrow s(x, y) - 4 \\ s(x, y \pm 1) &\rightarrow s(x, y \pm 1) + 1 \\ s(x \pm 1, y) &\rightarrow s(x \pm 1, y) + 1 \end{aligned} \tag{4}$$

When a sand particle drops from the boundary $x = R$ or $y = R$, it disappears from the system. The model can be readily extended to an abstract one with underlying spatial dimension being larger than 2. That is, we can consider the height and the slope of the sand pile as a function of more than two rectangular coordinates.

Bak et al.[4] used a different set of adding and falling rules in their simulations for the dynamics of the sand piles. They dealt with the variation of a variable $K(x, y)$ without giving its explicit relation to the height of a sand pile at (x, y). At each time step, they perturbed the system locally by adding 1 to the value of $K(x, y)$ at random position (x, y). Thus their adding rule is

Adding rules of Ref. [4]: at random position (x, y), do

$$K(x, y) \rightarrow K(x, y) + 1 \tag{5}$$

When $K(x, y) > K_c$, they made the following changes

Falling rules of Ref.[4]: $K(x, y) > K_c$, do

$$\begin{aligned} K(x, y) &\rightarrow K(x, y) - 4 \\ K(x, y \pm 1) &\rightarrow K(x, y \pm 1) + 1 \\ K(x \pm 1, y) &\rightarrow K(x \pm 1, y) + 1 \end{aligned} \tag{6}$$

Comparing with our model, it is interesting to see that the variation of $K(x, y)$ is partly like $h(x, y)$ (in adding rules) and partly similar to $s(x, y)$ (in falling rules). We will see below that the results of their model using Eqs.(5) and (6) are different from those of our model using Eqs. (3) and (4).

Let n be the number of sites affected due to addition of a sand particle to a sand pile, and $D(n)$ be the probability of the occurrence of cluster size n. Starting form arbitrary random stable sand pile we simulated the dynamics of the above two models and collected data for 100 thousands time steps. The results for the model according to Eqs. (5) and (6) averaged over 100 samples are plotted in Fig. 1. The log-log plot of $D(n)$ has approximate power law $D(n) \sim n^{-\tau}$ with τ = 0.98, 1.36, 1.49, and 1.65 when the underlying spatial dimension is 2D, 3D, 4D, and 6D respectively. Our results for 2D and 3D cases agree with the previous results of Bak et al.[4].

The results of $D(n)$ calculated from first 100,000 time steps of the simulation using Eqs. (3) and (4) are also close to a power law distribution. However, for this model we found that $D(n)$ has not reached its stability after first 100,000 time steps. The number of occurrence of each cluster size per 100,000 time steps decreases with time. The shape of the plot of the distribution $D(n)$ bents away from a power law as time increases (Fig. 2). It takes about 5,000,000 time steps before the distribution $D(n)$ stabilizes to a curve with a power law plus an exponential cutoff.

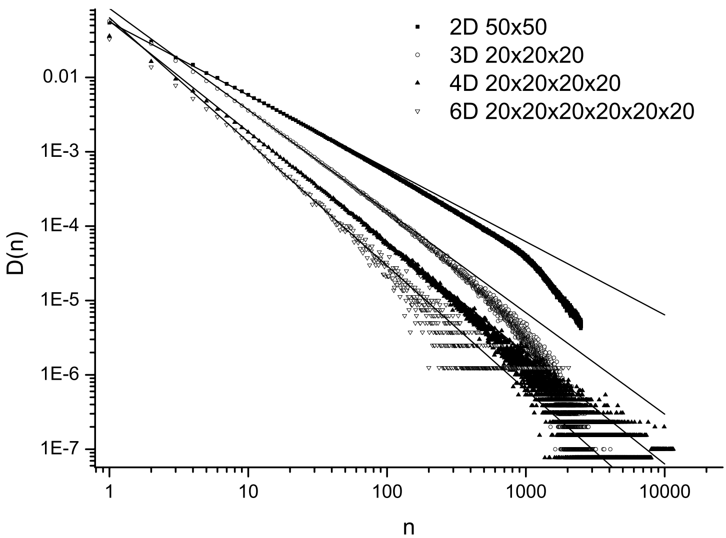

Fig. 1. Frequency distributions of cluster sizes using the rules described by Eqs. (5) and (6). The distribution is more close to a power law in higher dimension because the system size increases much faster than the level of complexity of the system. The straight lines indicate a power law with $\tau = 0.98$, 1.36, 1.49, and 1.65 respectively.

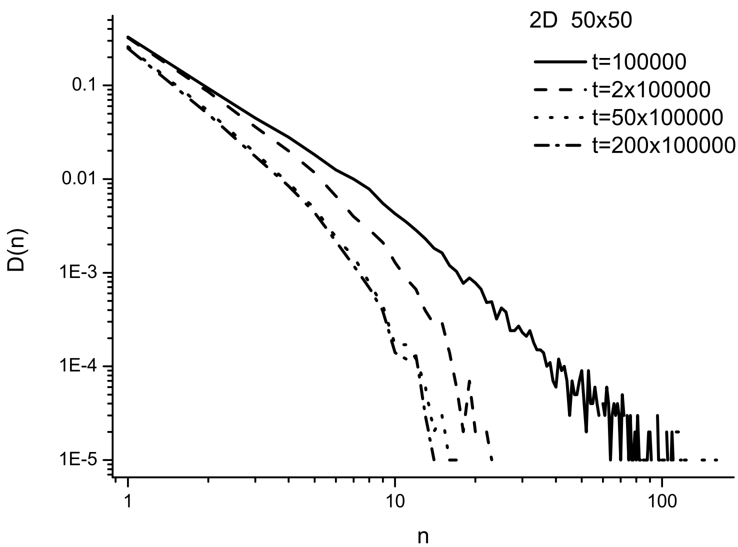

Fig. 2. Frequency distributions of cluster sizes at different time using the rules described by Eqs. (3) and (4). The distribution takes long time (about 5,000,000 time steps) to reach stabilization. The final stable distribution follows a power law plus an exponential cutoff.

If we keep adding sand particles always at the top the sand pile instead of at a random position in the model of Bak et al (Eqs. (5)(6)), the distribution $D(n)$ is also close to a power law similar to Fig. 1. However, when rules Eqs(3),(4) are used, keep adding sand particles at the top yields power law distribution for $D(n)$ too instead of a curve like Fig. 2. This is because the adding rule Eq. (3) is equivalent to Eq. (5) if only the top position (1,1) is concerned. While when sand particles are added randomly at any position (x, y), Eq. (3) makes a big difference with Eq. (5).

In Fig. 3 we plotted the results of our model for various system sizes and dimensions. The difference between one another is discussed in the following section.

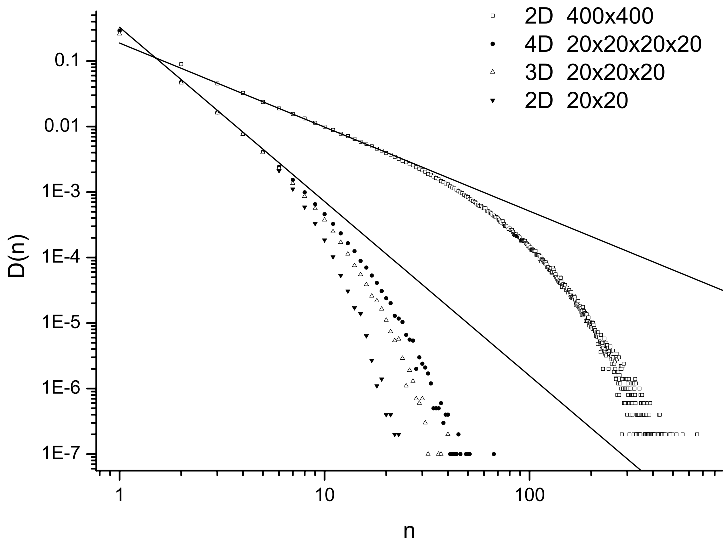

Fig. 3. Comparison of the frequency distributions of different system sizes and spatial dimensions. Results are obtained using the rules described by Eqs. (3) and (4). For a fixed system L=400´400=160000, the 2D case has smaller level of complexity than the 4D case so that its distribution is closer to a power law than the 4D case. On the other hand, when we decreases both the underlying spatial dimension and the system size from the 4D case, the distribution bent further away from a power law.

3 Discussion on the Forms of the Frequency Distributions

We have seen that the distribution $D(n)$ in sand pile models follows a power law or a power law plus exponential cutoff, depending on the dimension and the size of the sand pile, and the rules one used. These results are consistent with what was found in the variance distributions of a constrained Minority game [8]. In Ref. [8], it was found that there are two critical parameters in a complex system and the relative magnitude of them determines the distribution to be the form of a power law, a power law plus an exponential cutoff, or a log-normal. One of the critical parameters is the system size L. In current sand pile models discussed above, the system size L can be taken as total number of sites of a sand pile. The other parameter *M* is called *information* in Ref. [8]. It indicates how complicated a system is inter-connected so that we might as well call it the *level of complexity* of the system. It was shown in Ref. [8] that the frequency distribution of a complex system shows a power law behavior when *M* is small. As *M* increases for a fixed *L*, the frequency distribution changes from a power law to a power law plus an exponential cutoff, and then to a log-normal. It is not yet clear what the equivalence of *M* is in the sand pile models. It is however reasonable to assume that *M* is a monotonic increasing function of the number of ways that are generated by a perturbation to make changes to the system. For example, we see that the adding rule Eq. (3) has affected three sites while the rule Eq. (5) has perturbed one site only. Accordingly, the value of *M* for our model is expected to be larger than that of the Bak et al's model. The underlying dimension can change the value of *M* too. In 2D case, the falling rule affects 4 neighboring site of the perturbed position. That is, a perturbation has four different paths in changing the system. In 3D case, there will be 6 paths. In 4D, we have 8, etc. Thus when we simulate the dynamics of the sand pile in higher dimension, we have higher value of *M*. With these understandings, we can have a qualitative explanation for frequency distributions shown in Figs. 1, 2, and 3:

(i) For a given rule and a fixed dimension, say 2D, the value of *M* is fixed. When one increases system size *L*, e.g. increasing from 20´20 to 400´400, we obtain a better approximation to a power law.

(ii) For a fixed dimension and system size, say 2D with 50´50 sites, the value of *L* is fixed. Changing the adding rule from Eq. (5) to Eq. (3) will increase the value of *M*. Thus the frequency distribution changes from a power law to a power law plus an exponential cutoff.

(iii) For a fixed system size, say L=400´400=160000, increasing the underlying dimension, say from 2D to 4D as shown in Fig. 3, the curve will bent further away from a power law because *M* has increased.

(iv) Increasing both dimension and system size as the case shown in Fig. 1, although the value *M* increases when underlying dimension increases, the system size *L* has also increased in much larger amount (20 times larger when one goes from 2D 20´20 to 3D 20´20´20). Thus the frequency distribution is more close to a power law in higher dimension.

4 Conclusion

We have calculated the frequency distribution of the cluster sizes in two dynamic sand pile models situated at various spatial dimensions. We found that the distribution follows a power law when the perturbation to the sand pile is local at one site only. In a more realistic model the perturbation is not pure local so that the system is connected in a more complicated way (i.e., the system has a higher level of complexity), the distribution changes to a power law plus an exponential cutoff. This result is consistent with the previous finding in another complex system dealing with a complete different phenomenon related to the minority game. We have also simulated the sand pile models in higher spatial dimensions than two. In higher dimension a sand particle has more number of paths to fall so that the system is connected in a more complex way. We found, while keeping the system size fixed, the frequency distribution differs from a power law more for a sand pile situated at higher dimension. This result again supports the general property for the frequency distributions of complex systems: the distribution follows a power law when the system size is large enough, but changes to a power law with an exponential cutoff when the system size decreases or the level of complexity increases.

Acknowledgments. The author thanks Prof. Sy-Sang Liaw for many fruitful discussions. This work is supported by grants from National Science Council under the grant number NSC97-2112-M005-001, and National Center for Theoretical Sciences of Taiwan.

References

1. Newman, M.E.J.: Power laws, Pareto distributions and Zipf's law. Contemp. Phys. 46, 323–351 (2005)
2. Gutenberg, B., Richter, C.F.: Frequency of earthquakes in California. Bull. Seismol. Soc. Am. 34, 185–188 (1944)
3. Bak, P.: How nature works: the science of self-organized criticality. Copernicus, N.Y (1996)
4. Bak, P., Tang, C., Wiesenfeld, K.: Self-organized criticality: An explanation of the 1/*f* noise. Phys. Rev. Lett. 59, 381–384 (1987)
5. Kadanoff, L.P., Nagel, S.R., Wu, L., Zhou, S.M.: Scaling and universality in avalanches. Phys. Rev. A39, 6524–6537 (1989)
6. Olami, Z., Feder, H.J.S., Christensen, K.: Self-organized criticality in a continuous, nonconservative cellular automaton modeling earthquakes. Phys. Rev. Lett. 68, 1244–1247 (1992)
7. de Sousa Vieira, M.: Breakdown of self-organized criticality in sandpiles. Phys. Rev. E66, 051306 (2002)
8. Liaw, S.S.: Frequency distributions of complex systems. J. Stat. Mech. P05015 (2008)