Evolutionary Game in a Single Hub Structure

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Abstract. In this paper, we investigate the evolutionary game theory on a simplest heterogeneous network-a single hub structure. In order to describe the dynamics on structured populations, we firstly give a general form of a spatial replicator equation. Then according to it, the evolutionary equations describing the evolution of two strategies (cooperation and defection) are derived explicitly and the dynamics of the system is discussed theoretically and numerically. We found if judging the strategy according to its ability to resist the invasion of another, the cooperation does better than the defection. In some parameters when the population N is small, an initial D-hub system may evolve to an all-cooperator (AllC) state. All of these phenomena can be well explained by corresponding replicator equation.

Keywords: evolutionary game theory, structured population, single-hub structure, spatial replicator equation.

1 Introduction

Cooperation is an important aspect of all biological system. But ever since Darwin [1] the evolutionary and maintenance of it has been a major challenge in evolutionary biology. In spite of the risk of exploitation by cheaters, cooperation is abundant both in nature and in human society. This question has been addressed by studying the evolutionary replicator dynamics [2] of simple two-player games such as prisoner's dilemma, snowdrift game and stag hunter game, which model cooperation in different level. Replicator equation successfully describes the evolution of the frequencies of strategies in a well-mixed population in which every player interact with all the other ones. However, according to it cooperator can not exist in Prisoner's Dilemma.

In realistic multi-player systems players do not interact with all the other players. Population is often organized in certain structure. Since the pioneering work of Nowak and May [3], the study of the role of structure on cooperation have attracted considerable attention. Recently compelling evidence has been accumulated to show that many social, biology and technological real networks has small word or scale-free property [4,5]. Numerical studies on these networks show the heterogeneity of structure improves cooperation [6,7,8,9,10,11,12,13]. The most surprising result is that cooperation can even dominate over defection

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in a heterogeneous scale-free networks not only in PD game but also in snow drift game [9]. In order to understand the promotion in heterogenous network which is characterized by many vertices of small connectivity and a few highly connected vertices (hubs), in this paper we investigate the evolution of cooperation in a simplest heterogenous structure-single hub network. More complex network such as scale free network can been regarded as organized by different size of these networks. The update law we choose is similar as that in paper [6,7,8,9,10]. We give the detailed form of the master equation describing the evolution in structure population. From it, the replicator equation of single hub network is derived. In the help of the replicator equation, we analyze the dynamics on the network. The computer simulation shows it is a successful theoretical analysis.

2 The Theoretical Analysis

2.1 Micro Dynamics in Nodes

We place individuals (players) on the vertices of the single hub network. The network includes $N-1$ nodes with degree of $K=1$, and one hub node with degree of $K=N-1$. The hub can be a cooperator (C) or a defector (D). The edges of the network represent interactions among them. The interactions are modelled in terms of two-person games in which both players can either cooperate or defect when interacting with each other. Mutual cooperation leads to the reward R, whereas mutual defection leads to the punishment P. If one player cooperates and the other defects, the cooperator get payoff S (sucker's payoff) and the defector gets T (temptation). This can also be described by a payoff matrix as below.

	C	D
C	R	S
D	T	P

Different games depend on the relative ordering of these four payoffs: the Snowdrift game, for which $T > R > S > P$; the Stag-Hunt game, for which $R > T > P > S$; and the Prisoner's Dilemma game, for which $T > R > P > S$.

At each time step t, every individual i in the network plays with all its K_i neighbors and accumulates its obtained payoff P_i. Then, all individuals i synchronously update their strategies S_i by randomly choosing one of its K_i neighbors, and comparing their respective payoffs P_i and P_j. If the neighbors payoff is lower or equal , namely $P_j \leq P_i$, individual i keeps its strategy S_i for the next time step. Otherwise individual i adopts the strategy of j S_j for the next time step with probability $p = \frac{P_j - P_i}{max((K_i, K_i)}$.

2.2 From Micro to Macro Dynamics

The Master Equation. Let $x(t)$ denotes the frequency of cooperation in the network at time step t,

$$x(t) = \frac{\sum_i \delta(S_i = C, t)}{N}$$

where S_i denotes the state of node i, and $\delta(S_i = C, t)$ is defined as below:

$$\delta(S_i = C) = \begin{cases} 1, \ S_i = C \\ 0, \ \text{others} \end{cases}$$

When N is large enough, the value of $x(t)$ is equal to the value of the probability of the cooperation. According to the master equation, we get a spatial replicator equation

$$\frac{dx}{dt} = \frac{1}{N} \sum_i [\delta(S_j = C|S_i = D) w(D \to C, i) \\ -\delta(S_j = D|S_i = C) w(C \to D, i)] \tag{1}$$

where j denotes the randomly chosen neighbors of i, $\delta(S_j = C|S_i = D)$ ($\delta(S_j = D|S_i = C)$) denotes node i is a defector (cooperator) and having at least one cooperator (defector) neighbor j. Their particular forms are:

$$\delta(S_j = C|S_i = D) = \begin{cases} 1, & S_i = D \quad \text{and} \quad S_j = C \\ 0, & \text{others} \end{cases}$$

$$\delta(S_j = D|S_i = C) = \begin{cases} 1, & S_i = C \quad \text{and} \quad S_j = D \\ 0, & \text{others} \end{cases}$$

And $w(D \to C, i)$ and $w(C \to D, i)$ are the transition rates. Compare to the mean-field case in which the individual transition rate is easily written as $w(C \rightleftharpoons D)$, the individual transition rate here need be distinguished from place to place because of the introducing of the structure. So we mark the individual transition rate here $w(C \rightleftharpoons D, i)$. According to the update law, their forms are:

$$w(D \to C, i) = \rho_{C|D}(i) \max\left(\frac{P_C(j) - P_D(i)}{max(K_i, K_j)}, 0\right)$$

$$w(C \to D, i) = \rho_{D|C}(i) \max\left(\frac{P_D(j) - P_C(i)}{max(K_i, K_j)}, 0\right)$$

where

$$\rho_{C|D}(i) = \frac{\sum_j \delta(S_j = C|S_i = D)}{K_i}$$

$$\rho_{D|C}(i) = \frac{\sum_j \delta(S_j = D|S_i = C)}{K_i}$$

In the population with single hub structure, there are two types of players, one is the player on the hub with $N-1$ neighbors, and the other $N-1$ peripheral players with only one neighbor which is on the hub. Since there is no interaction between peripheral players, the replicator equation of the cooperator frequency can be given explicitly. Without loss of the generality, we set $R = 1$ and $P = 0$ and we pay attention to the parameter range of $T \in (0, 2), S \in (-1, 1)$.

The Replicator Equation. When the hub is occupied by a cooperator (C-hub), only the flips of the strategies for the cooperator on the hub and for the defectors on the peripheral nodes will contribute to the evolution of the cooperator frequency. It is straightforward to have the payoff for the cooperator on the hub as $P_h = (N-1)(x - \frac{1}{N}) + (N-1)(1-x)S$ and the payoff for each defector on the peripheral node as $P_p = T$. Similarly, when the hub is occupied by a defector (D-hub), only the flips of the strategies for the defector on the hub and the cooperators on the peripheral nodes will contribute to the evolution of the cooperator frequency. The payoff for the defector on the hub is $P_h = (N-1)xT$ and the payoff for each cooperator on the peripheral node is $P_p = S$. Substituting these payoffs into the transition rates, we may have the replicator equations for the model.

To see it clearly, we consider the situation with the rule-A in the updating stage. Assuming that the limit of $N \to \infty$ is taken, we have the replicator equation with C-hub

$$\begin{cases} \frac{dx}{dt} = (1-x)[(1-S)x + S] \\ x > \frac{-S}{1-S}, \end{cases} \tag{2}$$

To be noted, Eq.(2) only counts the contributions from the defectors on the peripheral nodes changing to be cooperators. Once the condition of $x > \frac{-S}{1-S}$ is violated, C-hub becomes a D-hub and the system will follow another equation which is discussed next. Let $f_c(x) = (1-x)[(1-S)x + S]$ and make $f_c(x) = 0$, we get two fixed points: $x^* = 1$ (an AllC state) and $x^* = \frac{-S}{1-S}$. Which one will the system evolve to? In order to get the answer, we investigate their stability.

$$\frac{\partial f_c(x)}{\partial x} = -2(1-S)x + 1 - 2S$$

then

$$\frac{\partial f_c(x)}{\partial x} |_{x^*=1} = -1, \quad \frac{\partial f_c(x)}{\partial x} |_{x^*=\frac{-S}{1-S}} = 1$$

So under the condition of C-hub, $x^* = 1$ always is a stable fixed points, which corresponds a evolutionary stable state (ESS), and $x^* = \frac{-S}{1-S}$ is unstable. Therefore for a system starting from a C-hub state with a given initial cooperator frequency x_i, the AllC state will be reached for $S > -\frac{x_i}{1-x_i}$.

Similarly, we have the replicator equation with D-hub

$$\begin{cases} \frac{dx}{dt} = -Tx^2 + \frac{S}{N-1}x \\ x \geq \frac{S}{(N-1)T}, \end{cases} \tag{3}$$

where only the contributions from the cooperators on the peripheral nodes changing to be defectors are counted. When the condition of $x \geq \frac{S}{(N-1)T}$ is not satisfied, the D-hub changes to be a C-hub (In the limit of $N \to \infty$, such a flip of the hub state seems impossible). Let $f_d(x) = -Tx^2 + \frac{S}{N-1}x$ and make $f_d(x) = 0$, we get two fixed points: $x^* = 0$ and $x^* = \frac{S}{(N-1)T}$. What about their stability?

$$\frac{\partial f_d(x)}{\partial x} = -2Tx + 1 + \frac{S}{N-1}$$

then

$$\frac{\partial f_d(x)}{\partial x}\Big|_{x^*=0} = \frac{S}{N-1}, \quad \frac{\partial f_d(x)}{\partial x}\Big|_{x^*=\frac{S}{(N-1)T}} = -\frac{S}{N-1}$$

The result clearly shows $x^* = \frac{S}{(N-1)T}$ is stable when $S > 0$ and $x^* = 0$ is stable when $S < 0$. For $S = 0$, two fixed points merge into one and the fixed point $x* = 0$ is neutrally stable. However, since N is always large, the system will spend much time to evolve to its stable fixed point. Also because in a real system the number of players N is an integer, if $\frac{S}{(N-1)T} < \frac{1}{N}$ the system will also evolve to AllD state. Therefore for a system starting from a D-hub with any initial frequency $x_i > 0$ we can observe the existence of cooperation in the area $\frac{S}{T} > 1$.

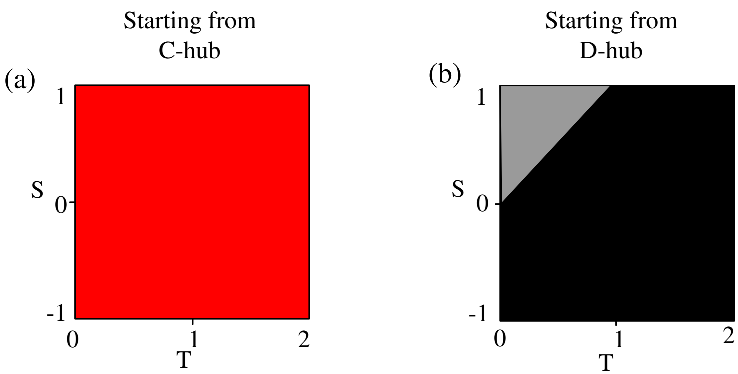

Fig. 1. Theoretical analysis: (a) and (b) respectively show the result of evolution under the conventional updating starting from an initial C-hub state and an D-hub state. The red color identifies the AllC region, and the black color identifies the AllD region. The gray color in (b) marks a low cooperator frequency region. All these result is get with same initial cooperator frequency $x_i = 0.5$.

To sum up, the system with the hub initially occupied by a defector evolves towards a state with almost every player being defector regardless of the parameter and initial cooperator frequency. However, the system with a C-hub develops into an ALLC state for $S > -\frac{x_i}{1-x_i}$; otherwise the system transits to a D-hub system. The theoretical results for a given initial cooperator frequency $x_i = 0.5$ are shown in Fig.1.

3 Numerical Results

In order to check the validity of our theoretical analysis, we create a single hub system with 100 nodes. According to the update law the system evolves. After a transient time we calculate the mean cooperation frequency. Since the system starting from different state will approach to fixed point in different rate,

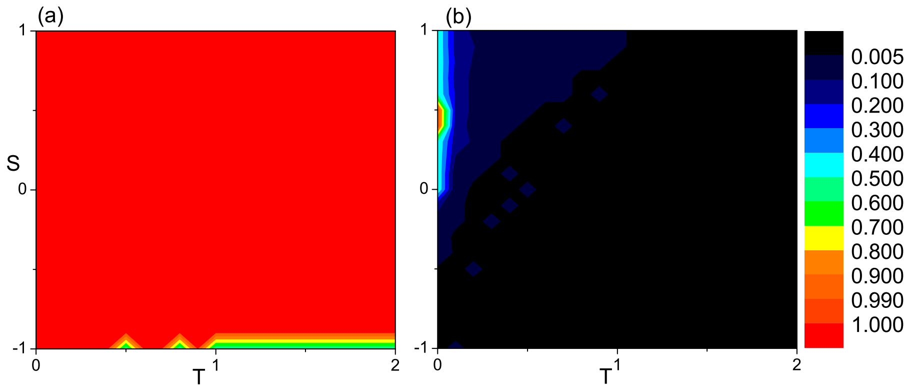

Fig. 2. Computer simulation result: (a) the stable mean cooperator frequency starting from C-hub with $x_i = 0.5$ (b) the stable mean cooperator frequency starting from D-hub Different colors represent the different value of x. The two graphs share a same color bar which is shown in right.

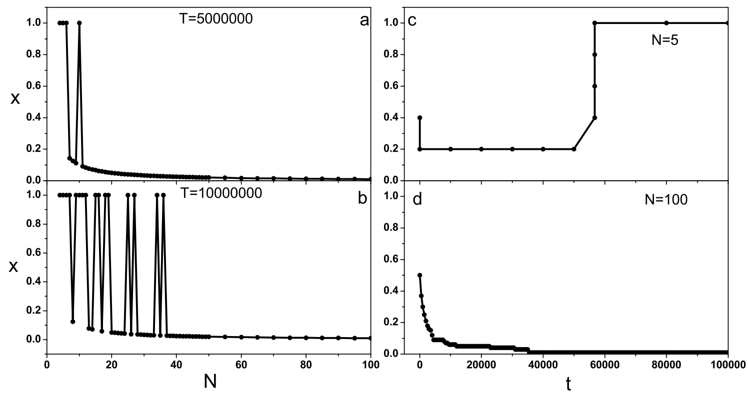

Fig. 3. Computer simulation result: the cooperator frequency x against N and t at parameter $S = 0.7$, $T = 0.5$. (a) x versus N with transition period setting $T = 5 \times 10^6$ (b) x versus N with transition period setting $T = 5 \times 10^7$ (c) the evolution of x in a population $N = 5$, (d) the evolution of x in a population $N = 100$.

we execute the calculation differently. For a system starting from a C-hub we execute the calculation after the system evolves 5×10^5 generation, and calculate the mean cooperation frequency over 5000 generations. If the system starts from a D-hub, it indeed spends much time to approach its fixed point as suggested by the theoretical analysis. So in this situation we execute the calculation after the system evolves 1×10^6 generation then calculate the mean cooperation frequency over 5000 generations. The result is showed in fig. 2.

Moreover, here we want to discuss an interesting phenomenon. In the parameter region $S > T > 0$ and $\frac{2}{N} > \frac{S}{(N-1)T} > \frac{1}{N}$, the system can evolve to the state that there is only one cooperator. Once in such state, the payoff of the only existing cooperator is larger than that of the hub. With a small probability $p = \frac{S-T}{N-1}$, the hub will take the cooperator's strategy. Supposing it happens, eventually the system will evolves to an AllC state. The expression $p = \frac{S-T}{N-1}$ shows the probability for a D-hub shifting to a C-hub is always existing and decreases with N. When N is small, after a not very long evolution this phenomenon can been observed. However, when N is large, the jump becomes rare. The computer simulation result in fig. 3 show after $N > 50$ the jump rarely happens. And it cost time to observe this phenomenon.

4 Conclusion and Hypothesis

In this paper, we deduce a spatial replicator equation describing the evolution of cooperation in a structured population. Since there has been many investigations on it, we think it is helpful to further research. As an example, according to the spatial replicator equation we investigate the evolution on a single hub network. The theoretical and numerical results agree with each other well.

The results on the single hub network shows this structure prefer the hub's strategy. The C-hub intends to popularize its strategy while the D-hub intends to popularize its. However, when we further to analyze the stability of the fixed points. They are different. The fixed point $x^* = 1$ for an initial C-hub system is always stable. Thus cooperation can resist the invasion of defection. However, for an initial D-hub system, the stability analysis shows the stability of the stable fixed point $x^* = 0$ for $S < 0$ becomes weaker and weaker when the population N increases. When $S > 0$, the stable fixed point is $x^* = \frac{S}{T(N-1)}$ not $x^* = 0$. And the stability is similar to that of $x^* = 0$. Thus, we find the strategy of defection can not resist the invasion of cooperation. When N is small, in some parameters an initial D-hub system can even change to be a C-hub system leading an AllC state.

Though the structure of population investigated here is simple, we think the results is still necessary for understanding the evolution of cooperation on complex networks. Since a complex network can be regarded as a network organized by different size of single hub structures. Thus single hub structure actually is the basic module of a complex network. Before understanding the behavior of cooperation in complex networks, it is essential to know the evolution on its module.

References

1. Darwin: the Origin of Specis: a Variorum Text. University of Pennsylvania Press, Pennsylvania (2006)
2. Hofbauer, J., Sigmund, K.: Evolutionary Games and Population Dynamics. Cambridge University Press, Cambridge (1998)
3. Nowak, M.A., May, R.M.: Evolutionary Games and Spatial Chaos. Nature 359, 826–829 (1992)

4. Albert, R., Barabasi, A.L.: Statistical Mechanics of Complex Networks. Rev. Mod. Phys. 74, 47–96 (2002)
5. Strogatz, S.H.: Exploring Complex Networks. Nature 410, 268–276 (2001)
6. Abramson, G., Kuperman, M.: Social Games in a Social Network. Phys. Rev. E 63, 030901–030904 (2001)
7. Ohtsuki, H., Hauert, C., Lieberman, E., Nowak, M.A.: A Simple Rule for the Evolution of Cooperation on Graphs and Social Networks. Nature 441, 502–505 (2006)
8. Gómez-Gardeñes, J., Campillo, M., Floría, L.M., Moreno, Y.: Dynamical Organization of Cooperation in Complex Topologies. Phys. Rev. Lett. 98, 108103–108106 (1996)
9. Santos, F.C., Pacheco, J.M.: Scale-Free Networks Provide a Unifying Framework for the Emergence of Cooperation. Phys. Rev. Lett. 95, 098104-1–098104-4 (2005)
10. Santos, F.C., Pacheco, J.M.: A New Route to the Evolution of Cooperation. J. Evol. Biol. 19, 726–733 (2006)
11. Zimmermann, M.G., Eguluz, V.M.: Cooperation, Social Networks, and the Emergence of Leadership in a Prisoner's Dilemma with Adaptive Local Interactions. Phys. Rev. E 72, 056118–056122 (2005)
12. Rong, Z., Li, X., Wang, X.F.: Roles of Mixing Patterns in Cooperation on a Scale-free Networked game. Phys. Rev. E 76, 027101–027104 (2007)
13. Pusch, A., Weber, S., Porto, M.: Impact of Topology on the Dynamical Organization of Cooperation in the Prisoner's Dilemma Game. Phys. Rev. E 77, 036120–036125 (2008)