Extinction and Coexistence in the Internet Market as Complex Networks

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Abstract. A model based on nonautonomous Lotka-Volterra system for web site growth is considered in this paper. Under the conditions that the parameters change with time and the competition conditions are dynamically evaluated, we show that the model exhibits some important characteristics, such as winning alliance and winner-take-all. It is shown that our results are improvement of those of Maurer and Huberman [Journal of Economic Dynamics & Control 27, 2195-2206(2003)], López and Sanjuán [Physica A 301, 512-534(2001)] and López et al. [Physica A 324, 754-758(2003)].

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1 Introduction

In [1], Maurer and Huberman used a dynamical model to analyze the competition dynamics of web sites in the Internet. Their model is the Lotka-Volterra competition equations

$$\frac{df_i}{dt} = f_i \left[\alpha_i \beta_i - \alpha_i f_i - \sum_{j=1, j \neq i}^{n} \gamma_{ij} f_j \right], \quad i = 1, \cdots, n, \tag{1}$$

where f_i is the fraction of the population that is a customer of web site i, α_i is the growth rate which measures the capacity of site i to growth, β_i is the maximum capacity which is related to the saturation value of f_i and γ_{ij} is the competition rate between sites i and j. Under general conditions, they showed

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that as the competition between web sites increases, these markets will exhibit winner-take-all characteristics [2] in which a few sites grab almost all the users, while most other sites go nearly extinct. López and Sanjuán [3] studied the three-variable Lotka-Volterra equations for three competition web sites, and gave different strategies for different sites: strong sites may look for high competition conditions, weak sites may look for noncompetitive markets or may ally with other sites if the markets are highly competitive. In [4], by theory of complex networks, López et al. got similar results as those of [1] and [3].

The parameters in the Lotka-Volterra competition equations (2), i.e. the growth rate α_i of web site i, the maximum capacity β_i of web site i and the competition rate γ_{ij} between sites i and j, are constant values that do not change with time. In fact, these parameters depend on the market evolution, therefore parameters changing with time would be interesting. The results of López et al. [4] are based on the assumption that the competition conditions are fixed, but the competition conditions dynamically evaluated must be taken into account in reality.

In order to overcome these drawbacks in [1], [3] and [4], in this paper, we consider the Lotka-Volterra competition equations with parameters dynamically evaluated

$$\frac{df_i(t)}{dt} = f_i(t)\left[\alpha_i(t)\beta_i(t) - \alpha_i(t)f_i(t) - \sum_{j=1,j\neq i}^{n} \gamma_{ij}(t)f_j(t)\right], \; i = 1, \cdots, n, \quad (2)$$

where $f_i(t)$ is the fraction of the population that is a customer of web site i at time t, $\alpha_i(t)$ is the growth rate at time t which measures the capacity of site i to growth, $\beta_i(t)$ is the maximum capacity which is related to the saturation value of $f_i(t)$ at time t and $\gamma_{ij}(t)$ is the competition rate between sites i and j at time t, $\alpha_i(t)$, $\beta_i(t)$ and $\gamma_{ij}(t)$ are continuous and nonnegative for $t \in [0, +\infty)$. By (2), under the conditions that the parameters change with time and the competition conditions are dynamically evaluated, our main objectives are to explore the extinction and coexistence of the web sites, and to improve the results of [1], [3] and [4].

The organization of this paper is as follows. In Section 2, we give some notations and definitions. In Section 3, We describe the Internet market as complex networks. We consider the coexistence and extinction of web sites in Section 4 and Section 5 respectively. In Section 6, we present our conclusions.

2 Some Notations and Definitions

Given a function $g(t)$ defined on $[0, +\infty)$, we set

$$g_M = \sup\{g(t) \mid 0 \le t < +\infty\}, \quad g_L = \inf\{g(t) \mid 0 \le t < +\infty\}.$$

According to Ahmad and Lazer [5], we define the lower and upper averages of a function g which is continuous and bounded above and below on $[0, +\infty)$. If $0 \le t_1 < t_2$, we set

$$A[g, t_1, t_2] = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} g(s) ds.$$

Definition 1. The *lower and upper averages* of g, denoted by $m[g]$ and $M[g]$, respectively, are defined by

$$m[g] = \lim_{s \to +\infty} \inf\{A[g, t_1, t_2] \mid t_2 - t_1 \geq s\}$$

and

$$M[g] = \lim_{s \to +\infty} \sup\{A[g, t_1, t_2] \mid t_2 - t_1 \geq s\}.$$

Since the set $\{A[g, t_1, t_2] | t_2 - t_1 \geq s\}$ gets smaller as s increases, the limits exist.

For the parameters with constant values, López and Sanjuán [3] defined different kinds of markets. Now, we give similar definitions for the parameters changing with time.

Definition 2. Site i is said to be in *strong competition* with site j at time t when $\gamma_{ij}(t) > \alpha_i(t)$.

Definition 3. Site j is said to be in *collaboration* with site i at time t when $\gamma_{ij}(t) < \alpha_i(t)$.

Remark: The model does not take into account any real collaboration phenomenon between sites, but under low competition rates.

Definition 4. A set of site is called an *alliance* at time t when they are under collaboration conditions between each other at time t, but under strong competition conditions with the rest of the rivals at time t.

Definition 5. A given market exhibits a *winning* site when there exists a site $i \in \{1, \cdots, n\}$ that satisfies $\limsup_{t \to +\infty} f_i(t) > 0$ and $\liminf_{t \to +\infty} f_j(t) = 0$ for all$j \neq i$. In the same way, a given market presents a *winning alliance* when there exists an alliance that satisfies $\limsup_{t \to +\infty} f_i(t) > 0$ for all the sites belonging to the alliance and $\liminf_{t \to +\infty} f_i(t) = 0$ for the rest.

Definition 6. A site is said to be *win* in a market when it is a winning site or when it belongs to a winning alliance.

In [3], only three particular kinds of markets are discussed:

(i) *Completely collaborative markets.* In this kind of market, all competitors collaborate, they are under weak competition condition.

(ii) *Completely competitive markets.* In this kind of market, all the competitors are under strong competition conditions.

(iii) *Mixed markets.* This kind of market presents the particularity that some competitors collaborate, but maintains, at the same time, strong competition conditions with the rest of the opponents.

Because of the constant parameters in the equations, the above three particular kinds of markets are absolutely fixed in the time evolution, i.e. Once one type of the three kinds markets is studied, the market is always in this type and does not change into another type in the discussion process. For the parameters changing with time, the competition conditions may be dynamically evaluated, thus the above three kinds of markets may transform each other in the time evolution, i.e. it presents completely collaborative markets (completely competitive markets or mixed markets) in some time, with the time going, it will transform into another kind of market. Therefore, in this paper, we do not discuss the above three particular kinds of markets separately, but consider the markets in general circumstances.

3 Competitive Networks in the Internet Market

In the framework of the theory of complex networks, the set of web sites can be regarded as a set of entities or nodes, and the competition relations can be regarded as the edges interconnecting the entities or nodes of the complex networks.

In equations (2), $\alpha_i(t)\beta_i(t)f_i(t)$ is the population growth of web site i at time t, $\alpha_i(t)f_i(t)f_i(t)$ represents the competition of site i with itself at time t, $-\sum_{j=1,j\neq i}^{n}\gamma_{ij}(t)f_i(t)f_j(t)$ represents the competitive interaction of site i with the rest of the rivals at time t. The set of $\gamma_{ij}(t)$ completely defines the interconnecting topology of a relational network. We write $\sum_{j=1,j\neq i}^{n}\gamma_{ij}(t)f_i(t)f_j(t)$ using the following matrix-based expression

$$\begin{bmatrix} f_1(t) & 0 & \cdots & 0 \\ 0 & f_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_n(t) \end{bmatrix} \begin{bmatrix} 0 & \gamma_{12}(t) & \cdots & \gamma_{1n}(t) \\ \gamma_{21}(t) & 0 & \cdots & \gamma_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n1}(t) & \gamma_{n2}(t) & \cdots & 0 \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix},$$

where the matrix in the middle can be regarded as the adjacency matrix of a directed weighted graph, and the weight of the connecting arrow between two nodes i and j is $\gamma_{ij}(t)$ at time t.

4 Coexistence of All Web Sites

In this section, we give the sufficient conditions for the coexistence of all web sites in the Internet market. Assume that

$$\gamma_{ij}(t) \geq 0, \quad \gamma_{ijM} < +\infty, \quad i,j = 1,\cdots,n, \quad i \neq j, \tag{3}$$

$$(\alpha_i\beta_i)_M < +\infty, \quad (\alpha_i\beta_i)_L > 0, \quad \alpha_{iL} > 0, \quad \alpha_{iM} < +\infty, \quad i = 1,\cdots,n. \tag{4}$$

Our paper [6] get the following result:

Theorem 1. (Zhao, Jiang, Lazer [6], Theorem 1.1) *Under assumptions* (3) *and* (4)*, suppose that there hold the averaged conditions*

$$M\left[\alpha_i\beta_i - \sum_{j=1,j\neq i}^{n} \gamma_{ij}U_{j0}\right] > 0, \quad i = 1,\cdots,n, \tag{5}$$

where $U_{j0}(t)$ *is any positive solution of the logistic equation*

$$\frac{dU(t)}{dt} = U(t)[\alpha_i(t)\beta_i(t) - \alpha_i(t)U(t)], \tag{6}$$

then for solutions of (2) *the following are true:*

(1) *If*

$$f(t) = \text{col}(f_1(t),\cdots,f_n(t))$$

is a solution of (2) *with* $f_i(t_0) > 0$*,* $1 \le i \le n$*, for some* $t_0 \ge 0$*, then there exist positive constants* l_1*,* l_2 *and* T *such that*

$$l_1 \le f_i(t) \le l_2$$

for $1 \le i \le n$ *and* $t \ge T$*, i.e., the system* (2) *is permanent;*
(2) *If* $f(t)$ *and*

$$h(t) = \text{col}(h_1(t),\cdots,h_n(t))$$

are two solutions of (2) *with* $f_i(t_0) > 0$*,* $h_i(t_0) > 0$*,* $1 \le i \le n$*,* $t_0 \ge 0$*, and if there exist positive constants* c_i*,* $i = 1,\cdots,n$*, and* $m > 0$ *such that*

$$c_j\alpha_j(t) > \sum_{i=1,i\neq j}^{n} c_i\gamma_{ij}(t) + m, \quad j = 1,\cdots,n, \tag{7}$$

then for $i = 1,\cdots,n$*,*

$$\lim_{t\to+\infty}(f_i(t) - h_i(t)) = 0,$$

i.e., the system (2) *is globally attractive.*

From Theorem 1, we know that all web sites will coexist forever under conditions (3)-(5) and (7), i.e. all web sites will present a winning alliance. Here, we do not consider one of the three kinds of markets (completely collaborative markets, completely competitive markets and mixed markets) separately, but the general condition. These three kinds of markets may be transformed each other in our discussion. For example, the market considered is completely collaborative markets at time $t_1 \ge 0$, it is completely competitive markets at time $t_2 > t_1$, and mixed markets at time $t_3 > t_2$.

This result shows that small web site will have the opportunity to be in a winning alliance in the competition with other powerful sites.

Remark: If we discuss completely collaborative markets, completely competitive markets and mixed markets separately, by the definitions of strong competition (where $\gamma_{ij}(t) > \alpha_i(t)$), collaboration (where $\gamma_{ij}(t) < \alpha_i(t)$) and alliance, it is easy to verify that we can choose suitable constants c_i $(i = 1, \cdots, n)$ such that condition (7) is satisfied. Therefore, condition (7) can be dropped off in studying one of the three particular kinds of markets.

5 Extinction of Web Sites in the Internet Market

In this section, we consider the extinction of web sites in the Internet market. Let $\gamma_{ii}(t) = \alpha_i(t)$. Assume that

Assume that

$$\gamma_{ijL} > 0,\ \gamma_{ijM} < +\infty, \quad i, j = 1, \cdots, n, \tag{8}$$

$$(\alpha_i\beta_i)_L > 0,\ (\alpha_i\beta_i)_M < +\infty, \quad i = 1, \cdots, n. \tag{9}$$

In addition, we need the following assumptions:

(I) For each $k > r$ there exists $i_k < k$ such that for any $j \leq k$ the inequality

$$\frac{M[\alpha_k\beta_k]}{m[\alpha_{i_k}\beta_{i_k}]} < \inf\left\{\frac{\gamma_{kj}(t)}{\gamma_{i_kj}(t)} |\ t \geq t_0\right\} \tag{10}$$

holds.

(II) For each $i \leq r$ the inequality

$$M\left[\alpha_i\beta_i - \sum_{j=1, j\neq i}^{r} \gamma_{ij}U_{j0}\right] > 0 \tag{11}$$

holds, where $U_{j0}(t)$ is any positive solution of the logistic equation (6).

(III) There exist positive constants b_i, $i = 1, \cdots, r$, and $d > 0$ such that for $t \geq 0$

$$b_i\alpha_i(t) > \sum_{j=1, j\neq i}^{r} b_j\gamma_{ji}(t) + d, \quad i = 1, \cdots, r. \tag{12}$$

For each $r \leq n$, let H^r denote the r-dimensional coordinate subspace on which $f_{r+1}, \cdots, f_n$ vanish. We use the variable v to denote the restriction of system (2) to H^r,

$$\frac{dv_i(t)}{dt} = v_i(t)\left[\alpha_i(t)\beta_i(t) - \alpha_i(t)v_i(t) - \sum_{j=1, j\neq i}^{r} \gamma_{ij}(t)v_j(t)\right],\ i = 1, \cdots, r. \tag{13}$$

From our paper [7], we have the following results:

Theorem 2. (Zhao, Jiang [7], Theorem 1.1) *Given system* (2) *satisfying* (8) *and* (9)*, suppose that* (10) *holds for* $r = 1$. *If* $\mathrm{col}(f_1(t), \cdots, f_n(t))$ *is any solution of* *the system* (2) *with* $f_i(t_0) > 0$, $i = 1, \cdots, n$, *for some* $t_0 \in [0, +\infty)$, *then for* $i = 2, \cdots, n$, $f_i(t) \to 0$ *exponentially as* $t \to +\infty$, *and* $f_1(t) - h_1^*(t) \to 0$ *as* $t \to +\infty$, *where* $h_1^*(t)$ *is the unique solution of the logistic equation*

$$\frac{dU(t)}{dt} = U(t)[\alpha_1(t)\beta_1(t) - \alpha_1(t)U(t)],$$

which is bounded above and below by strictly positive constants for all $t \in [0, +\infty)$ *and globally attractive.*

Theorem 3. (Zhao, Jiang [7], Theorem 1.2) *Given system* (2) *satisfying* (8) *and* (9), *suppose that* (I)-(III) *hold. If* $f(t) = \mathrm{col}(f_1(t), \cdots, f_n(t))$ *is any solution of system* (2) *with* $f_i(t_0) > 0$, $i = 1, \cdots, n$, *for some* $t_0 \in [0, +\infty)$, *then system* (13) *is permanent and globally attractive, and*

$$\lim_{t \to +\infty} (f_j(t) - v_j(t)) = 0, \ j = 1, \cdots, r,$$

$$\lim_{t \to +\infty} f_j(t) = 0, \ j = r + 1, \cdots, n,$$

where $v(t) = \mathrm{col}(v_1(t), \cdots, v_r(t))$ *is any solution of* (13) *with* $v_i(t_0) > 0$, $i = 1, \cdots, r$.

By Theorem 2, under the conditions (8), (9) and (10) for $r = 1$, we know that the Internet market will be transition from fair market share to a winner-take-all site, i.e. site 1 will become a winning site of the market making the other sites disappear.

By Theorem 3, under the conditions (8), (9) and (I)-(III), we know that the Internet market will be transition from fair market share to a winning alliance, i.e. r sites will become a winning alliance making the remaining $n - r$ sites disappear.

Remark: Similar to the remark of Section 4, if we discuss completely collaborative markets, completely competitive markets and mixed markets separately, condition (12) can be dropped off.

6 Conclusions

We have considered a model based on the nonautonomous Lotka-Volterra system for competition dynamics in the Internet market. Under the conditions that the parameters change with time and the competition conditions are dynamically evaluated, we have shown that the dynamical model exhibits some important characteristics, such as winning alliance and winner-take-all. The drawbacks of the constant parameters and fixed competition conditions in [1], [3] and [4] are overcome by our results to some extent.

The conditions in our results can not be easy to verify for the general case, but can be easy to verify for periodic case. The evolution of the Internet market is also affected by periodical forces, so the parameters in the model can be regarded as periodic function with common period. In the periodical case, the conditions of our results are very simple and more easy to compute. The web sites can take corresponding strategies according to our conditions.

In the non-periodical case, our results show that the small web site has the opportunity of surviving and the most powerful site may be driven to extinct. This requires that the web sites should take some measures to adjust their strategies in order to share the Internet market. Because we have difficulty in computing our conditions, this needs further study.

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