# A Stochastic Model for Layered Self-organizing Complex Systems

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**Abstract.** In this paper we study a problem common to complex systems that dynamically self-organize to an optimal configuration. Assuming the network nodes are of two types, and that one type is subjected to a an upward pressure according to a preferential stochastic model , we wish to determine the distribution of the active nodes over the levels of the network. We generalize the problem to the case of layered graphs as follows. Let $G$ be a connected graph with $M$ vertices which are divided into $d$ levels where the vertices of each edge of $G$ belong to consecutive levels. Initially each vertex has a value of 0 or 1 assigned at random. At each step of the stochastic process an edge is chosen at random. Then, the labels of the vertices of this edge are exchanged with probability 1 if the vertex on the higher level has the label 0 and the lower vertex has the label 1. The labels are switched with probability $\lambda$, if the lower vertex has value of 0 and the higher vertex has the value of 1. This stochastic process has the Markov chain property and is related to random walks on graphs. We derive formulas for the steady state distribution of the number of vertices with label 1 on the levels of the graph.

## 1 Introduction

Complex systems consisting of large aggregates of intelligent autonomous components frequently lend themselves to being modeled as acyclic graphs in general, and trees in a subset of cases. Examples of such systems can be found in engineering systems, human interaction networks, social behavior in the animal kingdom. A subset of these systems display interesting patterns emerging as the result of the reciprocal interaction of the component parts, each acting under the control of a predefined local behavior. For these systems it would be desirable to be able to derive general results about the dependence of system-wide patterns emerging as result of the local behavior. This is clearly a daunting task, and only limited results are available so far on this type of global-local relationships, often concerning global "properties" rather than true spatial "patterns". For example, in the field of network analysis a number of results are available on the dependence between generative models and the degree distribution of the resulting network (see for example [4]).

Given the difficulty of the the above stated objective in its general form, we have decide to study a relatively simple but general problem, which we believe is of interest in a number of applications. We are mainly concerned with studying self-organizing systems

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in which the component parts organize themselves using some form of point-to-point interactions, and the set of all system parts and their interactions can be represented by a graph. In its simplest form, the problem is as follows: in a layered representing the system each nodes is labeled as either one (”1”) or zero (”0”), these values representing two values of some relevant attribute of the node (i.e. high/low performance, dark/light color). Nodes are subjected to a random process, by which a parent-child pair occasionally exchange places, with different probability based on the value of the respective labels. Assuming that there is some form of ”pressure” pushing the 1’s toward the root in the form of higher probability of exchange in the case of (0,1) label values for the (parent, child) pair w.r.t. the (1,0) case, what is the resulting pattern of 1’ s distribution over the layers of the graph? We have solved the problem in the simple case of binary trees [8,9] , and we are now generalizing it for the case of different types of layered graphs, and for a number of label values larger than two.

In the remainder of the paper we first describe some complex systems whose analysis and/or design can benefit from our results. We then briefly outline some of the results we have obtained so far as part of our investigation.

## 2 The Notion of Pressure in Dynamic Complex Systems

*Overlay based systems over engineering networks.* Example of overlay-based systems over engineering networks are agent-based distributed computations [5,6,7], overlay-based data dissemination [13] and multicast systems over the Internet [12]. So called organic grid computations [5,6,7] are self-organizing networks which perform a large scale computation by subdividing it into smaller, autonomous computational units and distributing these computational units on the available nodes of the network. The units organize themselves around a layered overlay network in order to carry out the computation, acquire input data, and collect the results. The system also monitors the available computational resources on each node and continually reorganizes by applying a pressure (as described above) on the more powerful nodes so to push them closer to the root levels. The ability to predict the distribution of the high performing nodes over the layers would be very useful in fine tuning the self organizing behavior of the computational units, for example in order to achieve an optimal configuration based on the effective throughput available at every level. A similar optimization problem can be formulated for data dissemination and multicast systems relying on a layered overlay network.

*Human interactions.* The modeling of social networks according to a hierarchical structure of populations or groups of individuals has been proposed to model the spreading of epidemics and the routing of messages among people [18,19]. These forms of organization are often represented as balanced binary trees for simplicity, but they can obviously be generalized at the expense of a more complex forms of analysis. The most obvious form of upward pressure that can be conceived for these models is the social promotion of well-connected individuals toward the upper levels of the hierarchy; in this case our results could be used to describe the distribution of individuals at each hierarchy level.

For the sake of brevity in the following we focus our attention on overlay based systems over engineering networks, and specifically on organic grids, that were the complex systems that first motivated this work.

## 3 Preliminary Results

In previous work [8,9] we studied the behavior of organic grid networks and developed a model for the evolution of the computations in the nodes of the network. The networks that we studied in [8,9] were balanced binary trees. Here we extend our results to overlay networks with arbitrary shape and arbitrary number of nodes on each level. While the behavior of the organic grids is unpredictable and depends on many external factors in our experiments we observed a regular pattern in the long term behavior of the computational units. These experiments led to the above stochastic process as a model for the behavior of the computational units.

The labels 0 and 1 in the vertices of the graph $G$ correspond to the computational units in the network. The nodes which perform computations have label 1 while the nodes without a computation have label 0. The number of computational units varies and depends on the initial computation. We assign labels 0 or 1 to each node of the network at random. When $n$ nodes have value one and the remaining $M - n$ nodes have value zero the number of possible states with $n$ nodes with label 1 is $\binom{M}{n}$. We call these states configurations with $n$ ones. The probability to have $n$ nodes with number 1 is $\frac{\binom{M}{n}}{2^M}$. Let $r_t$ be the number of nodes of the network at level $t$. The numbers $r_t$ satisfy $r_1 + r_2 + \ldots + r_d = M$, because the total number ofvertices is $M$. We denote the configurations with $n$ ones as $\{v_1, v_2, \ldots, v_n\}$, where $v_i$ are the vertices of the graph $G$ with value 1. The configurations with $n$ ones form a graph where two configurations are adjacent if they are consecutive states of the system. We denote the graph of configurations with $n$ ones as $GC(n)$. Two configurations $C_1 = \{v_1, v_2, \ldots, v_n\}$ and $C_2 = \{u_1, u_2, \ldots, u_n\}$ are adjacent in $GC(n)$ if $u_i = v_i$ for all $i \neq j$ and $u_j \neq v_j$, where $u_j$ and $v_j$ are adjacent vertices of $G$. The vertices of the graph $G$ are divided into levels. Let $l(v)$ be the level of vertex $v$: $l(v) = i$ if $v$ is on level $i$. The highest level of $G$ is 1 and the lowest level is $d$. The vertices of $GC(n)$ are also divided into levels which are induced by the levels of the graph $G$. Let $L(C)$ be the level of configuration $C$ in $GC(n)$. We determine $L(C)$ in the following way: if $l(u_j) - l(v_j) = 1$ then $L(C_1) - L(C_2) = 1$. The graphs of configurations with $n$ ones $GC(n)$ are defined for all values of $n = 0, 1, \ldots, M$. When $n = 1$ the graph $GC(1)$ is the same as the graph $G$ because the configurations with only one node with value 1 are exactly the vertices of $G$. The graph $GC(M)$ has only one vertex which corresponds to all $M$ vertices of the graph $G$ with value 1. The graph $GC(0)$ also consists of a single vertex, where all nodes of $G$ have value 0. The graphs $GC(n)$ are connected for all values of $n$, because the graph $G$ is connected. Now we assign weights to the nodes of the graph $G$ and the configurations.

1. The weight $w(v)$ of the vertex $v$ which is on level $t$ of the graph $G$ is $w(v) = d - t + 1$.

2. The weight of the configurations in $GC(n)$ is the sum of the weights of its vertices with value 1.

   If $C$ is configuration $\{v_1, v_2, \ldots, v_n\}$ then $w(C) = \sum_{i=1}^{n} w(v_i)$.

The vertices of the graph $G$ which are on the same level have equal weight. The levels of the graphs of configurations $GC(n)$ are also determined by the weights of the configurations. Two configurations are on the same level of the graph $GC(n)$ when they have equal weight. We call the configurations on the bottom level of $GC(n)$ - *minimal configurations* and the configurations on the top level - *maximal configurations*. The maximal configuration in $GC(2)$ of $G_1$ is $\{1, 2\}$ and the minimal configurations are $\{3, 4\}, \{4, 5\}$ and $\{3, 5\}$ (Fig.1 and Fig. 3). The minimal configuration with three ones is $\{3, 4, 5\}$ and the maximal configurations are $\{1, 2, 5\}, \{1, 2, 3\}$ and $\{1, 2, 4\}$ (Fig. 4).

We also denote by $w_{min}(n)$ and $w_{max}(n)$ the weights of the minimal and maximal configurations with $n$ ones. The weights of the minimal and maximal configurations of $G_1$ with two ones are $w_{min}(2) = 2$ and $w_{max}(2) = 4$ and $w_{min}(3) = 3,\ w_{max}(3) = 5$. In general, if $n \leq r_d$, then $w_{min}(n) = n$, because the vertices of $G$ on level $d$ have weight 1. Let $M(t) = \sum_{i=t}^{d} r_i$ be the number of nodes on levels $t, t+1, \cdots, d$.

When $n > r_d$ we calculate the weight of the minimal configurations of $GC(n)$ in the following way:

**Let** $s$ be the number such that

$$M(s+1) < n \leq M(s)$$

**Then**

$$w(n) = \sum_{i=d}^{s+1} (d - i + 1) r(i) + (n - M(s+1))(d - s + 1)$$

Let $W = \sum_{i=1}^{M} w(v_i)$ be the weight of all nodes of $G$.

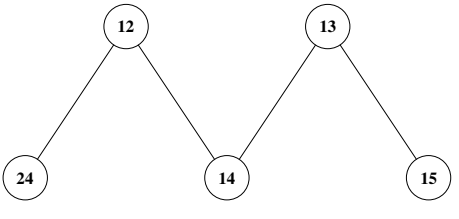


**Fig. 1.** The graph $G_1$

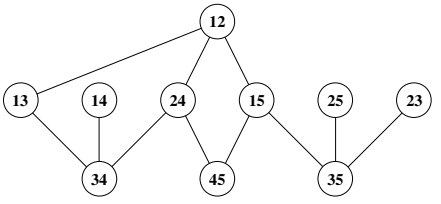


**Fig. 2.** The graph of configurations $GC(2)$ of $G_1$

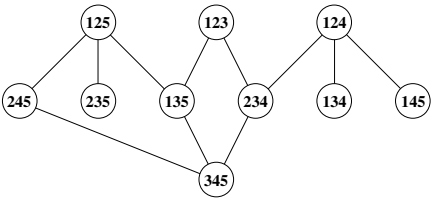


**Fig. 3.** The graph of configurations $GC(3)$ of $G_1$

The graphs $GC(n)$ and $GC(M-n)$ are dual to each other (Fig. 2 and Fig. 3). The are the same graph with their levels inverted. This property of the graph of configurations is explained in the following lemma.

**Lemma 1.** *Let $s(n,w)$ be the number of configurations of $GC(n)$ which have weight $w$. Then*

*(i)* $s(n,w) = s(M-n, W-w)$
*(ii)* $w_{max}(n) + w_{min}(M-n) = W$

The process that we study is a stochastic process on the graph of configurations $GC(n)$ which has the Markov chain property. Let $p(C)$ be the probability of configuration $C$ in the steady state distribution of the configurations of the network.

**Lemma 2.** *Let $C_1$ and $C_2$ be adjacent configurations in the graph of configurations and $L(C_1) = L(C_2) - 1$. Then*

$$p(C_2) = \lambda\, p(C_1)$$

**Corollary 1.** *Let $C_1$ and $C_2$ be any two configurations of $GC(n)$. Then*

$$p(C_2) = \lambda^{L(C_1)-L(C_2)}\, p(C_1)$$

The proofs of Lemma 1, Lemma 2 and Corollary 1 for networks for which the graph $G$ is a balenced binary tree are available in [9]. From Lemma 2 and Corollary 1 the probability of a configuration in $GC(n)$ is determined from its level in $GC(n)$. If two networks have the same number of nodes on each level their graphs of configurations also have the same number of configurations on their levels and the probability of the configurations in the steady state distribution is the same although the networks may have different topology. Now we determine the number of configurations on the levels of the graphs of configurations $GC(n)$. Let $k_t$ be the number of ones on level $t$ of the network. Then $0 \le k_t \le r_t$ for $t = 1, 2, \ldots, d$.

**Lemma 3.** *The number of configurations with $n$ ones and weight $w$ is*

$$s(n,w) = \sum_{k_j} \prod_{i=1}^{d} \binom{r_i}{k_i} \tag{1}$$

*where the numbers $k_t$ satisfy*

$$\sum_{i=1}^{d} k_t = n \quad \textit{and} \quad \sum_{i=1}^{d} (d-t+1)k_i = w \tag{2}$$

*Proof.* The summation in (1) is over all sequences $\{k_j\}_{j=1}^{d}$ which satisfy (2). Let $C$ be a configuration with $n$ ones and weight $w$. The equation $\sum_{i=1}^{d} k_i = n$ means that $C$ has $n$ ones and the equation $\sum_{i=1}^{d} (d-t+1)k_i = w$ guarantees that the weight of $C$ is $w$. The graph $G$ has $r_i$ nodes on level $i$, and there are $\binom{r_i}{k_i}$ arrangements of the $k_i$ ones on row $i$. Therefore the total number of configurations $s(n,w)$ with $n$ ones and weight $w$ is $s(n,w) = \sum_{k_j} \prod_{i=1}^{d} \binom{r_i}{k_i}$.

**Lemma 4.** *The number of configurations with $n$ ones and weight $w$ which have $s$ ones on row $t$ is*

$$c(n,w) = \sum_{k_j} \prod_{i=1}^{d} \binom{r_i}{k_i}$$

*where the numbers $k_j$ satisfy*

$$k_t = s, \quad \sum_{i=1}^{d} k_i = n \quad \textit{and} \quad \sum_{i=1}^{d} (d-i+1)k_i = w$$

*Proof.* Similarly to Lemma 3, the numbers $k_j$ satisfy $\sum_{i=1}^{d} k_i = n$ and $\sum_{i=1}^{d} (d-i+1)k_i = w$. Also $k_t = s$ because $k_t$ is the number of ones on level $t$.

## 4 Probability Distribution

In Section 2 we showed that the configurations of $GC(n)$ are divided into levels induced by the levels of $G$. Two configurations of $GC(n)$ are on the same level if they have the equal weight. Let $p_{n,t,s}(\lambda)$ be the probability to have $s$ ones on level $t$ in the steady state distribution of the network. Then

$$p_{n,t,s}(\lambda) = \frac{N_{n,t,s}(\lambda)}{D_n(\lambda)}$$

where $N_{n,t,s}(\lambda)$ and $D_n(\lambda)$ are polynomials in $\lambda$.

**Lemma 5.** *The polynomials $N_{n,t,s}(\lambda)$ and $D_n(\lambda)$ are determined as follows*

(i) $D_n(\lambda) = \sum_{w} s(n,w)\lambda^{W-w}$

(ii) $N_{n,t,s}(\lambda) = \sum_w c(n,w)\lambda^{W-w}$

*The lowest power of $\lambda$ in $D_n(\lambda)$ and $N_{n,t,s}(\lambda)$ is $w_{min}(M-n)$.*

*Proof.* Let $C$ be a configuration with weight $w$ in $GC(n)$. Its probability in the steady state distribution is inversely proportional to $w$. The polynomial $D_n(\lambda)$ consists of all configurations of $GC(n)$ with their probabilities while $N_{n,t,s}(\lambda)$ contains only the configurations which have $s$ ones on level $t$. Therefore (i) and (ii) hold. The lowest power of $\lambda$ is attained at the maximal configuration. From Lemma 1 we have that $W - w_{max}(n) = w_{min}(M-n)$.

Let $Q(x,y)$ and $Q_{t,s}(x,y)$ be the following polynomials in two variables

$$Q(x,y) = (y+x)^{r_d}(y+x^2)^{r_{d-1}}\cdots(y+x^d)^{r_1} = \prod_{i=1}^{d}(y+x^i)^{r_{d-i+1}}$$

$$Q_{t,s}(x,y) = \tbinom{r_t}{s} y^s x^{(r_t-s)(d-t+1)} \frac{Q(x,y)}{(x+y^{d-t+1})^{r_t}}$$

In Lemma 6 and Lemma 7 we express $D_n(x)$ and $N_{n,t,s}(x)$ as coefficients of $Q(x,y)$ and $Q_{t,s}(x,y)$.

**Lemma 6.** *The coefficient of $y^n$ in $Q(x,y)$ is $D_n(x)$.*

*Proof.* By substituting $i \leftrightarrow d-i+1$ we obtain

$$Q(x,y) = \prod_{i=1}^{d}(y+x^{d-i+1})^{r_i}$$

$$Q(x,y) = \prod_{i=1}^{d}\sum_{k_i=0}^{r_i}\tbinom{r_i}{k_i} y^{k_i} x^{(d-i+1)(r_i-k_i)}$$

Let's denote by $C(y^n)$ the coefficient of $y^n$ in $Q(x,y)$.

$$C(y^n) = \sum_{k_j}\prod_{i=1}^{d}\tbinom{r_i}{k_i} x^{(d-i+1)(r_i-k_i)}$$

The sum is over all sequences $\{k_j\}_{i=1}^{d}$ where $k_1+k_2+\ldots+k_d = n$.

$$C(y^n) = x^{\sum_{i=1}^{d}(d-i+1)r_i}\sum_{k_j}\prod_{i=1}^{d}\tbinom{r_i}{k_i} x^{-k_i(d-i+1)}$$

$$C(y^n) = x^W \sum_{k_j}\prod_{i=1}^{d}\tbinom{r_i}{k_i} x^{-k_i(d-i+1)}$$

We have that $w = \sum_{i=1}^{d} k_i(d-i+1)$. Then

$$C(y^n) = x^W \sum_{w} \sum_{k_j} \prod_{i=1}^{d} \tbinom{r_i}{k_i} x^{-w}$$

where $w$ takes values between $w_{min}(M-n)$ and $w_{max}(M-n)$ and $k_j$ satisfy

$$\sum_{t=1}^{d} k_i = n \quad \text{and} \quad \sum_{t=1}^{d} (d-t+1)k_i = w \tag{3}$$

$$C(y^n) = \sum_{w} x^{W-w} \sum_{k_j} \prod_{i=1}^{d} \tbinom{r_i}{k_i}$$

From Lemma 3 we obtain

$$C(y^n) = \sum_{w} s(n,w) x^{W-w}$$

Similarly $N_{n,t,s}(x)$ is a coefficient of the polynomial $Q_{t,s}(x,y)$.

**Lemma 7.** *The coefficient of $y^n$ in $Q_{t,s}(x,y)$ is $N_{n,t,s}(x)$.*

The proof of Lemma 7 is similar to the proof Lemma 6 and uses the formula for $c(n,w)$ from Lemma 4. Initially we assign zeros and ones to the nodes of the network at random. Let $p_{t,s}(\lambda)$ be the probability that the network has $s$ ones on level $t$ in the steady state distribution. Then

$$p_{t,s}(\lambda) = \frac{1}{2^M} \sum_{n=0}^{M} \tbinom{M}{n} p_{n,t,s}(\lambda) \tag{4}$$

We compute the values of $p_{n,t,s}(\lambda) = \dfrac{N_{n,t,s}(\lambda)}{D_n(\lambda)}$ from Lemma 6 and Lemma 7 where $N_{n,t,s}(\lambda)$ and $D_n(\lambda)$ are coefficients of the polynomials $Q(\lambda,y)$ and $Q_{t,s}(\lambda,y)$.

## 5 Conclusions and Future Work

In this paper we generalize our model from [8] to networks with arbitrary number of nodes at each level and arbitrary topology when the links of the network are between consecutive levels. We are working on extending our analysis to model the steady state distribution for a wider array of network topologies with links between nonconsecutive levels .

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