Almost Periodicity and Distributional Chaos in Banach Space

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Abstract. Let $(X, \| \cdot \|)$ be a Banach space, $f : X \to X$ continous Freche't differentiable map.Denote the set of almost periodic point by $A(f)$.In this paper,we prove that there exists an uncountable set Λ such that $f \mid_\Lambda$ is distributionally chaotic,and $\Lambda \subset A(f)$.

Keywords: distributional chaos, Banach space,almost periodic point.

1 Introduction

This paper is mainly concerned with distributional chaos in the following discrete dynamical system

$$x_{n+1} = f(x_n) \quad n \geq 0,$$

where $f : X \to X$ is a map and $(X, \| \cdot \|)$ is Banach space.

Research on chaos and chaotification in dynamical systems attracted interest from many scientists and mathematicians. In 1975, Li and Yorke [1] investigated a discrete dynamical system governed by a continuous map on the real line **R** and obtained the well-known result:"period 3 implies chaos."Since then,people have studied the chaotic system in various sense. The following definitions of chaos are often used in theoretical research at the moment. Chaos in the sense of Devaney [2,3], Smale horseshoe [4], Topological mixed system [5] and distributional chaos. The paper that period 3 implies chaos published by Li and Yorke in 1975 shows that periodicity and the properties of chaos are very important and are very closely related concepts in a non-linear system.

In 1994, Schweizer-smital investigated a discrete dynamical system governed by a continuous map on the real line **R** and obtained the result.If f has positive entropy,then there exists an uncountable distributional chaotic set in which each member is an $\omega-limit$ point. Later,many papers published thus for are concerned with chaos and chaotification for discrete dynamical systems in R^n [3,5,6-11] and infinite-dimensional system [6,10,11] which are very useful and applicable discussing existence of chaos for higher dimensional discrete dynamical systems and studying the chaotic behavior of partial differential equations (PDES) and

J. Zhou (Ed.): Complex 2009, Part II, LNICST 5, pp. 1435–1440, 2009.

ordinary differential equations (ODES) [7,12-19]. In this paper,we study distributional chaos of discrete dynamical systems governed by map in general Banach spaces,where maps are continuously differentiable in some domains.

The main results of this paper are stated as follows.

The Main Theorem. Let $(X, \| \cdot \|)$ be a Banach space and let $f : X \to X$ be map with a fixed point $z \in X$ and satisfies

1) f is continuously Freche't differentiable in a neighborhood of z. $Df(z)$ in an invertible linear map and satisfies

$$\| Df(z) \|^0 > 1$$

2) f has a homoclinic orbit Γ to z, is continuously differentiable in neighborhood of any point x on Γ, $Df(x)$ is an invariable linear map and satisfies

$$\| Df(x) \|^0 > 0$$

Then f has an uncountable distributional chaotic set in which each point is almost periodic point.

The rest of the paper is organized as follows. In section 2, we list some concepts and lemmas which are needed in the sequel. In section 3, we will give proof of the existence of distributional chaotic set.

2 Basic Definitions and Lemmas

In this section, we list some definitions and prepare several lemmas.

For a continuous map $f : X \to X$, we will denote the set of almost periodic points and the set of recurrent points of f by $A(f)$ and $R(f)$ respectively,with the usual definitions, f^n will denote the n-fold iterate of f.

For x, y in X, any real number t and positive integer n, let

$$\xi_n(f, x, y, t) = \#\{i \mid d(f^i(x), f^i(y)) < t, 1 \le i \le n\},$$

where we use $\#(\cdot)$ to denote the cardinality of a set. Let

$$F(f, x, y, t) = \liminf_{n\to\infty} \frac{1}{n}\xi_n(f, x, y, t), \qquad F^*(f, x, y, t) = \limsup_{n\to\infty} \frac{1}{n}\xi_n(f, x, y, t).$$

Definition 1. *Call $x, y \in X$ a pair of points displaying distributional chaos if*
(1) $F(f, x, y, t) = 0$ for some $t > 0$,
(2) $F^*(f, x, y, t) = 1$ for any $t > 0$.

Definition 2. *f is said to display distributional chaos if there exists an uncountable set $D \subset X$ such that any two different points in D display distributional chaos.*

Since the criteria of distributional chaos obtained in this paper are related to a symbolic dynamical system. We first recall its relevant properties as follows.

Let $S = \{0,1\}, \Sigma = \{x = x_1x_2\ldots \mid x_i \in S, i = 1,2,\ldots\}$ and define $\rho : \Sigma \times \Sigma \to \mathbf{R}$ as follows: for any $x, y \in \Sigma$, if $x = x_1x_2\ldots$ and $y = y_1y_2\ldots$, then

$$\rho(x,y) = \begin{cases} 0 & if\ x = y \\ \frac{1}{2^k}\ if\ x \neq y & and\ k = \min\{n|x_n \neq y_n\} - 1 \end{cases}$$

It is not difficult to check that ρ is a metric on Σ.The space (Σ, ρ) is compact and called the *one-sided symbolic space* on two symbols.

Define $\sigma : \Sigma \to \Sigma$ by $\sigma(x_1x_2\ldots) = x_2x_3\ldots$ for any $x = x_1x_2\ldots \in \Sigma$. Then σ is continuous and called the *shift* on Σ.

Definition 3. *Let (X, f) and (Y, g) be dynamical system, if there exists a homeomorphism $h : X \to Y$ such that $\forall x \in X$ satisfying $h \circ f(x) = g \circ h(x)$, then f and g topologically conjugate.*

Definition 4. *Let $f : X \to X$ and $g : Y \to Y$ be continuous, where X and Y are compact metric spaces. If there exists a continuous surjection $h : X \to Y$ such that $g \circ h = h \circ f$, then*
(1) $h(A(f)) = A(g)$,
(2) $h(R(f)) = R(g)$.

Definition 5. *(see[20]) There exists an uncountable set $\mathcal{T}$ on the one-sided symbolic space satisfying*
(1) $\mathcal{T} \subset A(\sigma)$,
(2) $\sigma|_{\mathcal{T}}$ is distributionally chaotic,

Now, we turn to study properties of continuous Freche't differentiable maps in Banach space. For convenience, denote a closed ball and an open ball centered at z in the Banach space $(X, \| \cdot \|)$, respectively by

$$\bar{B_r}(z) = \{x \in X :\| x - z \|\leq r\}, B_r(z) = \{x \in X :\| x - z \|< r\}$$

and introduce the following notion:

$$\| L \|^0 = inf\{\| Lx \|: x \in X\ with\ \ \| x \|= 1\}$$

for a linear map $L : X \to X$. If a linear map $L : X \to X$ has a bounded linear inverse map, then L is called an invertible linear map.

[see ref.[21] Definition 4.17]

Definition 6. *Let (X, d) be a metric space, $f : X \to X$ be a map.*

(1) A point $z \in X$ is called an expanding fixed point (or a repeller) of f in $\bar{B_r}(z)$ for some constant $r > 0$, if $f(z) = z$ and there exists a constant $\lambda > 1$ such that

$$d(f(x), f(y)) \geq \lambda d(x,y),\ \forall x, y \in \bar{B_r}(z),$$

where $\bar{B_r}(z) = \{x \in X : d(x,z) \leq r\}$ is a closed ball centered at z. The constant λ is called the expanding coefficient of f in $\bar{B_r}(z)$. Furthermore, z is called a regular expanding fixed point of f in $\bar{B_r}(z)$ if z is an interior point of $f(B_r(z))$.

(2) Assume that $z \in X$ is a regular expanding fixed point of f . Let U be the maximal open neighborhood of z in the sense that for each $x \in U$ with $x \neq z$ there exists $k \geq 1$ with $f^k(x) \notin U$ and for each $x \in U$ with $x \neq z$, $f^{-n}(x)$ is uniquely defined in U and $f^{-n}(x) \to z$ as $n \to \infty$. U is called the local unstable set of f at z and is denoted by $W^u_{loc}(z)$.

(3) Assume that $z \in X$ is a regular expanding fixed point of f. A point $x \in X$ is called homoclinic to z if $x \in W^u_{loc}(z), x \neq z$, and there exists $n \geq 1$ such that $f^n(x) = z$.

Lemma 1. *Let $(X, \| \cdot \|)$ be a Banach space and let $f : X \to X$ be a map with a fixed point $z \in X$, assume that*

1) f is continuously Freche't differentiable in a neighborhood of z, $Df(z)$ is an invertible linear map and satisfies

$$\| Df(z) \|^0 > 1$$

2) f has a homoclinic orbit Γ to z, z is continuously differentiable in a neighborhood of any point x on Γ, $Df(x)$ is an invertible linear map and satisfies

$$\| Df(x) \|^0 > 0$$

Then, for each neighborhood U of z, there exist a positive integer n and a cantor set $\Lambda \subset U$ such that $f^n : \Lambda \to \Lambda$ is topologically conjugate to the symbolic dynamical system $\sigma : \Sigma \to \Sigma$.

Proof. For a proof see [22].

Lemma 2. *Let X be a compact metric space with metric d, $f : X \to X$ a continuous map and $N > 0$ an integer. Then f is distributionally chaotic if and only if f^N is too.*

Proof. For proof see [23].

3 Proof of Main Theorems

In this section, we establish a criterion of distributional chaos for continuous differentiable map in some domain of a Banach space.

Proof. By the Lemma 2.3, there is a homeomorphism $h : \Lambda \to \Sigma$ and there exists a positive integer n such that for any $x \in \Lambda$

$$h \circ f^n(x) = \sigma \circ h(x)$$

where $\Lambda \subset X$ is a cantor set,for simplicity, let $g = f^n$. According to Lemma 2.2, there is an uncountable set $\mathcal{T} \subset A(\sigma)$ which is distributionally chaotic and $\sigma|_{\mathcal{T}}$ has the only ergodic measure u. For $y \in \mathcal{T}$, by Lemma 2.1, there exists $x \in A(g)$ such that $h(x) = y$. Let

$$D = \{x \mid x \in A(g), h(x) = y \text{ and } y \in \mathcal{T}\}.$$

Then $D \subset \Lambda$ and D is an uncountable set. To complete the proof, it suffices to show that D is a distributional chaotic set for g.

First of all, we prove that for any $x_1, x_2 \in \Lambda$, if $F(\sigma, h(x_1), h(x_2), t) = 0$ for some $t > 0$, then $F(g, x_1, x_2, s) = 0$ for some $s > 0$. Since $g = f^n$,then $g : \Lambda \to \Lambda$, $\sigma : \Sigma \to \Sigma$ be continuous map.For given $t > 0$, by uniform continuity of h, there exists $s > 0$ such that for any $p, q \in \Lambda$, $\rho(h(p), h(q)) < t$ provided $\| p - q \| < s$. since we easily see that $h \circ g^i = \sigma^i \circ h$, it follows that if $\rho(g^i(x_1), g^i(x_2)) < s$, then

$$\| \sigma^i(h(x_1)) - \sigma^i(h(x_2)) \| < t.$$

This implies

$$\xi_n(g, x_1, x_2, s) \leq \xi_n(\sigma, h(x_1), h(x_2), t)$$

for all $n \geq 0$. Thus by the definition of F, we immediately have the following result

$$F(g, x_1, x_2, s) = 0. \tag{1}$$

Secondly,we prove that if $F^*(\sigma, h(x_1), h(x_2), s) = 1$ for all $s > 0$, then $F^*(g, x_1, x_2, t) = 1$ for all $t > 0$. Since h is a homeomorphism, $h^{-1} : \Sigma \to \Lambda$ is a surjective continuous map.By the first proof, we have

$$\xi_n(\sigma, h(x_1), h(x_2), s) \leq \xi_n(g, x_1, x_2, t)$$

which gives

$$F^*(g, x_1, x_2, t) = 1. \tag{2}$$

By $(1), (2)$ and the arbitrariness of x_1 and x_2, we conclude that D is an uncountable distributionally chaotic set of g. Therefore, we also conclude that f^n is distributionally chaotic.

By Lemma (2.4) and $A(f^n) = A(f)$, we prove that f has an uncountable distributionally chaotic set D in $A(f)$.

The proof is completed.

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