A More Strict Definition of Steady State Degree Distribution

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Abstract. Accurate definitions of related concepts are prerequisite for further understanding of evolving network. To be an important concept, steady state degree distribution has been widely used. However, as we find out, all current definitions have a common default from mathematics point of view. In this paper, we first point out the shortcoming of current definitions through a special type of evolving network, and then provide a more strict definition of steady state degree distribution from stochastic process point of view.

Keywords: Steady state degree distribution, evolving network, stochastic process.

1 Introduction

Since Barabási, Albert and Jeong established scale-free network model in 1999 [1], extensive research on the evolution of complex network have been done in both theoretical and empirical areas. As a result, various evolving network models have been proposed such as power-law growth models [2], logarithmic growth model [3], exponential growth model [4], extended evolving model [5], evolving mortal network model [6], and local-world evolving network model [7]. For these models, much attention has been caught on how to calculate and derive the statistical characteristics of complex network like connectivity, clustering coefficient and degree distribution under the limit conditions. In particular, due to its importance in depicting the network topology and features like robustness and fragility, degree distribution especially the steady state degree distribution $P(k)$ has been widely explored by means of mean-field approach [1], rate-equation [8], master-equation [9] as well as Markov chains methods [3][10]. For example, Barabási, Albert and Jeong suggested that the steady state degree distribution of scale-free random network subjects to power-law distribution [1]. Dorogovtsev and Mendes [4] and Derek et al.[11] pointed out that steady state degree distribution of random evolving network subjects to exponent distribution. Meanwhile, Volz [12] provided an evolving network whose steady state degree distribution subjects to Poission distribution.

So far, the most representative definitions of steady state degree distribution $P(k)$ are provided by Barabási et al.[1], Krapivsky et al.[8], Dorogovtsev et al.[9], and Mohanty et al.[13] separately. Despite these definitions has been widely used in related

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literature, there is still a common default from point of mathematics, that is, they all think if $\lim_{t\to+\infty} P(t,k)$ exists, then $P(k)=\lim_{t\to+\infty} P(t,k)$ must be steady state degree distribution. However, in this paper, we give a kind of evolving network to show that even if their $P(k)=\lim_{t\to+\infty} P(t,k)$ exist, they may not be a distribution i.e. $\sum_k P(k)<1$. Furthermore, a more strict definition from stochastic process perspective is given at the end of the paper.

2 Evolving Network and Steady State Degree Distribution

Evolving network refers to the network whose topology structure is dynamic with time. The evolving rule denoted as R in this paper is defined as ways by which network evolves. The initial condition or state show the original network before the evolution starts. According to this definition, many common networks like random network, small-world network and scale-free network are all evolving networks, in which the first two are finite evolving networks and the last one is infinite evolving network.

Here, we define $\{G(t), t\geq 0\}$ as evolving process of a network, where t is the time parameter; $G(t)$ represents the network at t time step; $G(0)$ denotes the initial condition and $g_i(t), t\geq 0$ represents a sample function, which is the real evolving process of network. In the following section, we first explain some important concepts that have been used in the literature of evolving network.

Definition 1. Let $X_{t,1}, X_{t,2}, \cdots, X_{t,m_t}$ represent all possible degree distributions of evolving network $G(t)$ and $p_{t,1}$ denote the probabilities of all possible degree distributions. $KG(t)$ termed as degree distribution variable, shows a possible degree distribution of $G(t)$. Then the distribution law of $KG(t)$ is $P\{KG(t)=X_{t,j}\}=p_{t,j}$, $j=1,2,3,\cdots,m_t$ $\{KG(t), t\geq 0\}$ can be defined as degree distribution process of $\{G(t), t\geq 0\}$.

Definition 2. Let $K(t)$ represent the degree of a vertex in evolving network $G(t)$ and $P(t,k)$ denote the probability that the degree of this vertex is k, that is $P(t,k)=P\{K(t)=k\}$. Here we define $K(t)$ as average degree distribution and $\{K(t), t\geq 0\}$ as average degree distribution process.

For each k, we have

$$P(t,k)=\sum_{i=1}^{m_t} p_{t,i}\cdot P\{X_{t,j}=k\} \qquad (1)$$

so we can denote $K(t)=E[KG(t)]$;

Among all the definitions of degree distribution, the most representatives are transient degree distribution by Barabásī et al. [1], transient frequency by

Krapivsky et al. [8], average degree distribution by Dorogovtsev et al.[9]. In fact, all these definitions are equivalent to our average degree distribution $K(t)$. Based on these definitions, all literature above further define $P(k)=\lim_{t\to+\infty} P(t,k)$ as steady state degree distribution. Although the definition above has been widely accepted in evolving network literature, nevertheless, we argue that their definitions are not accurate from mathematics point of view because even if $P(k)=\lim_{t\to+\infty} P(t,k)$ exists, there are still possibility that $P(k)$ is not a distribution, i.e. $\sum_k P(k)<1$. Below some cases will be given to explain our findings.

3 Two Special Types of Evolving Network

3.1 Evolving Network That $\lim_{t\to+\infty} P(t,k)$ Does Not Exist

$R1$ (Evolving Rule 1): At the time t (t is a positive integer):

(1) add a new node with the probability $p(t)=\frac{1}{2}(1-\cos t\pi)$ and let it connect to other nodes in the network $G(t-1)$;

(2) delete a node in the network $G(t-1)$ with the probability $1-p(t)$.

And we denote the new network as $G(t)$.

Initial condition : $G(0)$ is a complete graph with m nodes.

At the time $2t-1$ $(t=1,2,3,\cdots)$ there is only one degree distribution $X_{2t-1,1}$, $P\{X_{2t-1,1}=m\}=1$;

$$P\{KG(2t-1)=X_{2t-1,1}\}=1 \tag{2}$$

at the time $2t$ $(t=1,2,3,\cdots)$ there is also only one degree distribution $X_{2t,1}$, $P\{X_{2t,1}=m-1\}=1$;

$$P\{KG(2t)=X_{2t,1}\}=1 \tag{3}$$

Hence $K(t)=X_{t,1}$ $(t=1,2,3,\cdots)$, that is

$$P(t,m)=\begin{cases}1 & t=2k-1\\ 0 & t=2k\end{cases} \quad (k=1,2,3,\cdots) \tag{4}$$

$$P(t,m-1)=\begin{cases}1 & t=2k\\ 0 & t=2k-1\end{cases} \quad (k=1,2,3,\cdots) \tag{5}$$

Neither $\lim_{t\to+\infty} P(t,m)$ nor $\lim_{t\to+\infty} P(t,m-1)$ exists.

3.2 Evolving Network That $\lim_{t\to+\infty} P(t,k)$ Exists But $P(k)$ Is Not a Distribution

Considering a evolving network according to the rule $R2$,
At time t (t is a positive integer):

(i) add a new node V_t to the network $G(t-1)$;

(ii) connect V_t to the first $\left[\frac{t-1}{2}\right]$ nodes ranked by degree with strict descend sort;

(iii) randomly connect V_t to a node with degree $\left[\frac{t}{2}\right]$.

where $\left[\frac{t-1}{2}\right]$ is the maximum integer no more than $\frac{t-1}{2}$.

And we denote the new network as $G(t)$.

Initial condition: $G(0)$ is an isolate node V_0.

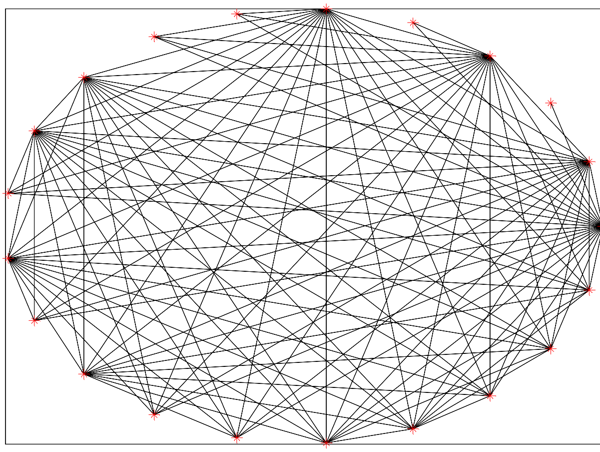

Fig. 1. Represents a realization simulated by $R2$ at $t = 20$

There are total 21 nodes in the network, in which only one node with degree $k(1 \le k \le 20, k \ne 10)$ and two nodes with degree 10.

When $t = 1$,the distribution of $X_{1,1}$ is $P\{X_{1,1} = 1\} = 1$;

$$P\{KG(1) = X_{1,1}\} = 1 \tag{6}$$

When $t = 2$,the distribution of $X_{2,1}$ is $P\{X_{2,1} = 1\} = \frac{2}{3}$, $P\{X_{2,1} = 2\} = \frac{1}{3}$;

$$P\{KG(2) = X_{2,1}\} = 1 \tag{7}$$

In the case $t \geq 3$,

$$P\{X_{t,1} = k\} = \frac{1}{t+1}, \left(1 \leq k \leq t, k \neq \left[\frac{t}{2}\right]\right), \quad P\left\{X_{t,1} = \left[\frac{t}{2}\right]\right\} = \frac{2}{t+1}$$

$$P\{KG(t) = X_{t,1}\} = 1 \tag{8}$$

then

$$P(t,k) = \begin{cases} 1/(t+1) & 1 \leq k \leq t, k \neq \left[\frac{t}{2}\right] \\ 0 & k > t \\ 2/(t+1) & k = \left[\frac{t}{2}\right] \end{cases} \tag{9}$$

For all $k \geq 1$, we have

$$P(k) = \lim_{t \to +\infty} P(t,k) = 0 \tag{10}$$

$$\sum_k P(k) = 0 \tag{11}$$

Obviously here $P(k)$ is not a distribution.

In fact, for any $0 \leq p < 1$, we can construct a kind of evolving network satisfying $\sum_k P(k) = p$. They can evolve according to the rule $R3$,

At time t (t is positive integer):

(i) add a new node V_t to the network $G(t-1)$;

(ii) connect V_t to the node V_0 with the probability p;

(iii) with complementary probability $1-p$, connect V_t to the first $\left[\frac{t-1}{2}\right]$ nodes ranked by degree with strict descend sort and randomly connect V_t to a node with degree $\left[\frac{t}{2}\right]$.

And we denote the new network as $G(t)$.

Initial condition: $G(0)$ is an isolate node V_0.

Thus for the average degree distribution $K(t)$, we have

$$\begin{aligned} P(t+1,1) &= P\{K(t+1) = 1\} \\ &= p\left[\frac{1}{t+1} + P\{K(t) = 1\} \cdot \frac{t}{t+1}\right] + (1-p) \cdot P\{K(t) = 1\} \cdot \frac{t}{t+1} \\ &= p \cdot \frac{1}{t+1} + P\{K(t) = 1\} \cdot \frac{t}{t+1} \\ &= p \cdot \frac{1}{t+1} + P(t,1) \cdot \frac{t}{t+1} \end{aligned}$$

For any $k > 1$, when t is sufficiently large ($t \geq 2k + 2$),

$$P(t+1,k) = P(t,k) \cdot \frac{t}{t+1} \tag{12}$$

Hence

$$P(k) = \lim_{t \to +\infty} P(t,k) = \begin{cases} p & k = 1 \\ 0 & k \neq 1 \end{cases} \tag{13}$$

$$\sum_k P(k) = p \tag{14}$$

As shown in the cases above, we can conclude that there is a default in present definition on steady state degree distribution. So in this paper, we provide a more strict definition from stochastic point of view (see Definition 3).

Definition 3. Let $\{K(t), t \geq 0\}$ represent the average degree distribution process of $\{G(t), t \geq 0\}$, the distribution law of $K(t)$ is $P\{K(t) = k\} = P(t,k)$; if there exists random variable K whose distribution law is $P(k)$ i.e. $\sum_k P(k) = 1$ and

$$P(k) = \lim_{t \to +\infty} P(t,k) \tag{15}$$

then K can be defined as steady state degree distribution of $\{G(t), t \geq 0\}$.

4 Conclusion

Accurate definitions of related concepts are prerequisite for further understanding of evolving network. To be an important concept, steady state degree distribution has been widely used. However, as we find out, all current definitions have a common default from mathematics point of view, which thinks that if $\lim_{t \to +\infty} P(t,k)$ exists, then $P(k) = \lim_{t \to +\infty} P(t,k)$ must be steady state degree distribution. However, our finding argues that in some cases, despite $\lim_{t \to +\infty} P(t,k)$ exists, $P(k)$ may not be a distribution (i.e. $\sum_k P(k) = 1$). Finally, we provide a more strict definition of steady state degree distribution from stochastic process point of view.

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