A Priority Queue Model of Human Dynamics with Bursty Input Tasks

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Abstract. The physics of human activities recently has been studied in the view point that they are dynamic processes of a complex system. The studies reveal that the human activities have bursty nature – occasional abrupt bursts of activity level for short periods of time, along with long periods of inactivity. Quantitative studies show that the distribution of the time, τ, between two consecutive activity events exhibits a power-law behavior with universal exponents $\sim \tau^{-1.5}$ or $\sim \tau^{-1.0}$. Such universal behaviors were explained by the universality in the waiting-time distribution of tasks in model queue systems, which operate based on priority. In the models, the rates of task input are presumed to follow a Poisson-type distribution. An empirical observation of human activities, however, shows that the task arriving rate for some people also has bursty nature – the number of tasks arrive to the people follows a power-law distribution. In this paper, a new model queue system for this case is introduced and studied by analytic and numerical methods. The waiting-time distribution for the new model is found also to follow a power law, but the exponent varies according to the parameters of the model and takes other values than 1.5 or 1.0. The analytic solution is obtained via the generating function formalism, different from the biased random walk approach used in the previous studies.

Keywords: complex systems, human dynamics, modelling, priority queue, power law, generating function.

1 Introduction

The queue is a sequence of any objects (usually called the *tasks*) which come into a system, are processed somehow, and then go out of the system. The queuing theory deals with diverse issues about managing such a sequence. Primary concern is the time spent by a task in the system (the waiting time), or the distribution of the waiting time. Consequently, the dynamics of queuing systems have been extensively studied in engineering aspects; diminishing the waiting time of the tasks, or making the distribution of waiting-time have a well-defined average and small deviation are considered to be desirable goals [1].

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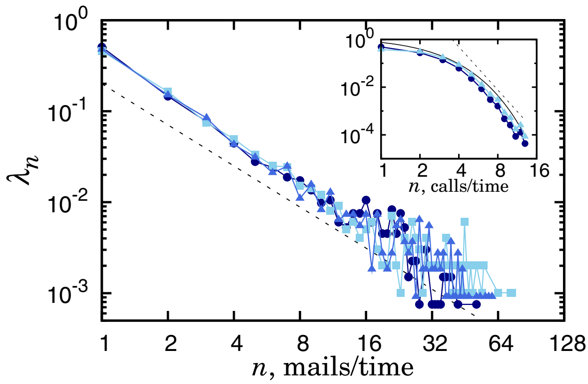

Fig. 1. (Color online) A size distribution of tasks arrive in a unit time. The main panel shows the number of e-mails delivered to an anonymous user in a unit time [2], which follows a power-law distribution with slope −1.5. Different lines correspond to different bin sizes, 500, 800, and 1000 seconds. The slope −1.5 nor the power-law distribution itself is universal but varies with users. The inset shows the number of wireless phone calls received by a cell station during 10 seconds in the peak time (i.e.,12:00-20:00) (△) and for the entire day (○). Both are well fit to a Poisson distribution (black solid line), which decays even faster than the power law with exponent −6 (dotted line).

Recently, on the other hand, several queue models with power-law waiting-time distribution gathered strong interests among the complex systems society [3,4,5,6]. Those series of queue models are first introduced to explain the pattern of correspondence between people: the time elapsed for a person to reply letters has heavy-tailed or power-law distributions. It is because each letter has different level of priority, that some urgent letters are replied immediately while others are postponed until one is unoccupied. Such priority concept is adopted to the queue model, and with it, Poissonian process of task input/output resulted in power-law distribution of waiting time. The emergence of the power law was considered to reflect the critical nature of the queuing dynamics.

While previous studies assumed that the task input/ouput are a Poissonian process, however, our analysis of empirical data shows that there are cases when the task input is bursty – the number of input-task has power-law distribution (see Fig. 1). In this paper, we study the waiting-time distribution of a queue model operating based on priority with bursty input tasks.

2 The Model

At each time step, n tasks arrive to the queue, where n is a number drawn from the power-law distribution $\lambda_n = \lambda n^{-\gamma}/\zeta(\gamma)(n > 0), \lambda_0 = 1-\lambda$, where $0 < \lambda < 1$ and $\zeta(\gamma) \equiv \sum_{i=1}^{\infty} i^{-\gamma}$ is the Riemann's zeta function. The task is randomly assigned with a priority x from the flat distribution in $[0, 1]$. At the same time, 1 highest priority task in the queue is executed with probability $0 < \mu < 1$. The maximum number of tasks in the queue is unlimited so that the queue accommodates all the incoming tasks of any sizes. This model is a generalized version of that in Ref. [5] and that is reproduced for $\lambda_0 = 1 - \lambda$, $\lambda_1 = \lambda$, and $\lambda_n = 0$ when $n \geq 2$. The average number of incoming tasks is

$$\langle n \rangle_\lambda = \sum_{n=1}^{\infty} \frac{\lambda n^{1-\gamma}}{\zeta(\gamma)} = \lambda \frac{\zeta(\gamma - 1)}{\zeta(\gamma)}, \tag{1}$$

which converge when $\gamma > 2$.

3 Generating Functions and Waiting Time Distribution

We are concerned about the waiting-time distribution $P_w(\tau)$ for tasks in the queue. To obtain it, we should study the transient dynamics of the number of tasks in the queue (the queue length). First, let $Q_x(m,t)$ be the probability that there are m tasks with priority higher than x in the queue at time t, and $\tilde{Q}_x(m)$ be its stationary state solution defined as $\tilde{Q}_x(m) \equiv \lim_{t\to\infty} Q_x(m,t)$. Also we define $G_x(m,\tau)$ as the probability that a given task with priority x which arrived to the queue at time $t = \tau_0$, is executed after time τ elapses. In other words, $G_x(m,\tau)$ is the first-passage probability of the queue length with priority larger than x, from initial condition m to destination origin $m = 0$. If a task with priority x arrives to the queue, there are already m of higher-priority tasks. Before that given task is to be executed, the pre-existing m tasks, together with all the task whose priorities are higher than x that arrives in the mean time, should be executed first. The transient dynamics is schematically illustrated in Fig. 2. Considering such dynamic process, the waiting time distribution is written by [5],

$$P_{\mathrm{w}}(\tau) = \sum_{m=0}^{\infty} \int_0^1 dx \tilde{Q}_x(m) G_x(m,\tau). \tag{2}$$

Since finding $Q_x(m,t)$ and $G_x(m,\tau)$ in explicit form is difficult, we obtain the implicit forms in terms of the generating functions.

First we consider the rate equation for the queue length with priority larger than x. The transition probability is given as

$$\begin{aligned} P_{m\to m+i} &= (1-\mu) \sum_{j=i}^{\infty} \lambda_j \binom{j}{i} (1-x)^i x^{j-i} \\ &\quad + \mu \sum_{j=i+1}^{\infty} \lambda_j \binom{j}{i+1} (1-x)^{i+1} x^{j-i-1} \qquad (i \geq 0), \end{aligned} \tag{3}$$

$$P_{m\to m-1} = \mu \sum_{j=0}^{\infty} \lambda_j x^j. \tag{4}$$

The transition $P_{m\to m+i}$, that the number of tasks with priority higher than x increases by i, occurs a) when totally j of tasks arrive, among which i of tasks have higher priority than x, and no task is executed nor removed, or b) when totally j of tasks arrive, among which $i+1$ of tasks have higher priority than x, and 1 task is

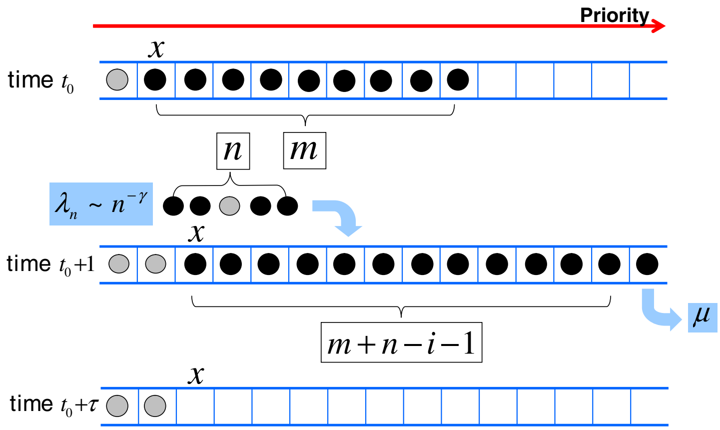

Fig. 2. (Color online) Schematic illustration of the queueing dynamics of the model. At time t_0, there are m tasks in the queue whose priority are larger than x. At time t_0+1, n tasks arrive to the queue with probability λ_n. Among them, $n-i$ tasks have priority $\geq x$ with probability $\binom{n}{n-i}(1-x)^{n-i}x^i$. A task with the largest priority is executed with probability μ. At time $t_0+\tau$, all tasks with priority $\geq x$ are executed for the first time.

executed and removed. Similarly, among the total j tasks, if there is no task with priority higher than x and 1 task is executed, $P_{m\to m-1}$ transition occurs.

For later uses, the generating function of this transition probability is obtained by multiplying z^i for each $P_{m\to m+i}$ and then sum them up for i, as,

$$\mathcal{P}(z) = \sum_{i=-1}^{\infty} P_{m\to m+i} z^i = \left(1-\mu+\frac{\mu}{z}\right) \Lambda\left((1-x)\,z+x\right), \tag{5}$$

where $\Lambda(z) \equiv \sum_{j=0}^{\infty} \lambda_j z^j$. Accordingly, the transition probability gives the rate equation for $Q_x(m,t)$: when $m \geq 1$,

$$\begin{aligned} Q_x(m,t+1) &\equiv \sum_{i=-1}^{m} P_{m-i\to m} Q_x(m-i,t) \\ &= \mu \sum_{j=0}^{\infty} \lambda_j x^j Q_x(m+1,t) \\ &+ \sum_{i=0}^{m} \left[(1-\mu) \sum_{j=i}^{\infty} \lambda_j \binom{j}{i} (1-x)^i \, x^{j-i} \right. \\ &\qquad \left. + \mu \sum_{j=i+1}^{\infty} \lambda_j \binom{j}{i+1} (1-x)^{i+1} \, x^{j-i-1} \right] Q_x(m-i,t), \end{aligned} \tag{6}$$

and when $m = 0$,

$$\begin{aligned} Q_x(0,t+1) &\equiv P_{1\to 0}Q_x(1,t) + P_{0\to 0}Q_x(0,t) \\ &= \mu \sum_{j=0}^{\infty} \lambda_j x^j Q_x(1,t) \\ &+ \left[(1-\mu)\sum_{j=0}^{\infty}\lambda_j x^j + \mu \sum_{j=1}^{\infty} \lambda_j j\,(1-x)\,x^{j-1} + \mu \sum_{j=0}^{\infty}\lambda_j x^j \right] Q_x(0,t). \end{aligned} \tag{7}$$

The generating function of $\tilde{Q}_x(m)$, defined as below, is obtained from the rate equation Eq.(6) and (7):

$$\tilde{\mathcal{Q}}_x(z) \equiv \sum_{m=0}^{\infty} \tilde{Q}_x(m) z^m = \frac{\left[\mu - \langle n \rangle_\lambda (1-x)\right](z-1)}{z - (\mu + z - \mu z)\Lambda\left((1-x)\,z + x\right)}. \tag{8}$$

On the other hand, the first passage probability is written in a recursive form using the transition probability. Similar recursive relation was used to obtain the analytic solution for the Bak-Sneppen's simple model of evolution [7,8,9].

$$G_x(1,t) = P_{1\to 0}\delta_{t,1} + P_{1\to 1}G_x(1,t-1) + P_{1\to 2}G_x(2,t-1) + P_{1\to 3}G_x(3,t-1) + \cdots \tag{9}$$

Here $G_x(1,t)$ is specifically defined as $f_x(t)$, and the definitions of generating functions of $G_x(m,t)$ and $f_x(t)$ are given as,

$$\mathcal{G}_x(m,s) \equiv \sum_{\tau} G_x(m,\tau) s^{\tau}, \tag{10}$$

$$\mathcal{F}_x(s) \equiv \sum_{t=1}^{\infty} f_x(t) s^t. \tag{11}$$

Note that $\mathcal{F}_x(s)$ is the generating function of the first-passage time to the origin starting at $x = 1$, while $\mathcal{G}_x(m,s)$ is that starting at arbitrary x. Since the transition to the left can occur only with the distance 1, the relation $\mathcal{G}_x(m,s) = \mathcal{F}_x(s)^m$ holds. From the recursive relation Eq.(9) and the fact above,

$$\mathcal{F}_x(s) = s \sum_{i=0}^{\infty} p_{1\to i}\mathcal{G}_x(i,s) = s \sum_{i=0}^{\infty} p_{1\to i}\mathcal{F}_x(s)^i = s\mathcal{F}_x(s)\mathcal{P}\left(\mathcal{F}_x(s)\right). \tag{12}$$

Using Eq.(5) for $\mathcal{P}(z)$ to the last expression, $\mathcal{F}_x(s)$ is,

$$\mathcal{F}_x(s) = s\left[(1-\mu)\,\mathcal{F}_x(s) + \mu\right] \Lambda\left((1-x)\,\mathcal{F}_x(s) + x\right). \tag{13}$$

If we recall the formalism for the waiting-time distribution and using the results above, the generating function for the waiting-time distribution is,

$$\begin{aligned}\mathcal{P}_{\rm w}(s) &\equiv \sum_{\tau=1}^{\infty} P_{\rm w}(\tau)s^{\tau} = \sum_{\tau=1}^{\infty}\sum_{m=0}^{\infty}\int_0^1 dx\tilde{Q}_x(m)G_x(m,\tau)s^{\tau} \\ &= \sum_{m=0}^{\infty}\int_0^1 dx\tilde{Q}_x(m)\mathcal{G}_x(m,s) = \sum_{m=0}^{\infty}\int_0^1 dx\tilde{Q}_x(m)\mathcal{F}_x(s)^m = \int_0^1 dx\tilde{\mathcal{Q}}_x(\mathcal{F}_x(s)),\end{aligned} \tag{14}$$

and now the problem is to find the closed-form of $\tilde{\mathcal{Q}}_x(\mathcal{F}_x(s))$ in series expansion.

Considering Eq.(8), $\tilde{\mathcal{Q}}_x(z)$ has different expansion according to the sign of $A_1 \equiv \mu - \langle n\rangle_\lambda(1-x)$. Moreover, all the generating functions above also have different expansions when $2 < \gamma \le 3$ and $\gamma > 3$. As a result, we have three different cases as below.

Case (i): $\mu > \langle n\rangle_\lambda$. In this case, to the leading singular term, $\tilde{\mathcal{Q}}_x(\mathcal{F}_x(s))$ is expanded as

$$\tilde{\mathcal{Q}}_x(\mathcal{F}_x(s)) \simeq (-1)^{\lceil\gamma\rceil}c_\gamma(1-x)^{\gamma-1}(1-s)^{\gamma-2}/A_1^{\gamma-1}, \tag{15}$$

to get $\mathcal{P}_{\rm w}(s) \sim (1-s)^{\gamma-2}$. Then by the Tauberian theorem [10], which states that if a function $F(x)$ can be written in a power-law form $F(x) \sim x^{-\alpha}$, then its generating function can also be written in a power-law with simple relation in the exponent as $\mathcal{F}(z) \sim (1-z)^{\alpha-1}$, we obtain

$$P_{\rm w}(\tau) \sim \tau^{-(\gamma-1)}. \tag{16}$$

Case (ii): $\mu = \langle n\rangle_\lambda$. For $2 < \gamma < 3$, it is obtained that

$$\begin{aligned}\mathcal{P}_{\rm w}(s) &\simeq \int_0^{(1-s)^{\frac{\gamma-2}{\gamma-1}}} dx\langle n\rangle_\lambda x c_\gamma^{-\frac{1}{\gamma-1}}(1-s)^{\frac{1}{\gamma-1}-1} + \int_{(1-s)^{\frac{\gamma-2}{\gamma-1}}}^1 dx + \cdots \\ &= 1 + \left(\frac{\langle n\rangle_\lambda c_\gamma^{-\frac{1}{\gamma-1}}}{2} - 1\right)(1-s)^{\frac{(\gamma-2)}{(\gamma-1)}} + \cdots.\end{aligned} \tag{17}$$

and $P_{\rm w}(\tau) \sim \tau^{-(2\gamma-3)/(\gamma-1)}$. While $\gamma > 3$,

$$\begin{aligned}\mathcal{P}_{\rm w}(s) &\simeq \int_0^{\frac{\sqrt{A_2(1-s)}}{\langle n\rangle_\lambda}} dx\frac{\langle n\rangle_\lambda x}{\sqrt{A_2(1-s)}} + \int_{\frac{\sqrt{A_2(1-s)}}{\langle n\rangle_\lambda}}^1 dx + \cdots \\ &= 1 - \frac{\sqrt{A_2(1-s)}}{2\langle n\rangle_\lambda} + \cdots.\end{aligned} \tag{18}$$

and $P_{\rm w}(\tau) \sim \tau^{-3/2}$.

Case (iii): $\mu < \langle n \rangle_\lambda$. The analysis shows that this case can be described by similar process as that of case (ii) saving the fact that only the task with priority $x > (\langle n \rangle_\lambda - \mu)/\langle n \rangle_\lambda$ are executed. Thus, $P_{\rm w}(\tau) \sim \tau^{-(2\gamma-3)/(\gamma-1)}$ for $\gamma \leq 3$ and $P_{\rm w}(\tau) \sim \tau^{-3/2}$ for $\gamma > 3$.

4 Numerical Results

As we have seen in the previous section, the average number of input/output tasks governs the waiting-time behavior. When $\langle n \rangle_\lambda < \mu$, the queue length fluctuates around zero. When $\langle n \rangle_\lambda > \mu$, on the other hand, the queue length grows perpetually. The tasks with $x > x_M \equiv (\langle n \rangle_\lambda - \mu)/\langle n \rangle_\lambda$ only are executed, while the others are remained in the queue for good and evenly distributed in priority (Fig. 3). Such a situation is similar to that of the Bak-Sneppen (BS) model of evolution. It has been already pointed out that the similar recursive relation as Eq.(9) was used in Ref. [7]. It is also noteworthy that the fitness (or barrier) in BS model has clear threshold and flat distribution. More specifically, the similarities are originated from the similarity in microscopic dynamical rules of the models. In the mean field BS model, the species with the lowest fitness is substituted by a new species with random fitness; that corresponds to the execution of the task with the largest priority and then input of a new task with random priority. Repetition of such a process yields well-defined threshold and flat distribution in priority (or fitness). Additionally, in this aspect, the waiting time of a task in the queue model corresponds to the life-time of a species in the BS model.

The plot of the waiting-time distributions is shown in Fig. 4 for the three different cases: (a) Case $\mu > \langle n \rangle_\lambda$. Given $\lambda = 0.3$ and $\mu = 1.0$, shown are numerically obtained $P_{\rm w}(\tau)$ for $\gamma = 2.5$ (top,red), 3.0 (blue), 3.5 (purple), and 4.0 (bottom,black), yielding $\langle n \rangle_\lambda \approx 0.58$, 0.41, 0.36, and 0.33, respectively. Solid

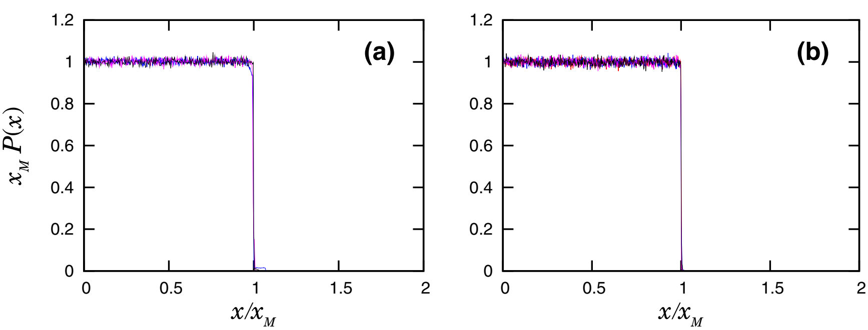

Fig. 3. (Color online) The distribution of tasks in the queue versus priority x in steady state, for different sets of $\langle n \rangle_\lambda$ and μ is shown. x_M is the threshold, so that only the tasks with priority $x > x_M$ are executed. The data are for the cases (a) $\lambda = 0.5$ and $\mu = 0.5$, $\gamma = 2.1$, 2.5, 2.8 and 3.0, (b) $\lambda = 0.5$ and $\mu = 0.3$, $\gamma = 3.3$, 3.8, 4.0 and 4.5. Thus, $x_M =$ (a) 0.85, 0.48, 0.33 and 0.26 and (b) 0.52, 0.47, 0.46 and 0.43.

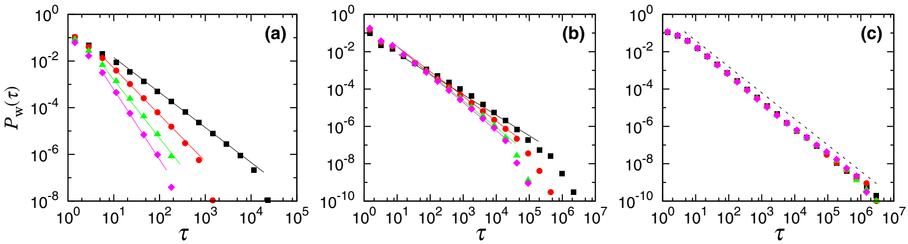

Fig. 4. (Color online) Plot of the waiting-time distribution of the tasks

lines indicate $P_{\mathrm{w}}(\tau) \sim \tau^{-(\gamma-1)}$. (b) Case $\mu \le \langle n \rangle_\lambda$ with $2 < \gamma \le 3$. Given $\lambda = 0.5$ and $\mu = 0.5$, shown are numerically obtained $P_{\mathrm{w}}(\tau)$ for $\gamma = 2.1$ (top, red), 2.5 (blue), 2.8 (purple), and 3.0 (bottom, black), yielding $\langle n \rangle_\lambda \approx 3.39$, 0.97, 0.75, and 0.68, respectively. Solid lines indicate $P_{\mathrm{w}}(\tau) \sim \tau^{-(2\gamma-3)/(\gamma-1)}$. (c) Case $\mu \le \langle n \rangle_\lambda$ with $\gamma > 3$. Given $\lambda = 0.5$ and $\mu = 0.3$, shown are numerically obtained $P_{\mathrm{w}}(\tau)$ for $\gamma = 3.3$ (red), 3.8 (blue), 4.0 (purple), and 4.5 (black), yielding $\langle n \rangle_\lambda \approx 0.62$, 0.57, 0.56, and 0.53, respectively. The dotted line is a guideline with slope -1.4, close to the theoretical value -1.5.

5 Summary and Discussion

In summary, we introduced the human dynamics model of priority queue with bursty input tasks based on the empirical observation, and studied the waiting-time distribution of the model analytically and numerically. The power-law exponent α of the waiting-time distribution $P_{\mathrm{w}}(\tau) \sim \tau^{-\alpha}$ varies having general values, while previous models only produce $\alpha = 1$ or $3/2$.

The most interesting result is $\alpha = (2\gamma - 3)/(\gamma - 1)$ for the case of $\langle n \rangle_\lambda \ge \mu$ and $2 < \gamma \le 3$. Consider the Lévy's flight random walks (LFRW) in one dimension: In this problem, a random walker can jump by distance x with the rate $r(x) \sim 1/|x|^\gamma$, where $2 < \gamma < 3$. Chechkin *et al.* [11] studied the first passage time (FPT) distributions and the first arrival time (FAT) of the LFRW, where a random walker is allowed (disallowed) to cross the destination point in a long jump for the FAT (FPT) walks. They obtained that the FAT distribution decays as $p_{\mathrm{fa}}(\tau) \sim \tau^{-(2\gamma-3)/(\gamma-1)}$, whereas FPT distribution $p_{\mathrm{fp}}(\tau) \sim \tau^{-3/2}$. On the other hand, the jump distance and rate in this paper is asymmetric according to the direction: while the jump distance is unbounded to the right, it is at most 1 to the left. By this fact, the FAT and FPT are the same in our model and in turn, the asymmetry does not affect the distribution of the FPT. As we have already seen, the FPT distribution has the same exponent as that of the LFRW. In conclusion, the anomalous behavior of the waiting-time distribution for the case $\langle n \rangle_\lambda > \mu$ when $2 < \gamma < 3$ is rooted from the FAT distribution of LFRW in one dimension.

Note: The original work reviewed in this paper is submitted to Phys. Rev. E and in review process. The preprint can be found in the arXiv [12].

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