Chaotic and Hyperchaotic Attractors in Time-Delayed Neural Networks

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Abstract. It is well known that complex dynamic behaviors exist in time-delayed neural networks. Infinite positive Lyapunov exponents can be found in time-delayed chaotic systems since the dimension of such systems is infinite. This paper presents an infinite-dimension hyperchaotic time-delayed neuron system with sinusoidal activation function. The hyperchaotic neuron system is studied by Lyapunov exponent, phase diagram, Poincare section and power spectrum. Numerical simulations show that the new system's behavior can be convergent, periodic, chaotic and hyperchaotic when the time-delay parameter varies.

Keywords: neuron system, time delay, hyperchaos.

1 Introduction

Chaos theory has been extensively studied in various research fields after Lorenz's observation of the chaotic phenomenon. In recent years, the synchronization of chaotic systems has become an active research area [1-4] because of its application to secure communication. However, theoretical and experimental models studied thus far concern mainly with low dimensional systems with only one positive Lyapunov exponent. Consequently, the messages masked by such chaotic systems are shown to be readily extracted sometimes once intercepted [5] and so the security of the communication system generated by low dimensional dynamical system is fragile. For enhancing the security, it is necessary to use high dimensional system to obtain hyperchaos with multiple positive Lyapunov exponents. However, on the one hand, the cost of equipment would increase substantially as the increasing dimensions of system, and on the other hand, the dimension of a high dimensional system is still bounded as the degree of freedom is constrained by its dimension.

Time delay exists in neural network systems inevitably. It is found that simple time-delayed neural networks can exhibit very chaotic dynamical behavior [6-8]. The dimension of solution space of a time-delayed dynamical system is infinite and so more than one positive Lyapunov exponents could be produced just by some low-dimension delayed neural networks. Therefore, communication system with a higher security level can be designed by means of the time-delayed neuron systems. In recent years, some investigations about that have been reported [9-10]. In these studies, each neuron has a sigmoid transfer function. Neuron networks with various nonlinear activation

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functions have also been found to possess chaotic and hyperchaotic behaviors. In this paper, hyperchaos is generated from an artificial two-neuron model with activation function $f(x)=\sin(x)$. The new infinite-dimension system's behavior is studied by Lyapunov exponent, phase diagram, Poincare section and power spectrum when the time-delay parameter varies.

2 Background of the Model

A class of artificial neural network with single delay considered in this Letter is described by the following differential equations with delay:

$$\frac{dx_i(t)}{dt}=-a_i x_i(t)+\sum_{j=1}^{n} b_{ij} f\left(x_j(t)\right)+\sum_{j=1}^{n} c_{ij} f\left(x_j(t-\tau)\right)+I_i \ ,\ i=1,2,\cdots,n\ . \tag{1}$$

where all $a_i>0$, b_{ij} and c_{ij} are real numbers, $\tau\geq 0$ represents the time delay, I_i is external inputs. The input-output transfer function $f(x)$ is a continuous nonlinear function. The model can be rewritten with matrix as

$$\frac{dx(t)}{dt}=-Ax(t)+Bf\left(x(t)\right)+Cf\left(x(t-\tau)\right)+I\ , \tag{2}$$

The difference differential equation (2) can be categorized as a kind of functional differential equations [11], and rewritten as

$$\frac{dx(t)}{dt}=-Ax_t(0)+Bf\left(x_t(0)\right)+Cf\left(x_t(-\tau)\right)+I\ , \tag{3}$$

where x_t is a continuous mapping defined on $[-\tau,0]$ as $x_t(\theta)=x(t+\theta)$, the right-hand side of Eq.(3) defines a functional mapping $C\left([-\tau,0],R^2\right)$ to R^2, where $C\left([-\tau,0],R^2\right)$ denotes the set of all continuous mapping from $[-\tau,0]$ to R^2. The solution space of Eq.(3) is infinite-dimensional, with initial conditions as any continuous functions defined on the closed interval $[-\tau,0]$.

3 Existence of Equilibrium Point and Hopf Bifurcation

We consider a two-neuron network, in this case the variables in the Eq.(2) can be written as

$$\begin{aligned}
&x(t)=(x_1(t),x_2(t))^T\ ,\\
&f(x(t))=(f_1(x_1(t)),f_2(x_2(t)))^T\ ,\\
&f(x(t-\tau))=(f_1(x_1(t-\tau)),f_2(x_2(t-\tau)))^T\ ,\\
&I=(I_1,I_2)^T\ .
\end{aligned} \tag{4}$$

and we assume

$$A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ B=\begin{pmatrix}0 & p\\ q & 0\end{pmatrix},\ C=\begin{pmatrix}0 & q\\ p & 0\end{pmatrix},I=\left(0,0\right)^{T}. \tag{5}$$

The model is rewritten as

$$\begin{cases}\dot{x}_1(t)=-x_1(t)+pf\left(x_2(t)\right)+qf\left(x_2(t-\tau)\right),\\ \dot{x}_2(t)=-x_2(t)+qf\left(x_1(t)\right)+pf\left(x_1(t-\tau)\right),\end{cases} \tag{6}$$

where the parameters p,q are positive real numbers.

First of all, we prove that the system possesses an equilibrium point (x_1^*,x_2^*) with a proper parameter set.

Theorem 1. Suppose the function $f(\bullet)$ satisfies the Lipschitz condition (i.e., there exists a $0<L<+\infty$, such that $\|f(x_1)-f(x_2)\|\le L\|x_1-x_2\|$ for arbitrary x_1 and x_2), and $\max\left(\left\|p(1+\frac{q}{p})L\right\|,\left\|q(1+\frac{p}{q})L\right\|\right)<1$, then the system (6) possesses an equilibrium point (x_1^*,x_2^*), which is unique.

Proof: We define an iterating map

$$F: R^2\to R^2:(x_1^{k+1},x_2^{k+1})=\left(p(1+\frac{q}{p})f(x_2^k),q(1+\frac{p}{q})f(x_1^k)\right),k\ge 0.$$

Set the distance between the arbitrary two points in the series as

$$d_{m,n}=\left\|x_1^m-x_1^n\right\|+\left\|x_2^m-x_2^n\right\|,$$

then we have

$$\begin{aligned}d_{m,n}&=\left\|p(1+\frac{q}{p})[f(x_2^{m-1})-f(x_2^{n-1})]\right\|+\left\|q(1+\frac{p}{q})[f(x_1^{m-1})-f(x_1^{n-1})]\right\|\\ &\le\left\|p(1+\frac{q}{p})L\left\|x_2^{m-1}-x_2^{n-1}\right\|\right\|+\left\|q(1+\frac{p}{q})L\left\|x_1^{m-1}-x_1^{n-1}\right\|\right\|\\ &\le\max\left(\left\|p(1+\frac{q}{p})L\right\|,\left\|q(1+\frac{p}{q})L\right\|\right)d_{m-1,n-1}.\end{aligned}$$

If $\max\left(\left\|p(1+\frac{q}{p})L\right\|,\left\|q(1+\frac{p}{q})L\right\|\right)<1$, then $\left\{(x_1^k,x_2^k)\right\}_{k=0,1,2,\cdots}$ is a Cauchy series. As a result, there exists a limit vector (x_1^*,x_2^*), and $x_1^*=p(1+\frac{q}{p})f(x_2^*)$, $x_2^*=q(1+\frac{p}{q})f(x_1^*)$.

Therefore (x_1^*, x_2^*) is the equilibrium point of system (6). It is obvious that the equilibrium point is unique. The proof of Theorem 1 is complete.

Next, we will show that there exists a Hopf bifurcation point in system (6) with a proper parameter set. We set $\left(\Delta x_1(t), \Delta x_2(t)\right) = (x_1 - x_1^*, x_2 - x_2^*)$, and then we obtain the following dynamical system:

$$\begin{cases} \Delta\dot{x}_1(t) = -\Delta x_1(t) + pf\left(\Delta x_2(t) + x_2^*\right) + qf\left(\Delta x_2(t-\tau) + x_2^*\right) - x_1^* \ , \\ \Delta\dot{x}_2(t) = -\Delta x_2(t) + qf\left(\Delta x_1(t) + x_1^*\right) + pf\left(\Delta x_1(t-\tau) + x_1^*\right) - x_2^* \ . \end{cases} \tag{7}$$

Suppose that the function $f(\bullet)$ is differentiable and $f'(x_i^*) \neq 0 \ (i = 1,2)$, then the linearized system of (7) is

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + pf'(x_2^*)x_2 + qf'(x_2^*)x_2(t-\tau) \ , \\ \dot{x}_2(t) = -x_2(t) + qf'(x_1^*)x_1 + pf'(x_1^*)x_1(t-\tau) \ , \end{cases} \tag{8}$$

which possesses the equilibrium point $(0,0)$. The characteristic equation for system (8) is

$$\det\begin{bmatrix} \lambda+1 & -pf'(x_2^*) - qf'(x_2^*)\exp(-\lambda\tau) \\ -qf'(x_1^*) - pf'(x_1^*)\exp(-\lambda\tau) & \lambda+1 \end{bmatrix} = 0 \ . \tag{9}$$

i.e.

$$h^2(\lambda+1)^2 - \left[1 + \exp(-2\lambda\tau) + r\exp(-\lambda\tau)\right] = 0 \ . \tag{10}$$

where $h = \sqrt{1/pqf'(x_1^*)f'(x_2^*)}$, $r = \dfrac{p^2+q^2}{pq}$. We can prove the following result.

Theorem 2. If $\tau = \tau_c = \dfrac{1}{\omega}\left[k\pi + arc\cot(\dfrac{\omega^2-1}{2\omega})\right]$ $(k \in Z, \omega \in R, \omega \neq 0)$, $r \neq 2\left|\dfrac{\omega^2-3}{\omega^2+1}\right|$, then Eq.(10) has a pure imaginary root $i\omega$, and $\left.\dfrac{\partial \operatorname{Re}(\lambda)}{\partial\tau}\right|_{\tau=\tau_c} \neq 0$, where $\operatorname{Re}(\lambda)$ denotes the real part of λ. Hence, there exist Hopf bifurcation phenomena in system (10).

Proof: Let $\lambda = i\omega$, we substitute it into the Eq.(10):

$$h^2(1+i\omega)^2 - \left[1 + \exp(-2i\omega\tau) + r\exp(-i\omega\tau)\right] = 0 \ ,$$

i.e.,

$$h^2(1-\omega^2) - \cos(2\omega\tau) - r\cos(\omega\tau) - 1 + i\left[2h^2\omega + \sin(2\omega\tau) + r\sin(\omega\tau)\right] = 0 \ .$$

Separating the real and imaginary parts, we obtain

$$\begin{cases} r\cos(\omega\tau) + \cos(2\omega\tau) = h^2(1-\omega^2) - 1 \ , \\ r\sin(\omega\tau) + \sin(2\omega\tau) = -2h^2\omega \ . \end{cases}$$

The solution of above equation is $\tau = \frac{1}{\omega}\left[k\pi + arc\cot(\frac{\omega^2-1}{2\omega})\right]$ $(k \in Z)$.

We take the derivative with respect to τ at both sides of Eq.(10) and obtain

$$\frac{\partial\lambda}{\partial\tau} = \frac{-2\lambda\exp(-2\lambda\tau) - r\lambda\exp(-\lambda\tau)}{2h^2(1+\lambda) + 2\tau\exp(-2\lambda\tau) + r\tau\exp(-\lambda\tau)}.$$

Let $\lambda = i\omega$, we rewrite it as follows

$$\frac{\partial\lambda}{\partial\tau} = \frac{-2i\omega\exp(-2i\omega\tau) - ir\omega\exp(-i\omega\tau)}{2h^2(1+i\omega) + 2\tau\exp(-2i\omega\tau) + r\tau\exp(-i\omega\tau)},$$

i.e.,

$$\frac{\partial\lambda}{\partial\tau} = \frac{-2\omega\sin(2\omega\tau) - r\omega\sin(\omega\tau) - i\left[2\omega\cos(2\omega\tau) + r\omega\cos(\omega\tau)\right]}{2h^2 + 2\tau\cos(2\omega\tau) + r\tau\cos(\omega\tau) - i\left[2\tau\sin(2\omega\tau) + r\tau\sin(\omega\tau) - 2h^2\omega\right]}.$$

The real part of $\frac{\partial\lambda}{\partial\tau}$ can be expressed as

$$\mathrm{Re}\left(\frac{\partial\lambda}{\partial\tau}\right) = -2h^2\omega\left[2\omega\cos(2\omega\tau) + 2\sin(2\omega\tau) + r\omega\cos(\omega\tau) + r\sin(\omega\tau)\right] \Big/ \Big\{(r^2+4)\tau^2 +$$
$$4h^4(\omega^2+1) + 4r\tau(h^2+\tau)\cos(\omega\tau) + 4h^2\tau\left[2\cos(2\omega\tau) - \omega\left(r + 4\cos(\omega\tau)\right)\sin(\omega\tau)\right]\Big\}.$$

$\left.\frac{\partial\,\mathrm{Re}(\lambda)}{\partial\tau}\right|_{\tau=\tau_c} = \mathrm{Re}\left.\left(\frac{\partial\lambda}{\partial\tau}\right)\right|_{\tau=\tau_c}$, if $\left.\frac{\partial\,\mathrm{Re}(\lambda)}{\partial\tau}\right|_{\tau=\tau_c} \neq 0$, we obtain

$$2\omega\cos(2\omega\tau_c) + 2\sin(2\omega\tau_c) + r\omega\cos(\omega\tau_c) + r\sin(\omega\tau_c) \neq 0.$$

Solving the inequation and simplifying the result, we obtain $r \neq 2\left|\frac{\omega^2-3}{\omega^2+1}\right|$. So the proof of Theorem 2 is also complete.

4 Chaotic and Hyperchaotic Behavior

Eq.(6) is a simple two-neuron system with time delay. Yet, it seems that the system is capable of generating a wide variety of interesting dynamical behaviors. We will mainly concern with chaos and hyperchaos phenomena.

According to the paper [12], if we want to obtain hyperchaos, three requirements must be satisfied:

1) The model is a dissipative system.
2) The number of terms in the equations giving rise to instability is at least two, of which at least one has a nonlinear function.
3) The dimension of the phase space of autonomous system is at least four.

Because $\nabla V = \frac{\partial \dot{x}_1}{\partial x_1} + \frac{\partial \dot{x}_2}{\partial x_2} = -2 < 0$, there are four terms about the nonlinear function f in the equations and the dimension of a time-delayed system is infinite, the model discussed by us can satisfy above necessary conditions easily.

It is difficult to test if there exists chaos and hyperchaos in a time-delayed system, but numerical simulation analysis is a valid method for such a system. In the following computer simulation, we fix p, q and let τ vary. We also adopt the nonlinear activation function $f(x) = \sin(x)$. The consequence of increasing τ will lead to a limit cycle which arises through a Hopf bifurcation. In our simulation of Eq.(6) with activation function $\sin(x)$, $p = 3, q = 5$ and the numerical integration was performed using fourth-order Runge-Kutta technique.

For discovering the chaos and hyperchaos phenomena, we firstly calculate and plot the two largest Lyapunov exponent of system (6) for τ by employing the method of Wolf.

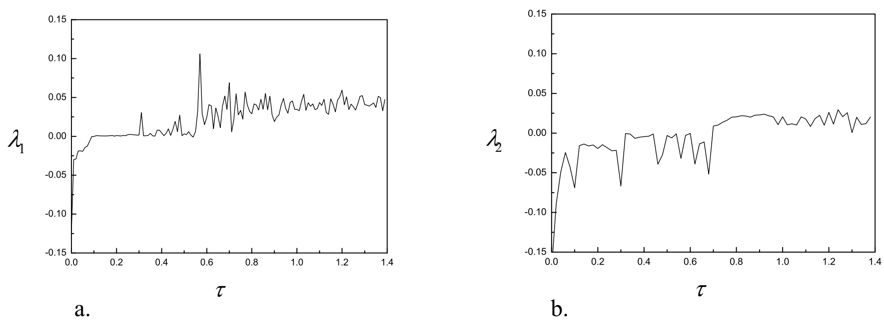

Fig. 1. The two largest Lyapunov exponents λ_i

The Fig.1 (a) and (b) show us some important information. According to them, we can conclude that:

1) When the time delay parameter $\tau = 0$, $\lambda_1 < 0$ and $\lambda_2 < 0$, the Eq.(6) possesses an equilibrium point.
2) When τ varies from 0 to 0.1, λ_1 comes near to zero from a negative value gradually and λ_2 is still less than zero. When τ is in the neighborhood of 0.1, a Hopf bifurcation maybe occurs and then the system will possess a periodic solution.
3) When τ varies from 0.55 to 0.7, they are either $\lambda_1 > 0, \lambda_2 < 0$ or $\lambda_1 > 0, \lambda_2 = 0$, the system has and only has one positive Lyapunov exponent. We should discover the chaos phenomenon.
4) When $0.75 < \tau < 1.4$, $\lambda_1 > 0$ and $\lambda_2 > 0$, there exist more than one positive Lyapunov exponents in the model. We are likely to see the hyperchaos phenomenon, which is more complex than general chaos.

In the following, we will look for the chaotic and hyperchaotic attractors of system with the parameter set given by above paragraphs and validate the program designed by us with Wolf algorithm by phase diagram, Poincare section and power spectra.

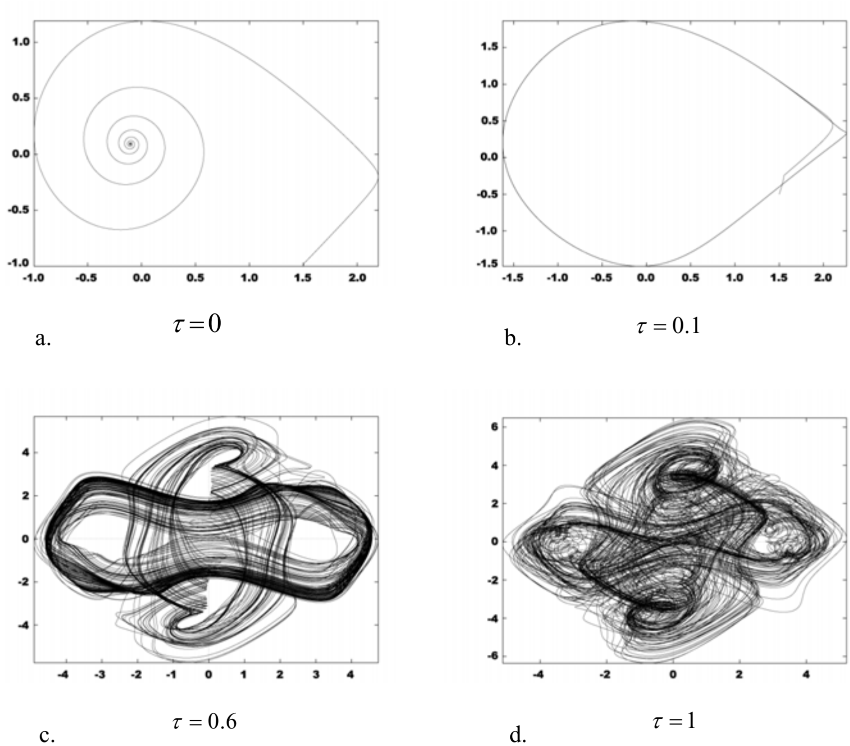

Fig. 2. Projections in the plane $x_2(t) - x_1(t)$ of attractor of system (6)

The Fig.2 shows us that from (a) to (d), with the increase of time delay parameter, the attractor of system experience equilibrium point, periodic solution, chaos and hyperchaos successively. They do consist with the above conclusions. For better distinguishing the hyperchaos from chaos, it is necessary to plot the projections of attractor in three-dimension space.

With the Fig.3, we can observe the difference more clearly between a chaotic attractor and a hyperchaotic attractor. In the direction of $x_1(t-\tau)$ or $x_2(t-\tau)$, the trajectory of hyperchaotic one is irregular and complicated, as shown in figures (b) and (d). For further testifying the existence of chaos and hyperchaos in the system, it will be very helpful and useful to plot the Poincare sections and the power spectra.

We set $x_1(t-\tau) = 0$ and plot the Poincare sections of attractors in Fig.3 (a) and (b). The Fig.4 exhibits that they are neither several numerable dots nor a closed line, so they are

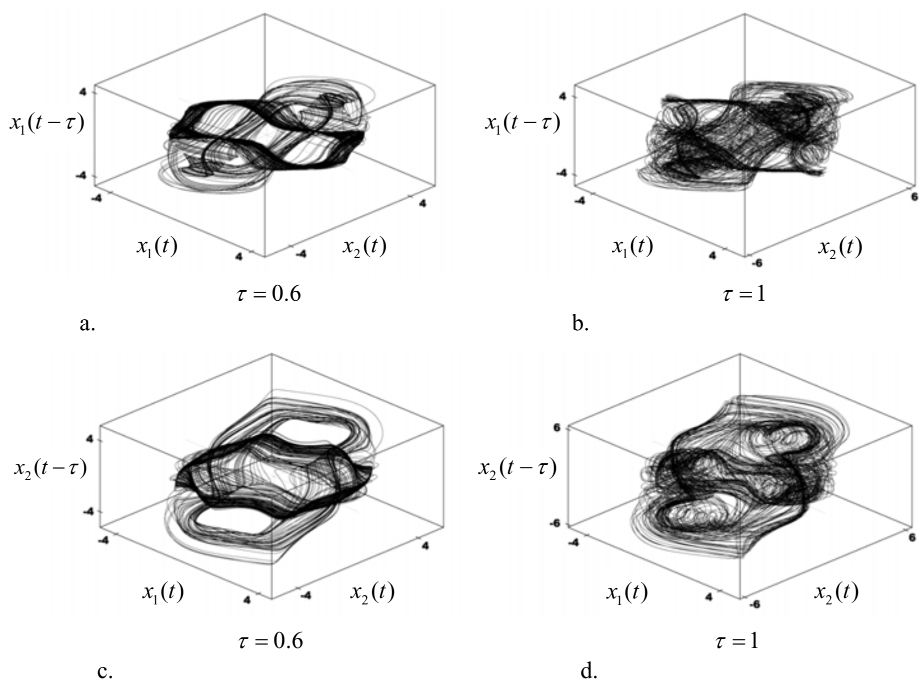

Fig. 3. Projections in 3-dimension space of chaotic and hyperchaotic attractors

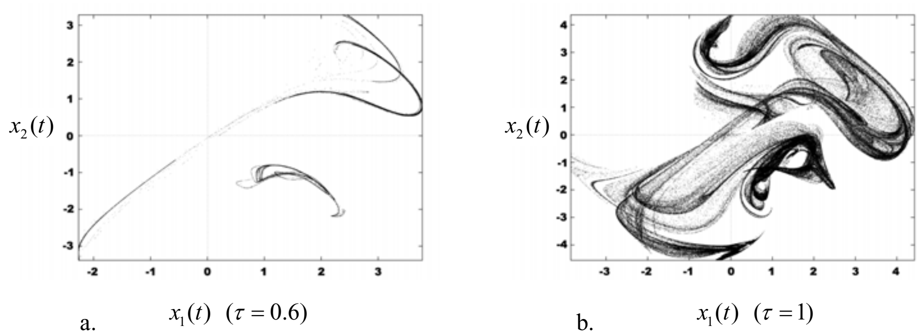

Fig. 4. Poincare sections of attractors in Fig.3 (a) and (b)

neither periodic solution nor quasi-periodic solution. Great deals of discrete-distributed dots imply the existence of chaos. And obviously, the figure (b) is much more complex than (a).

The Fig.5 show that there exist continuous peak values and noise background in signals. These characters are possessed by chaos uniquely. By now, we have obtained quite enough proofs to validate the existence of chaos and hyperchaos in the model.

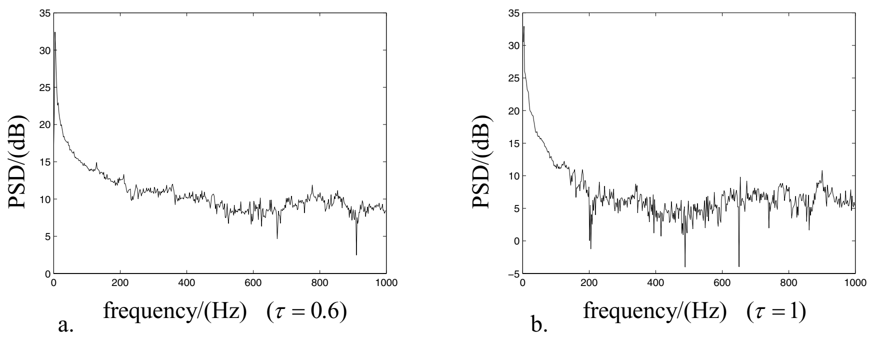

Fig. 5. Power spectral density

5 Conclusion

In this paper, we show that even a simple neural network model can exhibit chaotic and hyperchaotic behaviors. From the plots of the two largest Lyapunov exponents, phase diagram, Poincare section and power spectra, we observe the existence of chaotic and hyperchaotic phenomena in a two-neuron model with time delay. As the variation of time delay parameter τ , the model demonstrates the equilibrium point, periodic oscillation, chaotic and hyperchaotic output with different characteristics. Therefore, we can control the dynamical behaviors and the degree of chaos in this neural network simply by changing the values of parameter.

The importance of chaotic and hyperchaotic dynamics for the purpose of higher-order information processing and secure communication is emphasized in recent years [13-16]. The simple artificial neural networks considered here may serve as a basic element for the private secure communication which is realized by chaos synchronization, chaos masking and so on.

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