Stability of Non-diagonalizable Networks: Eigenvalue Analysis

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Abstract. The stability of non-diagonalizable networks of dynamical systems are investigated in detail based on eigenvalue analysis. Pinning control is suggested to stabilize the synchronization state of the whole coupled network. The complicated coupled problem is reduced to two independent problems: clarifying the stable region of the modified system and specifying the eigenvalue distribution of the coupling and control matrix. The dependence of the stability on both pinning density and pinning strength is studied.

Keywords: non-diagonalizable network, pinning control, stability, eigenvalue analysis.

1 Introduction

The last decade has eyewitnessed the birth of a new movement of research interest in the study of complex networks [1-7], which is pervading all disciplines of sciences today, ranging from physics to chemistry, biology, information science, mathematics, and even social sciences. Recently, the interplay between the complexity of the overall topology and the collective dynamics of complex networks gives rise to a host of interesting effects. Especially, there are attempts to control the dynamics of complex networks and guide it to a desired state [8-16]. Previous work on this problem [8-16] has typically focused on diagonalizable networks, where the corresponding Laplacian matrix is assumed to be diagonalizable. However, most optimal networks are non-diagonalizable [17], in particular when the networks are directed.

In this paper, the pinning control problem is further visited for nondiagonalizable networks with identical node dynamics. The main contribution of this paper is to developing a new stability analysis scheme, named eigenvalue analysis, for non-diagonalizable networks of dynamical systems. Briefly, from this approach, the network stability problem is operationalized in two independent tasks: one is to characterize the stable region of the modified system and the other to analyze the eigenvalue distribution of the coupling and control matrix. The former is determined by the dynamical rules governing the isolated node whereas the latter is determined by both the topology of the network and the control scheme.

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An outline of this paper is as follows. The design of pinning controllers of a non-diagonalizable network and the stability of the pinned network are discussed in Sec. 2, in which stable and unstable regions are identified. Various essentially different structures of stable regions are shown and eigenvalue distributions of different coupling matrices are investigated. The dependence of the stability on both pinning density and pinning strength is discussed based on eigenvalue analysis and numerical simulation. Finally, Sec. 3 gives a few concluding remarks.

2 Stability of Synchronization State by Pinning Control

Consider a network of N coupled dynamical systems whose state equations are written in the following form:

$$\dot{x}_i(t) = f(x_i(t)) - \sigma \sum_{j=1}^{N} L_{ij} \Gamma x_j(t), \quad i = 1, 2, \cdots, N, \tag{1}$$

where $x_i \in R^m$ represents the state vector of the i-th node, and the nonlinear function $f(\cdot)$, describing the local dynamics of the nodes, is continuously differentiable and capable of producing various rich dynamical behaviors, including periodic orbits and chaotic states. The parameter σ is positive ruling the overall coupling strength. Also, $\Gamma \in R^{m\times m}$ is a constant matrix linking coupled variables, while the real matrix $L = (L_{ij})$ is called the Laplacian matrix of the non-diagonalizable network, satisfying zero row-sum. The topological information on the network in terms of the connections and the weights is contained in the Laplacian matrix L, whose entries L_{ij} are zero if node i is not connected to node j $(j \neq i)$, but are negative if there is a directed influence from node j to node i. In addition, L is not necessarily diagonalizable and symmetric because the network is not constrained to be undirected and unweighted.

Supposing the isolated node accepts a chaotic solution, our central task is to synchronize network (1) onto a prescribed state $\bar{x}$, which is the solution of the individual system $\dot{x}(t) = f(x(t))$ and satisfies $f(\bar{x}) = 0$. To do so, feedback pinning control is acted on the network (1) and the controlled network can be described as

$$\dot{x}_i(t) = f(x_i(t)) - \sigma (\sum_{j=1}^{N} L_{ij} \Gamma x_j(t) + d_i \Gamma (x_i(t) - \bar{x})), \quad i = 1, 2, \cdots, N, \tag{2}$$

where $d_i = d > 0$ if control is applied to the i-th node and $d_i = 0$ otherwise. Without loss of generality, we rearrange the order of nodes in the network such that the pinned nodes $i = 1, \cdots, l$ are the first l nodes in the rearranged network. Note that, l corresponds to the "pinning density", while d is the feedback gain to be designed and furthermore determines the "pinning strength".

We suggest that the pinned system (2) is stable at $\bar{x}$, if $\lim_{t\to\infty} \|x_i(t) - \bar{x}\| = 0$ for all $i = 1, 2, \cdots, N$.

The stability of the system (2) can be analyzed exactly by setting $e_i(t) = x_i(t) - \bar{x}$ and linearizing it at state $\bar{x}$. This leads to

$$\dot{E} = EJ^T(\bar{x}) - \sigma C E \Gamma^T, \tag{3}$$

where $J(\bar{x})$ is the Jacobian matrix of f evaluated at $\bar{x}$. $E^T = [e_1, e_2, \cdots, e_N] \in R^{m\times N}$ and $C = L + D$ with feedback gain matrix $D = diag(d_1, d_2, \cdots, d_N)$.

For convenience, we denote the matrix C as the "coupling and control matrix". It is easy to prove that the matrix C is non-diagonalizable while all the real parts of its eigenvalues are strictly positive.

Consider C written in Frobenius normal form [18], i.e.,

$$C = PBP^T = P \begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_n \end{bmatrix} P^T, \tag{4}$$

where P is a permutation matrix and B_k are blocks of the form

$$B_k = \begin{bmatrix} \lambda_k & & & \\ 1 & \lambda_k & & \\ & \ddots & \ddots & \\ & & 1 & \lambda_k \end{bmatrix}, \tag{5}$$

where λ_k is one of the eigenvalues of C.

Introducing a transformation

$$E = P\eta, \tag{6}$$

along with (3), leads to

$$\dot{\eta} = \eta J^T(\bar{x}) - \sigma B \eta \Gamma^T. \tag{7}$$

Each block of the Jordan canonical form corresponds to a subset of these columns in η, which obeys a subset of equations in (7). If block B_k is $n_k \times n_k$, then the equations take the form

$$\dot{\eta}_1 = (J(\bar{x}) - \sigma\lambda_k\Gamma)\eta_1, \tag{8}$$

$$\dot{\eta}_2 = (J(\bar{x}) - \sigma\lambda_k\Gamma)\eta_2 - \sigma\Gamma\eta_1, \tag{9}$$

$$\cdots$$

$$\dot{\eta}_{n_k} = (J(\bar{x}) - \sigma\lambda_k\Gamma)\eta_{n_k} - \sigma\Gamma\eta_{n_k-1}. \tag{10}$$

Here $\eta_1, \eta_2, \cdots, \eta_{n_k}$ represent the modes of perturbation in the generalized eigenspace associated with eigenvalue λ_k. Clearly, η_1 converges exponentially to zero as $t \longrightarrow \infty$, if and only if all the real parts of the eigenvalues of the matrix $(J(\bar{x}) - \sigma\lambda_k\Gamma)$ are less than 0. If this condition holds and the norm of Γ is bounded, then the second term in Eq.(9) is exponentially small as well, which results in exponential convergence of η_2 to zero as $t \longrightarrow \infty$. The same argument when applied repeatedly shows that $\eta_3, \cdots, \eta_{n_k}$ must also converge to zero if all the real parts of the eigenvalues of the matrix $(J(\bar{x}) - \sigma\lambda_k \Gamma)$ are less than 0. Therefore, when all the Jordan blocks are taken into account, we see that the stability condition for the synchronous solution in the general non-diagonalizable case is

$$Re(J(\bar{x}) - \sigma\lambda_k \Gamma) < 0, \quad k = 1, 2, \cdots, n, \tag{11}$$

where $Re(\cdot)$ denotes the real part of an eigenvalue.

The significance of (11) is that the stability problem of the controlled network (2) can be separated into two independent tasks: one is to analyze the stable regions of the modified systems (8)-(10), which depends on the dynamics of the isolated node such as the Jacobian matrix $J(\bar{x})$ and the inner linking structure Γ; the other task is to analyze the eigenvalue distribution of σC, which is independent of the inner dynamics including $J(\bar{x})$ and Γ.

This criterion as shown in (11) provides the stability boundary in the complex plane. Of course, this stability boundary depends on the dynamics of the isolated node and the inner linking matrix Γ. In the following subsection we clarify various essentially different structures of stable regions.

2.1 Stable and Unstable Regions

Now we specify the well-known Rössler model as an example. A single Rössler oscillator [19] is described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -(x_2 + x_3) \\ x_1 + \alpha x_2 \\ x_3(x_1 - \gamma) + \beta \end{bmatrix}, \tag{12}$$

which has a chaotic attractor when $\alpha = \beta = 0.2$ and $\gamma = 5.7$. With this set of system parameters, one unstable equilibrium point is $\bar{x} = [0.007, -0.0351, 0.0351]^T$.

Here we consider the full diagonal coupling $\Gamma = diag(1, 1, 1)$ and the partial diagonal couplings $\Gamma = diag(1, 0, 0)$, $\Gamma = diag(0, 1, 0)$, and $\Gamma = diag(1, 1, 0)$ respectively. In Fig. 1 we plot the stable regions of the synchronization state $\bar{x}$ of Rössler model in the complex plane for different coupling links. The curves represent the critical condition at which the largest real part of the eigenvalues of the matrix $(J(\bar{x}) - \sigma\lambda_k \Gamma)$ is equal to zero. In the region marked by "S" (stable), the largest real part of the eigenvalues of the matrix $(J(\bar{x}) - \sigma\lambda_k \Gamma)$ is negative, while it is positive in the region marked by "U" (unstable). It is interesting to notice that the structure of the stable regions in Fig. 1 can be classified into three groups. Class (i), shown in Fig. 1(a): the critical curve is a straight line, and then larger $Re(\lambda)$ is favorable for stable synchronization of the homogenous states. Class (ii), shown in Fig. 1(b): the critical curve forms a closed circle, and then the stable region is localized in a certain finite $Re(\lambda) - Im(\lambda)$ region. Too large and too small $Re(\lambda)$ and too large $|Im(\lambda)|$ can definitely destroy the stability of the

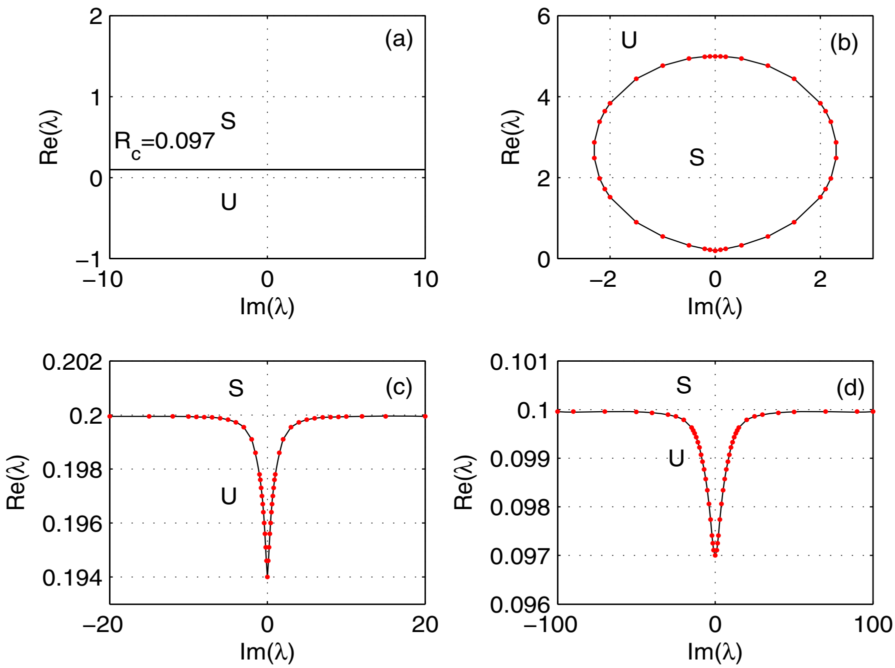

Fig. 1. Distributions of stable ("S") and unstable ("U") regions of the synchronization state $\bar{x}$ for the Rössler model. (a) $\Gamma = diag(1, 1, 1)$. (b) $\Gamma = diag(1, 0, 0)$. (c) $\Gamma = diag(0, 1, 0)$. (d) $\Gamma = diag(1, 1, 0)$. The black solid lines represent the zero maximum real part of the eigenvalues.

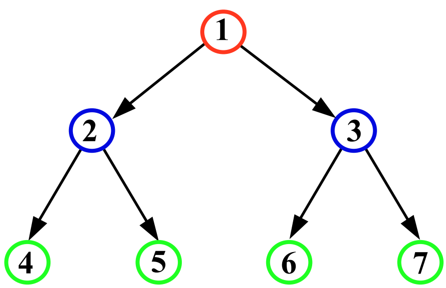

Fig. 2. A non-diagonalizable network with 7 nodes

synchronization state. Class (iii), shown in Figs. 1(c) and 1(d): the critical curve is V-shaped, then larger $Re(\lambda)$ and smaller $|Im(\lambda)|$ are favorable for stabilizing the homogenous states. Note that for the case of the full diagonal coupling, the stability is controlled by the value of $Re(\lambda)$ only [not $Im(\lambda)$], and the threshold value of the stable-unstable boundary at the imaginary axis $R_c = 0.097$.

2.2 Eigenvalue Distribution

To stabilize the synchronization state, the key point is to move all the unstable eigenvalues of L to the stable region by adding suitable control signal. In the following we consider a non-diagonalizable network with 7 nodes as shown in Fig. 2. The network involves three levels. The top level contains the unique node without input, which is the root node. The Laplacian matrix L is

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

whose eigenvalues are $0, 1, 1, 1, 1, 1, 1$. The root node 1 must be pinned in order to synchronize the network.

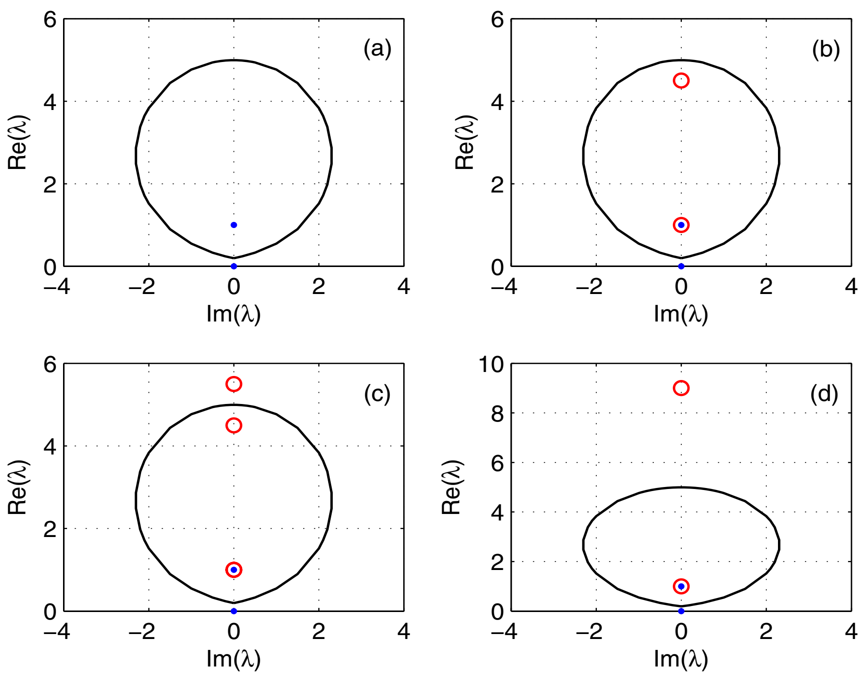

Fig. 3. Eigenvalue distribution of the matrix C: $\sigma = 1$. (a) $l = 0, d = 0$. (b) $l = 1, d = 4.5$. (c) $l = 2, d = 4.5$. (d) $l = 1, d = 9$. The blue dots represent the eigenvalues of C with $l = d = 0$, while the red empty circles represent the eigenvalues of C with $l \neq 0$ and $d \neq 0$. These notations are valid for Fig.4. The black lines denote the critical curves of the stable region in the case of Fig. 1(b).

In Fig. 3 we plot various eigenvalue distributions at $\sigma = 1$ for different d and l. It is observed from Fig. 3(a) that without control (i.e., $l = 0$ and $d = 0$), there is one unstable eigenvalue, 0, located in the unstable region. Keeping all parameters unchanged except setting $l = 1$ and $d = 4.5$ ($l = 1$ means the root node 1 is pinned), the unstable zero eigenvalue moves up and crosses the critical line and finally enters the stable region, as shown in Fig. 3(b). Continuously increasing the pinning density until $l = 2$, an interesting phenomenon occurs: one of the bottom eigenvalues first crosses the upper critical curve and then enters the unstable region, which leads to desynchronization. This feature is still observed in Fig. 3(d) when the feedback gain is increased to $d = 9$ from Fig. 3(b). It is concluded that too large d and/or l can definitely destroy the stabilization of the synchronization state.

In Fig. 4, we do the same as in Fig. 3 except that $\sigma = 0.25$ and the stable region distribution Fig. 1(c) are considered. Consider $l = 1$ and $d = 0.65$, the origin zero eigenvalue moves up as shown in Fig. 4(b). Continuously increasing pinning density until $l = 3$, there is still one nonzero eigenvalue sitting in the unstable region. Increasing pinning strength until $d = 1.95$ from Fig. 4(c), the unstable eigenvalue crosses the critical line and enters the stable region [see Fig. 4(d)].

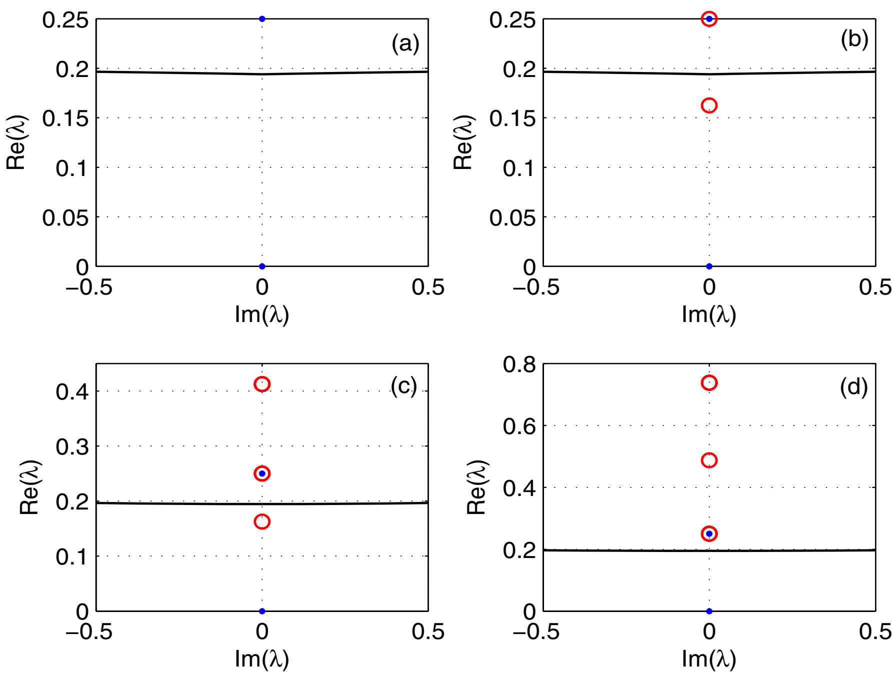

Fig. 4. Eigenvalue distribution of the matrix C: $\sigma = 0.25$. (a) $l = 0, d = 0$. (b) $l = 1, d = 0.65$. (c) $l = 3, d = 0.65$. (d) $l = 3, d = 1.95$. The black lines denote the critical curves of the stable region in the case of Fig. 1(c).

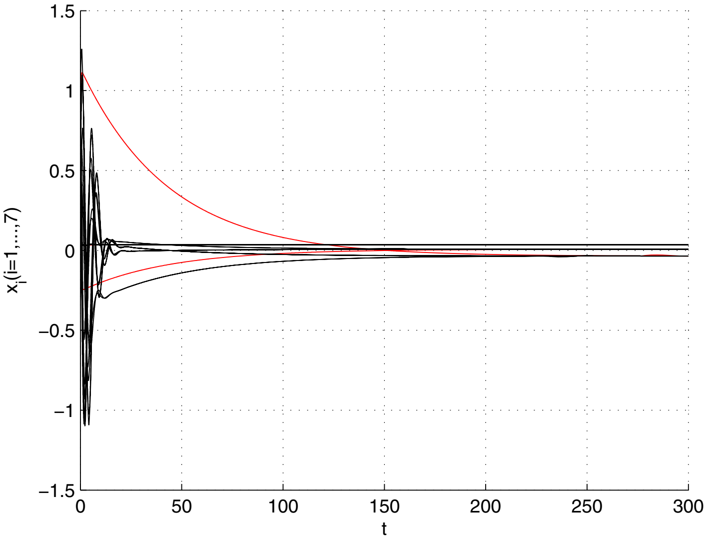

Fig. 5. The evolution of network states corresponding to Fig.3(b). The red lines denote the states of the pinned nodes. These notations are valid for Figs. 6-8.

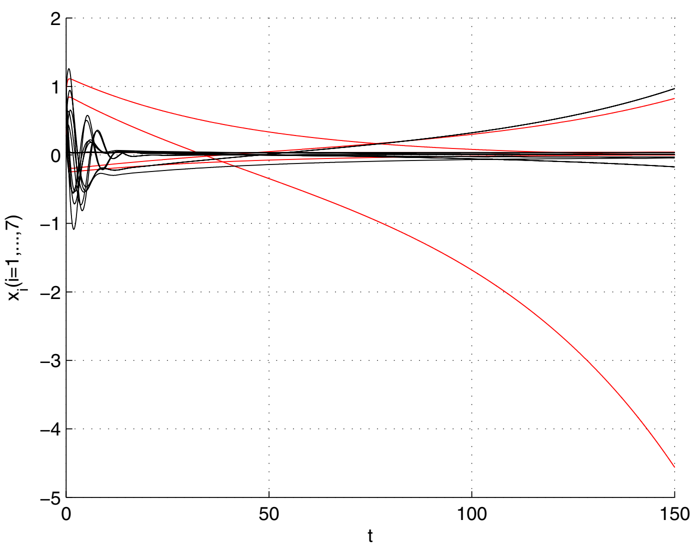

Fig. 6. The evolution of network states corresponding to Fig. 3(c)

It is concluded that increasing the pinning density and/or pinning strength can considerably enhance the controlling efficiency in the case of V-shaped region.

We should emphasize that the stabilization effect depends sensitively on the structure of stable region. For the case of the full diagonal coupling, only if the condition $\sigma\lambda > R_c$ holds, the pinned system can be stable. A different case

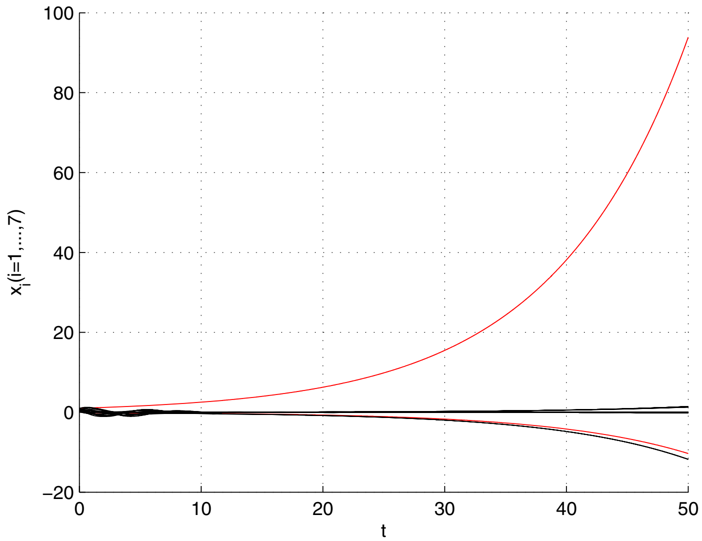

Fig. 7. The evolution of network states corresponding to Fig. 3(d)

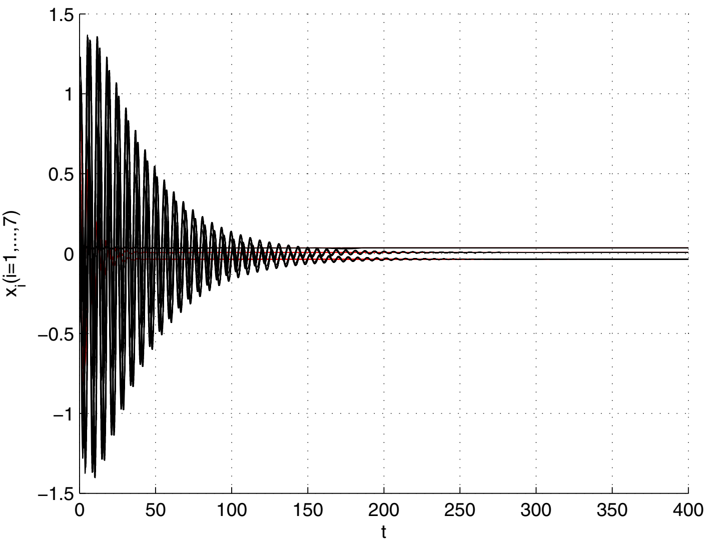

Fig. 8. The evolution of network states corresponding to Fig. 4(d)

is class (ii) structure, in which increasing sufficiently the pinning density or pinning strength definitely spoils stabilization, because some eigenvalues will be pushed upward by the control to the unstable region. This behavior is essentially different from the one displayed in Fig. 4.

Figures 5, 6, 7 and 8 show the process of controlling the 7-node network corresponding to Figs. 3(b), 3(c), 3(d) and Fig. 4(d), respectively. It is clear that the aim state is stabilized well after control, which is consistent with Figs. 3(b) and 4(d). Also, the stability is destroyed by larger l or larger d as shown in Figs. 6 and 7 when the stable region is a closed circle.

3 Conclusions

In this paper, the stability of non-diagonalizable networks under pinning control is examined by applying eigenvalue analysis. The effects of pinning density and pinning strength are investigated in detail.

Based on eigenvalue analysis, the stabilization problem of complicated high-dimensional systems can be divided into two independent problems: One is the description of stable and unstable regions of the isolated node modified by an eigenvalue forcing $\sigma\lambda_k\Gamma$ [see (8)-(10)]; the other is the eigenvalue analysis of the node coupling and control matrix C. The former is independent of the node interaction scheme and the control mechanism, whereas the latter is independent of the inner dynamics, the synchronization state and the inner linking matrix. Both problems have been solved easily. They, together, provide definite answers to the problems of stability of non-diagonalizable coupled networks. For instance, one can easily reveal and classify the stability of the synchronization state by examining whether and how some unstable eigenvalues enter into the stable region. Moreover, one can apply control matrix D to stabilize a synchronization state by moving all the unstable eigenvalues into the stable region. Therefore, the investigation of the distribution of stable and unstable regions of the isolated system becomes extremely important and widely significant for the stability problems of coupled networks with large size. The ideas in this paper can be applied to general coupled extended networks: by changing $f(x), \bar{x}$ and Γ, we can obtain different distributions of stable and unstable regions; by adjusting the control matrix D, we can flexibly change the distribution of the matrix C; and by combining all these manipulations we can stabilize the homogenous stationary state of a general coupled network.

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References

1. Watts, D.J., Strogatz, S.H.: Collective Dynamics of ‘Small World’ Networks. Nature 393, 440–442 (1998)
2. Barabási, A.-L., Albert, R.: Emergence of Scaling in Random Networks. Science 289, 509–512 (1999)

3. Li, C.G., Chen, G.: Synchronization in General Complex Networks with Coupling Delays. Physica A 343, 263–278 (2004)
4. Lü, J., Chen, G.: A Time-varying Complex Dynamical Network: Model and Its Controlled Synchronization Criteria. IEEE. Trans. Autom. Control. 50, 841–846 (2005)
5. Motter, A.E., Zhou, C., Kurths, J.: Network Synchronization, Diffusion, and the Paradox of Heterogeneity. Phys. Rev. E. 71, 016116 (2005)
6. Wu, C.W.: Synchronization and Convergence of Linear Dynamics in Random Directed Networks. IEEE. Trans. Autom. Control. 51, 1207–1210 (2006)
7. Zhu, J.J.H., Meng, T., Xie, Z.M., Li, G., Li, X.M.: A Teapot Graph and Its Hierarchical Structure of the Chinese Web. In: Proceedings of the 17th International Conference on World Wide Web, pp. 1133–1134 (2008)
8. Wang, X.F., Chen, G.: Pinning Control of Scale-Free Dynamical Networks. Physica A 310, 521–531 (2002)
9. Li, X., Wang, X.F., Chen, G.: Pinning a Complex Dynamical Network to Its Equilibrium. IEEE. Trans. Circuits. Syst-I 51, 2074–2086 (2004)
10. Xiang, L.Y., Liu, Z.X., Chen, Z.Q., Yuan, Z.Z.: Pinning Control of Complex Dynamical Networks with General Topology. Physica A 379, 298–306 (2007)
11. Xiang, L.Y., Chen, Z.Q., Liu, Z.X., Chen, F., Yuan, Z.Z.: Stabilizing Weighted Complex Networks. J. Phys. A: Math. Theor. 40, 14369–14382 (2007)
12. Sorrentino, F., di Bernardo, M., Garofalo, F., Chen, G.: Controllability of Complex Networks via Pinning. Phys. Rev. E. 75, 046103 (2007)
13. Sorrentino, F.: Effects of the Network Structural Properties on Its Controllability. Chaos 17, 033101 (2007)
14. Chen, T.P., Liu, X.W., Lu, W.L.: Pinning Complex Networks by a Single Controller. IEEE. Trans. Circuits. Syst-I 54, 1317–1326 (2007)
15. Xiang, J., Chen, G.: On the V-stability of Complex Dynamical Networks. Automatica 43, 1049–1057 (2007)
16. Duan, Z.S., Chen, G., Huang, L.: Complex Network Synchronizability: Analysis and Control. Phys. Rev. E. 76, 056103 (2007)
17. Nishikawa, T., Motter, A.E.: Synchronization is Optimal in Nondiagonalizable Networks. Phys. Rev. E. 73, 065106 (2006)
18. Brualdi, R.A., Ryser, H.J.: Combinatorial Matrix Theory. Cambridge University Press, Cambridge (1991)
19. Rössler, O.E.: An Equation for Continuous Chaos. Phys. Lett. A. 57, 397–398 (1976)