Synchronization in Complex Networks with Different Sort of Communities

Ming Zhao[1], Tao Zhou[1], Hui-Jie Yang[1], Gang Yan[2], and Bing-Hong Wang[1]

[1] Department of Modern Physics, University of Science and Technology of China, Hefei 230026 P.R. China
zhaom17@mail.ustc.edu.cn
[2] Department of Electronic Science and Technology, University of Science and Technology of China, Hefei Anhui, 230026, P.R. China

Abstract. In this paper, inspired by the idea that many real networks are composed by sorts of communities, we investigate the synchronization property of oscillators on such community networks. We identify the communities by two ways, one is by the structure of individual community and the other by the intrinsic frequencies probability density $g(\omega)$ of Kuramoto oscillators on different communities. For the two sorts of community networks, when the community structure is strong, only the oscillators on the same community synchronize. With the weakening of the community strength, an interesting phenomenon appears: although the global synchronization is not achieved, oscillators on the same sort of communities will synchronize independently. Global synchronization will appear with the further weakening of community structure.

Keywords: complex network, community structure, synchronization.

1 Introduction

Many social, physical and biological systems have the structure of networks. It is found recently that these networks are not simply regular lattices or random networks but bear some common and important characters, such as short average distance, large clustering coefficient and the power-law degree distribution [1]. Networks possessing these characters are often called complex networks. Besides these characters, many real-world networks have the so-called community structure [2,3]. Community network can be divided into several subsets of nodes, where the edges within a subset are much denser than those between them. It is worth notice that communities in networks may not be the same sort: they may be different in the structure or in the characters of nodes.

Collective synchronization phenomena have been observed for hundreds of years and also exist in a variety of field, including natural, physical, chemical and biological systems [4]. Because of the limitation of knowledge of networks, the studies of collective synchronization are restricted to either on the regular lattices or the random networks for a long time. Recently, with the development of physical networks, the focus of the studies of collective synchronization shifts

J. Zhou (Ed.): Complex 2009, Part I, LNICST 4, pp. 924–933, 2009.

to dynamical networks with complex structures. Without surprise, it is found that the property of collective synchronization on complex networks is much different from that in regular lattices or random networks. To date, scientists have investigated the relationship between some topological coefficients and the network synchronizability, and found that only the combination of short average distance and homogeneous degree distribution that ensures better network synchronizability [5,6,7,8]. Furthermore, the synchronization on community networks is studied [9,10,11], and community structure is proved to inhibit the global synchronization of oscillators. However, the synchronization property of dynamical networks having different sorts of communities is still unclear.

In this paper, with the help of Kuramoto model [12,13,14,15,16], we investigate the synchronization property of complex networks with different sorts of communities from two aspects: we take the famous Watts-Strogatz (WS) small-world [17] model with different rewiring probability p to represent different community structure, and use the natural frequencies probability density $g(\omega)$ of Kuramoto oscillators to identify the communities.

This paper is organized as follows. In section 2, the Kuramoto model and the order parameter are introduced. In section 3 and section 4, we will give the simulation results of synchronization properties of Kuramoto oscillators on complex networks with the communities different in the structure and the characters of nodes separately. The conclusion remarks are drawn in section 5.

2 Kuramoto Model and the Order Parameter

In this paper, we use the coupled phase oscillators, Kuramoto model, to analyze the collective synchronization on complex networks. A modified Kuramoto model is as follow,

$$\frac{d\phi_i}{dt} = \omega_i - \frac{\sigma}{k_i} \sum_{j \in \Lambda_i} \sin(\phi_i - \phi_j), \tag{1}$$

where ϕ_i, ω_i and Λ_i are the phase, the intrinsic frequency and the neighbor set of node i, respectively, and σ is the coupling strength. ω_i is chosen from the probability density $g(\omega)$.

The order parameter M is defined as

$$M \equiv \left[\left\langle \left| \frac{1}{N'} \sum_{j=1}^{N'} e^{i\phi_j} \right| \right\rangle \right], \tag{2}$$

where $\langle \cdots \rangle$ and $[\cdots]$ denote the average over time and over different configurations, respectively. N' is the number of nodes. In this paper, we not only consider the order parameter of all the nodes in the networks, but also take into account the order parameter of oscillators on individual community and that of the group of nodes composed by the same sort of communities. For different situation, the sum goes over the group of nodes we interested in, and N' is taken accordingly.

3 Communities Identified by Structure

WS small-wold model is built on a low-dimensional regular lattice and then rewire one end of each edge with probability p to create some “shortcuts”. that join remote parts of the lattice to one another. In this procedure, the number of edges in the network keeps fixed, and it has been found that with the rewiring probability p's increasing, the network synchronizability gets stronger [18,19,20]. Thus, take some WS networks with different rewiring probabilities as different sorts of communities, and then add a few edges randomly among them, a community network is composed.

In this paper, we take a one-dimensional lattice of 500 nodes with periodic boundary conditions, and join each node to its neighbors 3 lattice spacings away, rewire one end of all the 1500 edges with probability 0.15, then a small-world model with about 225 short cuts is created. With the same procedure, create another small-wold model with rewiring probability 0.5. These two networks we created have the same size (the same number of nodes and edges), but different structures, and the latter is much easier to synchronize than the former. Create five WS small-world networks of each kind, and take the ten networks as community units, then add some edges among them, a network of 5000 nodes with two sort of communities are composed. The *community strength* C is defined as the ratio of the number of edges between communities (external edges) and in the communities (internal edges), it is used to measure the strength of the community structure. Clearly, a smaller C corresponds to spars external edges thus a stronger community structure.

In the simulation, the phase ϕ_i and the intrinsic frequency ω_i are randomly and uniformly distributed in the intervals $[0, 2\pi]$ and $[0, 1]$ initially. The numerical results are obtained by integrating Eqs. (1) using the Runge-Kutta method with step size 0.01. After 2000 time steps to allow for relaxation to a steady state, the order parameter M are obtained from the average over 2000 time steps. All the presented data are the average over 100 realizations of configurations.

Figure 1 displays the relationship between the order parameter m and the coupling strength σ for different community strength C. From the figure one can see that no matter how strong the community structure is, individual community synchronize soon with the coupling strength's increasing, but the community with larger rewiring probability shows stronger synchronizability than with the smaller one. However, when the community structure is very strong (C is small), although some communities of the network are composed by similar structure units (WS small-world networks with the same p), because of the external edges is very sparse, the coherent phenomenon is not appear among them, and the network part composed by the easier to synchronize community (WS small-world networks with smaller p) show better synchronizability than the other part. When the community strength C is large, the above-mentioned phenomenon is not evident.

When the oscillators are coupled, after some times' iteration, they will rotate at some stable frequencies, which are not equal to their intrinsic frequencies usually. We define them as stable frequencies Ω_i. Figure 2 shows the stable

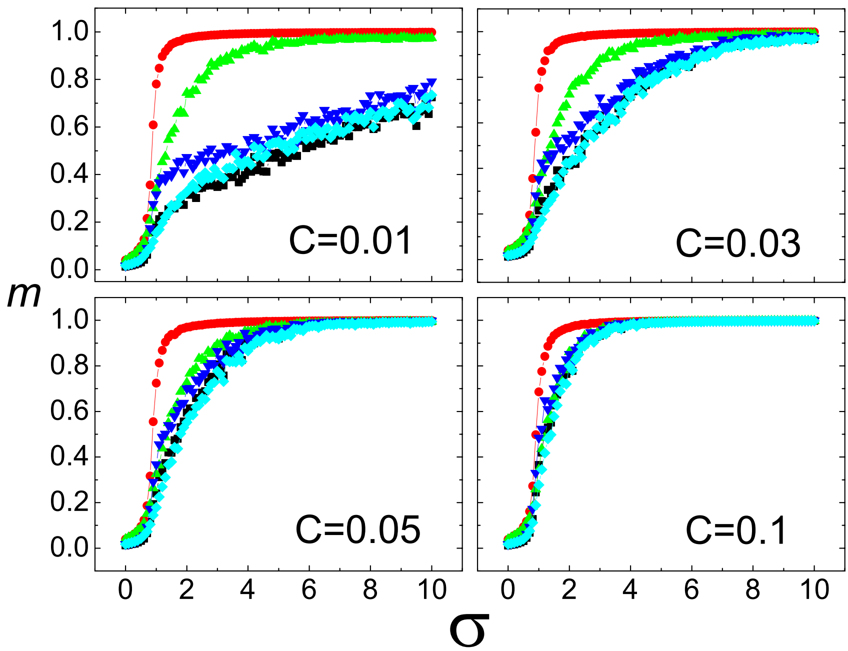

Fig. 1. (color online) The relationship between order parameter m and the coupling strength σ for the community network identified by the community structure. Red circles and green triangles represents the order parameter of the individual community with rewiring probability $p = 0.5$ and $p = 0.15$ separately, blue up triangles and bright blue diamonds represents that of the group of communities that composed by the units of similar structure, and the black squares represent the order parameter of the whole network. There are two sort of communities in the network and the number of communities of each sort are equal.

frequencies Ω_i vs the intrinsic frequencies ω_i at community strength $C = 0.03$.It can be seen that with the coupling strength's increasing, individual community synchronize first, and then the communities composed by the same structure unit and then the whole network, but the communities composed by easier synchronized units also synchronize easier than composed by harder synchronized units, which is consistent with the former results.

Community networks with different size and community number and community sort number have been verified, and there is no essential difference.

4 Communities Identified by the Intrinsic Frequencies Probability Density

Communities can also be identified by the sort of oscillators on the communities. In this paper, intrinsic frequencies probability density $g(\omega)$ are used to brand the communities, i.e., the intrinsic frequencies of the oscillators in different

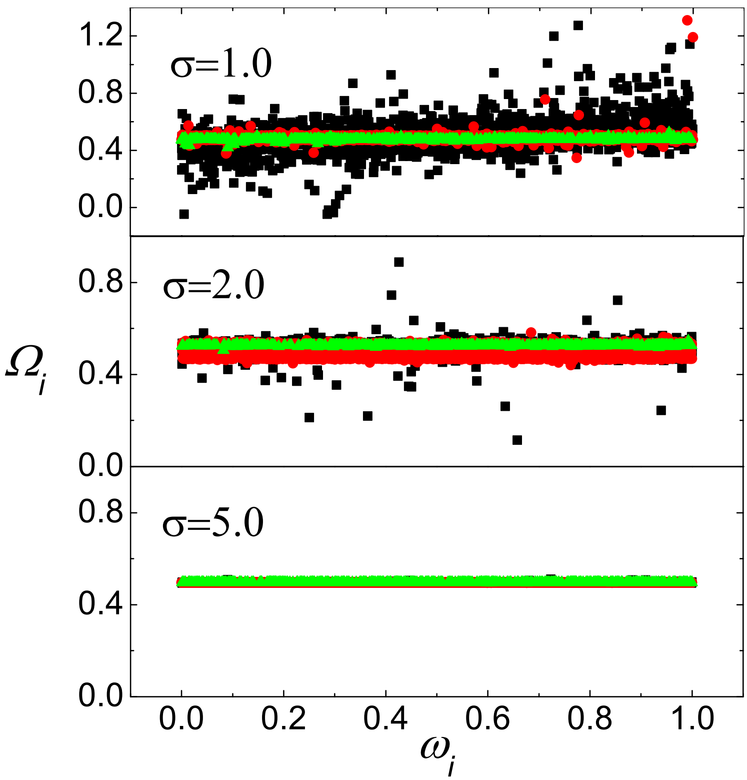

Fig. 2. (color online) The relationship between stable frequencies Ω_i and the intrinsic frequencies ω_i for the community network identified by the community structure. Green triangles, red circles and black squares represent a community that composed by a WS small-world network with $p = 0.5$, the communities that composed by similar structure units with $p = 0.5$ and the communities that composed by similar structure units with $p = 0.15$, respectively. There are two sort of communities in the network and the number of communities of each sort are equal.

communities are distributed with different probability density $g(\omega)$. In this section, we use a toy model that has power-law degree distribution and tunable community strength to simulate the synchronization process.

The model is created as follows. Start from n community cores, each core contains m_0 fully connected nodes. Initially, there are no connections among different community cores. Thus, there is in total n new nodes being added in one time step. Each node will attach m edges to existing nodes within the same community core, and simultaneously m' edges to existing nodes outside this community core. The former are internal edges, and the latter are external edges. Then the network's community strength $C = \frac{m'}{m}$. Similar to the evolutionary mechanism of Barabási-Albert (BA) networks [21], we assume the probability of choosing an existing node i to connect to is proportional to is degree k_i.

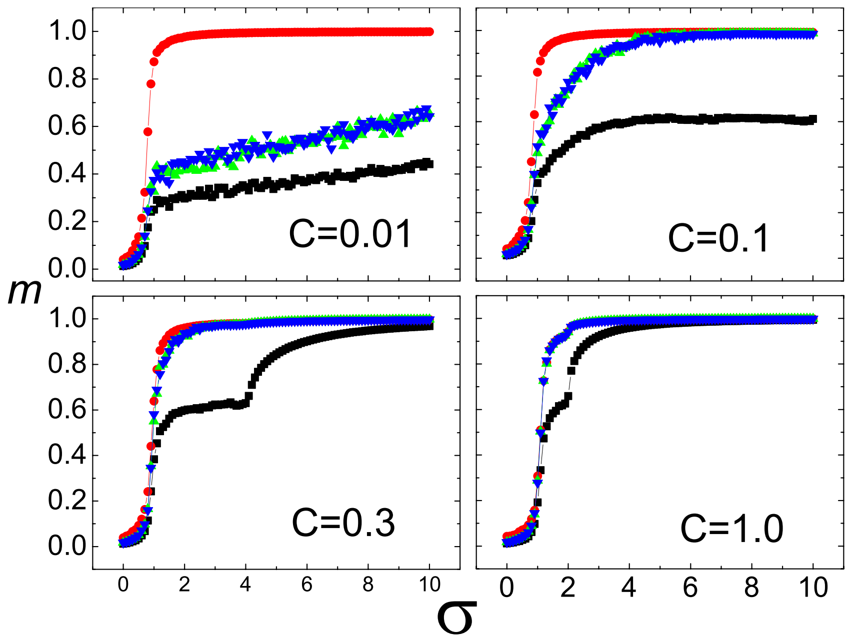

Fig. 3. (color online) (color online) The relationship between order parameter m and the coupling strength σ for the community network identified by the intrinsic frequency distribution probability $g(\omega)$. Red circles represent the order parameters that a community with the oscillators' intrinsic frequencies distributed in $[0, 1]$. Green triangles and blue up triangles represent that of communities that the oscillators intrinsic frequencies distributed in $[0, 1]$ and $[1, 2]$, respectively. And the black squares represent the order parameter of the whole network. There are two sort of communities in the network and the number of each sort are equal.

Each community core will finally become a single community of size N_c, and the network size $N = nN_c$. By using the rate-equation approach [22], one can easily obtain the degree distribution of the whole network, $p(k) \propto k^3$.

From 10 community cores, and with $m + m' = 3$, network of 5000 nodes are created and the average degree is $\bar{k} = 6$. Five communities are located Kuromoto oscillators with intrinsic frequencies distributed randomly and uniformly distributed in the interval $[0, 1]$, and the other five in $[1, 2]$. Figure 3 shows the simulation results. It is easily drawn that for stronger community structure, the oscillators on each community are phase synchronized but the phase synchronization dose not emerge between different communities. With the community structure's weakening, phase synchronization appear among the same sort of communities but not the whole network and then all the oscillators are synchronized.

Figure 4 shows the relationships between stable frequencies and intrinsic frequencies of each oscillator at different coupling strength for $C = 0.3$. From the figure it can be seen that for a weaker coupling strength, the stable frequencies Ω_i

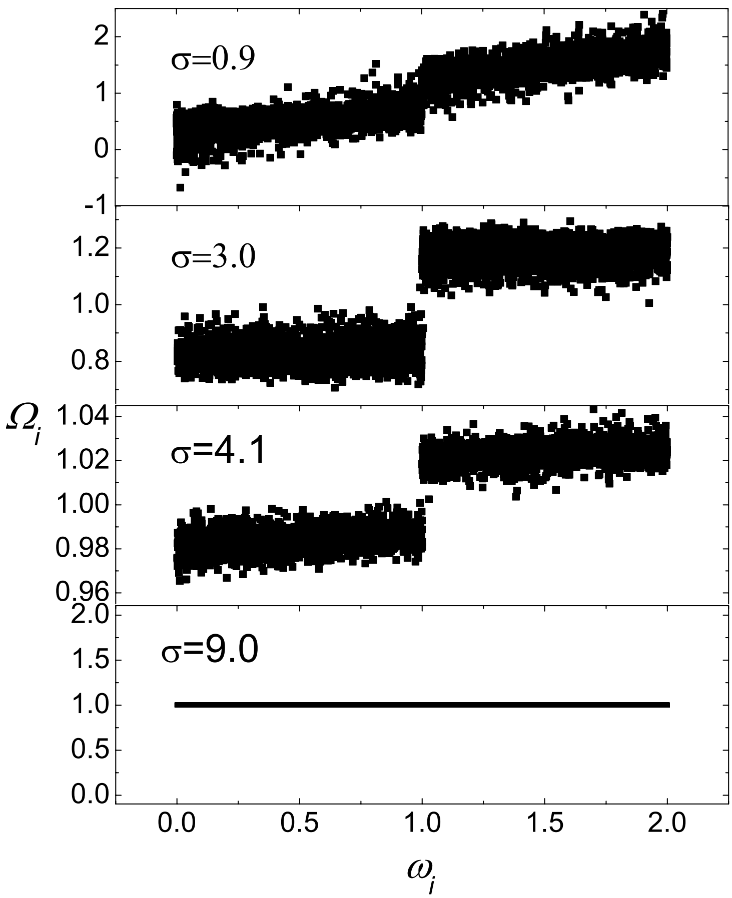

Fig. 4. The relationship between stable frequencies Ω_i and the intrinsic frequencies ω_i for the community network identified by the community structure. There are two sort of communities in the network and the number of each sort are equal.

are almost equal to the intrinsic frequencies ω_i, increasing the coupling strength σ will make oscillators on communities with the same intrinsic frequency distribution rotate at about their own average frequencies separately. Increasing the coupling strength furthermore will make all the oscillators rotate at almost the same frequency and then the synchronization is realized.

We also test the community networks with different size and composed by communities with more structure sorts but with each sort the same number of communities, and no essential difference is found. However, we notice that when the number of communities are not equal between different sort of communities, the synchronization property is not the same as the conclusion drawn above. Figure 5 shows the order parameter m vs coupling strength σ in network with two sort of communities. The meaning of geometry configurations in the figure

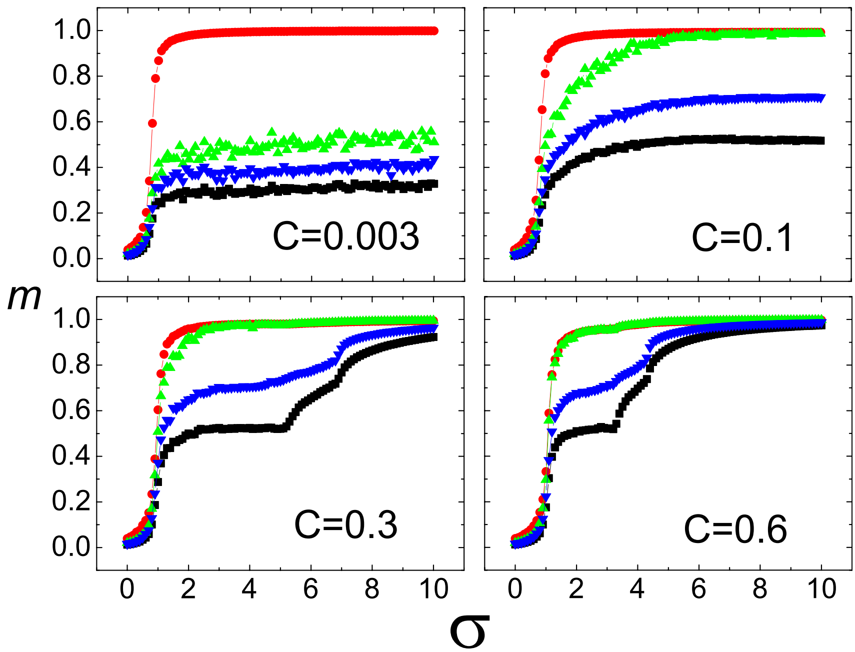

Fig. 5. (color online) (color online) The relationship between order parameter m and the coupling strength σ for the community network identified by the intrinsic frequency distribution probability $g(\omega)$. There are two sort of communities in the network and the ratio of the number of each sort are 2:3.

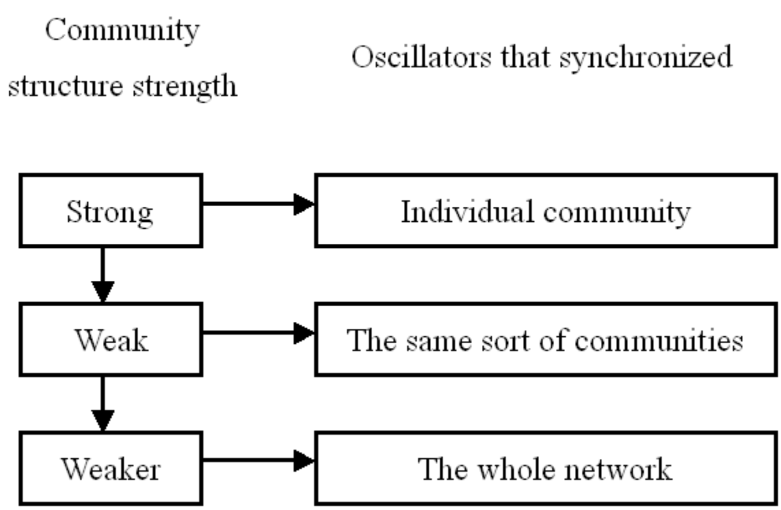

Fig. 6. With the community structure strength changes from strong to weaker, the synchronization phenomenon appears on the individual community, the same sort of communities and then the whole network.

is same to figure 3, but the number of oscillators whose intrinsic frequencies distribute in $[0, 1]$ are 2000, and the number of the others are 3000. It is clearly seen that the sort of communities with small number synchronize much easier.

5 Conclusion and Discussion

In conclusion, we investigated the synchronization properties in complex networks with different sorts of communities and found that when the community structure is strong, only the oscillators on the same community synchronize, with the weakening of the community strength, oscillators on the same sort of communities will synchronize independently, only when the community structure is not evident that all the oscillators on the network can synchronize, this is clearly shown in figure 6. Furthermore, when the communities are identified by their structure, communities composed by easier to synchronize unit also show strong synchronizability. However, when the communities are identified by the intrinsic frequency distribution probability, the sort of communities that have smaller number of unit are more synchronizable.

Acknowledgments. This work is supported by the Specialized Research Fund for the Doctoral Program of Higher Education (SRFD No. 20070420734), and the National Natural Science Foundation of China (Grant No. 10805045).

References

1. Albert, R., Barabási, A.L.: Statistical mechanics of complex networks. Rev. Mod. Phys. 74, 47 (2002)
2. Girvan, M., Newnam, M.E.J.: Community structure in social and biological networks. Proc. Natl. Acad. Sci. U.S.A. 99, 8271 (2002)
3. Palla, G., Derényi, I., Farkas, I., Vicsek, T.: Uncovering the overlapping community struc e ture of complex networks in nature and society. Nature 435, 814 (2005)
4. Strogatz, S.H.: SYNC-How the emerges from chaos in the universe, nature, and daily life. Hyperion, New York (2003)
5. Nishikawa, T., Motter, A.E., Lai, Y.-C., Hoppensteadt, F.C.: Heterogeneity in Oscillator Networks: Are Smaller Worlds Easier to Synchronize? Phys. Rev. Lett. 91, 14101 (2003)
6. Hong, H., Kim, B.J., Choi, M.Y., Park, H.: Factors that predict better synchronizability on complex networks. Phys, Rev. E 69, 067105 (2004)
7. Donetti, L., Hurtado, P.I., Muñoz, M.A.: Entangled Networks, Synchronization, and Optimal Network Topology. Phys. Rev. Lett. 95, 188701 (2005)
8. Zhao, M., Zhou, T., Wang, B.-H., Yan, G., Yang, H.-J., Bai, W.-J.: Relations between average distance, heterogeneity and network synchronizability. Physica A 371, 773–780 (2006)
9. Oh, E., Rho, K., Hong, H., Kahng, B.: Modular synchronization in complex networks. Phys. Rev. E 72, 047101 (2005)
10. Zhou, T., Zhao, M., Chen, G., Yan, G., Wang, B.-H.: Phase synchronization on scale-free networks with community structure. Phys. Lett. A 368, 431 (2007)

11. Park, K., Lai, Y.-C., Gupte, S., Kim, J.-W.: Synchronization in complex networks with a modular structure. Chaos 16, 015105 (2006)
12. Kuramoto, Y.: Araki, H. (ed.) Proceedings of the International Symposium on Mathematical Problems in Theoretical Physics. Springer, New York (1975)
13. Kuramoto, Y.: Chemical Oscillations, Waves, and Turbulence. Springer, Berlin (1984)
14. Kuramoto, Y., Nishikawa, I.: Statistical Macrodynamics of Large Dynamical Systems. Case of a Phase Transition in Oscillator Communities. J. Stat. Phys. 49, 569 (1987)
15. Daido, H.: Population Dynamics of Randomly Interacting Self-Oscillators. I. Prog. Theor. Phys. 77, 622 (1987)
16. Daido, H.: Quasientrainment and slow relaxation in a population of oscillators with random and frustrated interactions. Phys. Rev. Lett. 68, 1073 (1992)
17. Watts, D.J., Strogatz, S.H.: Collective dynamics of small-world networks. Nature 393, 440 (1998)
18. Lago-Fernández, L.F., Huerta, R., Corbacho, F., Sigüenza, J.A.: Fast Response and Temporal Coherent Oscillations in Small-world Networks. Phys. Rev. Lett. 84, 2758 (2000)
19. Wang, X.F., Chen, G.: Synchronization in small-world dynamical networks. Int. J. Bifurcation Chaos Appl. Sci. Eng. 12, 187 (2002)
20. Barahona, M., Pecora, L.M.: Synchronization in Small-World Systems. Phys. Rev. Lett. 89, 054101 (2002)
21. Barabási, A.-L., Albert, R.: Emergence of scaling in random networks. Science 286, 509 (1999)
22. Krapivsky, P.L., Render, S., Leyvraz, F.: Connectivity of Growing Random Networks. Phys. Rev. Lett. 85, 4629 (2000)