Scaling Relations in Absorbing Phase Transitions with a Conserved Field in One Dimension

Sang-Gui Lee and Sang Bub Lee*

Department of Physics, Kyungpook National University,
Daegu, 702-701, Republic of Korea
sblee@knu.ac.kr

Abstract. Validity of two scaling relations $\beta = \nu_{\parallel}\theta$ and $z = \nu_{\parallel}/\nu_{\perp}$ widely known in absorbing phase transitions is studied for the conserved lattice gas (CLG) model and the conserved threshold transfer process CTTP) both in one dimension. For the CLG model, it is found that both relations hold when the critical exponents calculated from the all-sample average density of active particles are considered. For the CTTP model, various exponents are calculated via Monte Carlo simulations and they are confirmed by the off-critical scaling and the finite-size scaling analyses. The exponents estimated from the all-sample averages again satisfy both relations. These observations are in strict disagreement with earlier observations in two dimensions [Phys. Rev. Lett. **85**, 1803 (2000); Phys Rev. E **68**, 056102 (2003)] but support the more recent observation for the CLG model [Phys. Rev. E **78**, 040103(R) (2008)].

Keywords: absorbing phase transition, conserved lattice gas, conserved threshold transfer process, critical exponents, scaling relations.

1 Introduction

Nonequilibrium, continuous phase transition from a fluctuating active phase to single or multiple absorbing states has attracted great attention during last several decades [1,2,3,4]. So far, only few universality classes were identified; i.e., the directed percolation (DP) class [5,6,7,8] and the parity conserving (PC) class [9,10,11,12] were firmly established, but the pair-contact process with diffusion (PCPD) class is still under controversy [13,14,15,16]. The triplet and quadruplet reaction-diffusion models were also claimed to belong to the new universality class [17,18]. Besides these classes, a new universality class was proposed by Rossi *et al.* for the models with a conserved field, generated from the symmetry that the order parameter is locally coupled to a short-range nondiffusive conserved field [19]. The conserved lattice gas (CLG) model, the conserved threshold transfer process (CTTP), and the stochastic sandpile model were found to belong to this universality class [20,21,22].

In usual critical phenomena, many physical quantities near criticality are known to be described by the power-law behaviors [23,24]. For example, for the

* Corresponding author.

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models with a conserved field, the density of active particles exhibits a power-law decay against the evolution time, and the steady-state density in the supercritical region also yields a power-law behavior against the distance from criticality. The universality classes are classified by the powers of such behaviors, known as critical exponents. The critical exponents are not independent but they are linked according to the scaling relationships derived from the scaling hypothesis. The two scaling relations widely known in absorbing phase transitions are

$$\beta = \theta \nu_{\parallel} \tag{1}$$

and

$$z = \nu_{\parallel} / \nu_{\perp}, \tag{2}$$

where β and θ are the exponents associated with, respectively, the order parameter against the distance from criticality and the density of active particles in time, $\nu_{\parallel}$ and $\nu_{\perp}$ are the exponents characterizing temporal and spatial correlation lengths, and z is the dynamic exponent.

Rossi *et al.* [19] claimed that the former relation broke the "simple" scaling for the CLG model in two dimensions. Failure of the scaling was first reported on a sandpile model by Vespignani *et al.* [25] and was also discussed in other works [22]. Lübeck and his collaborators also reported supported results [26,27]. Very recently, however, the present authors studied the CLG model in one dimension and found that the former scaling relation held precisely but the latter relation was questioned [28]. Further extensive works in two dimensions revealed that both relations held but the exponent $\nu_{\perp}$ calculated from the surviving-sample averages were found to be inconsistent with the value in the thermodynamic limit of an infinite size system. Similar results were also found for the CTTP model [29].

In the CLG model, each lattice site may be occupied by at most one particle, and a particle is defined to be active if it has at least one particle in the nearest-neighbor sites; otherwise, it is inactive. The dynamics proceeds with the hopping of active particles to one of the nearest-neighbor empty sites. In the CTTP model, on the other hand, each lattice site may be occupied by up to two particles, and the doubly occupied sites are assumed to be active sites. In each process, particles on each active site attempt to hop to randomly selected nearest-neighbor inactive sites. If such inactive site is not available, the particle on an active site does not move. The density of active particles (sites) are considered to be the order parameter for the CLG (CTTP) model. The critical behaviors of the CLG model and the CTTP model in one dimension are particularly important by two reasons. Firstly, because of the simplicity of the models by dimensional reduction, the critical density and some of the critical exponents for the CLG model are known exactly, thus, enabling one to examine the scaling relations. Indeed, de Oliviera calculated the exponents θ, β, and $\nu_{\perp}$ analytically [30], and the present authors estimated the rests of the exponents very accurately by Monte Carlo simulations [28]. Secondly, the universality split between the CLG model and the CTTP model is known to occur in one dimension, whereas in higher dimensions the two models are known to belong to the same universality class [31]. The cause of the universality split is known to be the different hopping mechanisms for the two models. For the CLG model, since only one of the two nearest-neighbor sites is empty, the hopping is deterministic in one dimension, while for the CTTP model the hopping is stochastic because the direction of hopping may be selected randomly when both neighboring sites are inactive and at least one of them is empty.

In the previous work of the present authors [28], the scaling relations in Eqs. (1) and (2) were examined with the known exponents for the CLG model in one dimension, and the former relation was found to be satisfied while the latter was not. This observation is in disagreement with the more recent work in two dimensions, where both relations were found to hold precisely [29]. In this paper, the scaling relations will be reexamined for the CLG model and the CTTP model both in one dimension. For the CLG model, the known, presumably exact, results will be used in the scaling analyses. For the CTTP model, various critical exponents will be calculated and crosschecked via the off-critical and finite-size scaling analyses. Since the scaling relations were examined in the previous work only for the CLG model and the CLG model and the CTTP model are known to exhibit different behaviors in one dimension, the present work completes examinations of the validity of the scaling relations for the models with a conserved field. (Note that in two and higher dimensions all variant models with a conserved field are known to belong to the same universality class.) It is found that both scaling relations appear to hold precisely for both models, but the critical exponent $\nu_\perp$ obtained from the surviving-sample averages appears to be invalid in the thermodynamic limit, in agreement with the earlier observation [29].

In Sec. 2, the known scaling theory will be reviewed briefly, introducing the off-critical and finite-size scaling functions. In Secs. 3 and 4, the results for the CLG model and the CTTP model will be presented, with appropriate discussions. The concluding remarked will be made in the last section.

2 Scaling Theory

In both the CLG model and the CTTP model, initially ρL particles are distributed randomly, following the rules for each model as described in Sec. 1, in a given system of size L. If the selected site is already occupied by the allowed number of particles, the particle is not added and a new site is selected. As the dynamics proceeds, the density of active particles, ρ_a, decreases in time due to the repulsive contribution of hopping of active particles. If the density of particles, ρ, is too small, all particles eventually become inactive and, if ρ is sufficiently large, ρ_a saturates. Therefore, there exists a critical density ρ_c at which ρ_a decreases, following the power law

$$\rho_a \sim t^{-\theta}, \tag{3}$$

θ being the decay exponent of the density of active particles. For $\rho > \rho_c$, ρ_a converges to the steady-state density $\rho_{\rm sat}$, which exhibits the power-law behavior against the distance from criticality, i.e.,

$$\rho_{\text{sat}} \sim (\rho - \rho_c)^{\beta} \tag{4}$$

for $\rho > \rho_c$, where β is the order-parameter exponent. Therefore, the off-critical values of ρ_a depend on the evolution time and the distance from criticality via the correlation time $\tau \sim |\rho - \rho_c|^{-\nu_\parallel}$. Thus,

$$\rho_a(t) = t^{-\theta}\mathcal{F}(t/\tau) \equiv t^{-\theta}\mathcal{F}(t|\rho - \rho_c|^{\nu_\parallel}), \tag{5}$$

where $\mathcal{F}(x)$ is the universal off-critical scaling function. Since $\rho_a \to \rho_{\text{sat}}$ in the $t \gg \tau$ limit, the scaling relation in Eq. (1) follows. On the other hand, for a finite system, since the correlation length cannot exceed the size of system L near criticality, i.e., $\xi \sim |\rho - \rho_c|^{-\nu_\perp} \sim L$, it is obtained that $|\rho - \rho_c| \sim L^{-1/\nu_\perp}$. Therefore, Eq. (5) becomes

$$\rho_a(t) \sim t^{-\theta}\mathcal{G}(t/L^z) \tag{6}$$

with the aid of Eq. (2), where $\mathcal{G}(x)$ is the finite-size scaling function. If, on the other hand, one focuses on the late time, Eq. (6) can be rewritten as

$$\rho_a(t) \sim L^{-\beta/\nu_\perp}\mathcal{H}(t/L^z), \tag{7}$$

where $z\theta = \beta/\nu_\perp$ has been used.

The scaling in Eq. (7) is the simple scaling claimed to be broken by an anomalous exponent θ. This scaling is particularly useful when ρ_a saturates in the $t \gg L^z$ region; thus, at ρ_c, Eq. (7) yields $\rho_a(t) \to \rho_{\text{sat}} \propto L^{-\beta/\nu_\perp}$ as $t \to \infty$, which enables one to estimate the value of $\beta/\nu_\perp$.

3 Conserved Lattice Gas Model

For the CLG model in one dimension, the absorbing phase is one of the two states $010101\cdots$ and $101010\cdots$ at the critical density of $\rho_c = \frac{1}{2}$. Using the simplicity of dynamics, de Oliveira obtained $\beta = 1$ by considering the density of nearest-neighbor pairs of occupied sites close to ρ_c. He also obtained $\nu_\perp$ using the transfer matrix approach; although he proposed the classical result of $\nu_\perp = \frac{1}{2}$, the present authors pointed out that his result was erroneous and the correct one was $\nu_\perp = 1$ (see Ref. [25] of Ref. [28]). The rests of the exponents were obtained by the present authors via the numerical simulations. Summarizing the results, they are

$$\theta = \frac{1}{4}, \quad \beta = 1, \quad \nu_\parallel = 4, \quad \nu_\perp = 1, \quad z = 2.$$

It should be noted that the value of θ was obtained by direct simulations of ρ_a against the evolution time at ρ_c, and the value of $\nu_\parallel$ was obtained from the best data collapse for the scaling function in Eq. (5) and was also confirmed by the scaling of the persistence distribution, i.e., the distribution of average time that the system persists in one of the phases, *e.g.*, in the phase that the density of active particles is larger than the mean density of active particles. With these results, it is clear that the relation in Eq. (1) holds.

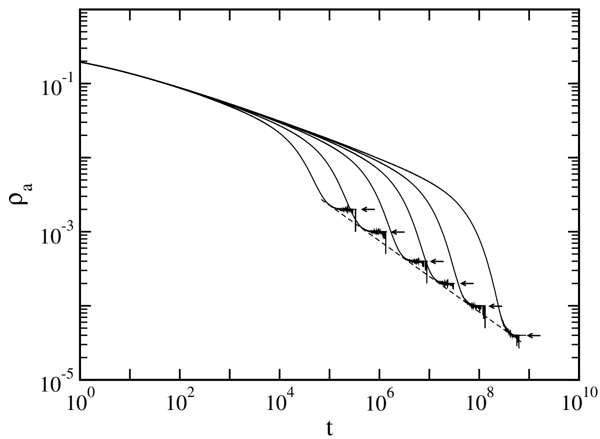

Fig. 1. The surviving-sample averages of $\rho_a(t)$ against the evolution time at the critical density $\rho_c = 0.5$ for the CLG model in one dimension. Data are for, from left to right, $L = 1000$, 2000, 5000, 10000, 20000, and 50000.

The value of z was obtained from the best collapse of the data for the finite-size scaling analysis of Eq. (6) using the all-sample average data. [Note that the all-sample average is the average over all samples attempted, whereas the surviving-sample average is the average over the samples remaining in an active phase up to the time of interest.] The same result was also conjectured from the spreading of the active particles in the close vicinity of the absorbing state [28]. On the other hand, the exponent $\nu_\perp = 1$ was obtained by transfer matrix approach by de Oliveira [30] and was also obtained from the data for $\rho_{\rm sat}$ for systems of various sizes L at ρ_c [29]. Since Eq. (7) focuses the steady-state density of active particles in the late-time limit, the surviving-sample average should be used to estimate the value $\beta/\nu_\perp$ because the all-sample average densities decay in the long-time limit. Figure 1 shows the data for $\rho_a(t)$ for L ranging from $L = 1000$ to $L = 50000$ at ρ_c. The density of active particles for a given size system yields the power-law behavior in the early time, decays sharply, then saturates in the long time region, and eventually falls into the absorbing state. In the close vicinity of an absorbing state, the active particles consist of a single dimer, i.e., $\rho_{\rm sat} = \frac{2}{L}$, which implies $\beta/\nu_\perp = 1$ or, equivalently, $\nu_\perp = 1$. The saturation values for various L (marked as arrows in Fig. 1) are precisely $\frac{2}{L}$. With the estimate of $\nu_\perp$ and the values of $\nu_\parallel$ and z, the scaling relation in Eq. (2) appears to be violated. On the contrary, if one assumes that the scaling relation in Eq. (2) is correct, then, $z = \frac{\nu_\parallel}{\nu_\perp} = 4$ would be obtained.

In the previous works in which the simple scaling was claimed to be broken by an anomalous value of θ, the value of z was obtained from the finite-size scaling analysis of Eq. (6), the value of $\nu_\perp$ was measured from the data for $\rho_{\rm sat}$ at ρ_c against the size of system L, and the scaling in Eq. (7) was examined with the surviving-sample average data. We here test the scaling in Eq. (7) using both

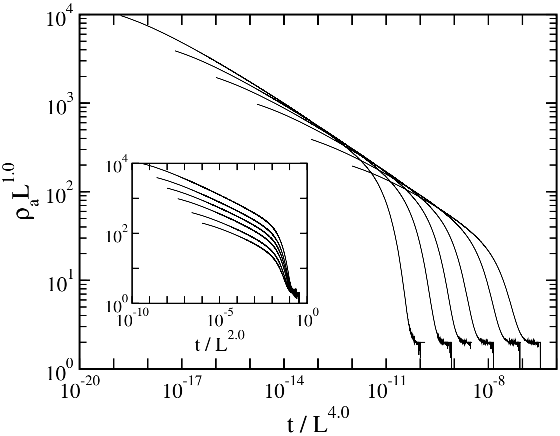

Fig. 2. The scaled density of active particles $\rho_a L^{\beta/\nu_\perp}$ for the surviving-sample averages against the scaled time t/L^z, using $\beta/\nu_\perp = 1$ and $z = 4$. The inset is the same data scaled with $\beta/\nu_\perp = 1$ and $z = 2$.

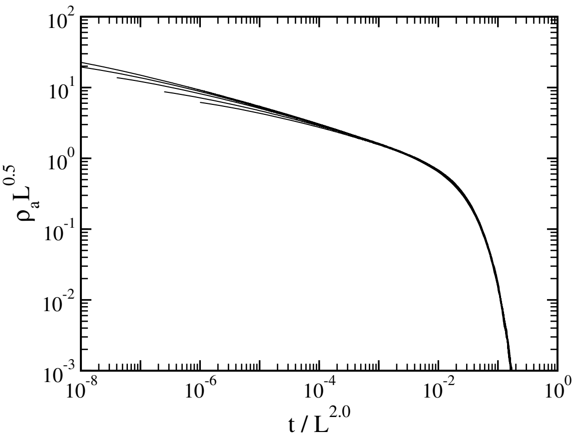

Fig. 3. The scaled density of active particles $\rho_a L^{\beta/\nu_\perp}$ for the all-sample averages against the scaled time t/L^z, using $\beta/\nu_\perp = \frac{1}{2}$ and $z = 2$

$z = 2$ and $z = 4$. In Fig. 2, the main plot is the scaling function $\rho_a(t)L^{\beta/\nu_\perp}$ against the evolution time scaled with $z = 4$ and the inset is the same data against the time scaled with $z = 2$. It is clear that both plots do not show data collapsing, implying that at least one or more critical exponents used in the scaling analysis might be incorrect. In the previous work of the present authors in two dimensions, it was found that the scaling was failed due to the two time scales for the surviving-sample averages, one at the first inflection point (which appears on both the all-sample data and the surviving-sample data) at $t = L^z$

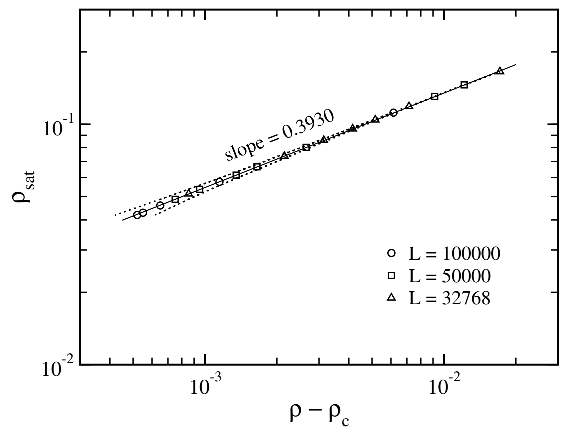

Fig. 4. The double logarithmic plot of the steady-state density $\rho_{\rm sat}$ against the distance from criticality, with $\rho_c = 0.98285$. Two dotted lines which veer up and veer down are the trial plots for $\rho_c^t = 0.98295$ (above) and for 0.98275 (below), and the solid line is the regression fit over the data for $\rho_c = 0.98285$.

with $z = 2$ and the other at the time when saturation sets in, i.e., at $t = L^{z'}$ with $z' \neq z$. The value of z' may be estimated from the point that the data of ρ_a touches the dashed line in Fig. 1 and is found to be $z' \approx 2.15$.

When the same scaling is examined with the all-sample data, the value $\nu_\perp = 1$ should not be used because it was measured with the surviving-sample data. Instead, it should be obtained from Eq. (2), i.e., $\nu_\perp = \nu_\parallel / z = 2$. (Note that the value $z = 2$ was obtained from the finite size scaling of the all-sample data.) Figure 3 shows the all-sample averages of $\rho_a(t)$ scaled by $L^{-\beta/\nu}$ against the scaled time, using $\beta/\nu_\perp = \frac{1}{2}$ and $z = 2$. It is clear that scaling holds perfectly, indicating that the exponents used in this analysis are correct. This assures us that the exponent $\nu_\perp$ obtained from the surviving-samples is not valid.

4 Conserved Threshold Transfer Process

For the CTTP model in one dimension, the critical exponents β, θ, $\nu_\parallel$, $\nu_\perp$, and z were calculated by Lübeck and Heger [27] and, with the values, the scaling relation in Eq. (1) was found to be violated. (In fact, Lübeck and Heger assumed that the relation in Eq. (1) was invalid and they obtained $\nu_\parallel$ from Eq. (2) using the estimates of $\nu_\perp$ and z.) However, since most of the estimates were not crosschecked by alternative methods, such as the scaling analyses in Eqs. (5), (6), and (7), it is necessary to recalculate the exponents in order for the close examination of the scaling relations.

The critical density was estimated by Lübeck and Heger [27] from the best power-law fit of $\rho_{\rm sat}$ against $\rho - \rho_c$ using the surviving-sample average data up to predetermined time steps. Since for any finite size systems there remain some

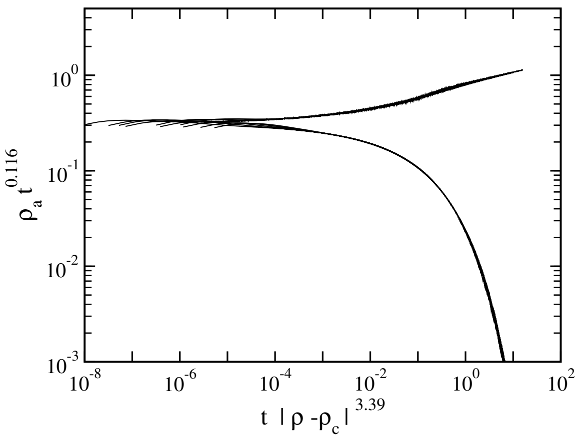

Fig. 5. The off-critical scaling function of the active site density $\rho_a t^\theta$ against the scaled time $t|\rho - \rho_c|^{\nu_\parallel}$ for $\rho = 0.955$, 0.96, 0.965, 0.97, and 0.975 (below) and for $\rho = 0.987$, 0.989, 0.99, 0.995, and 1.0 (above) with $\theta = 0.116$ and $\nu_\parallel = 3.39$, and $\rho_c = 0.98285$, for the CTTP model in one dimension.

surviving samples even below criticality, the value of ρ_c estimated in this way with surviving samples would be underestimated, because the densities for which samples remain in an active phase would be assumed supercritical. In this work, ρ_c is predetermined from the power-law behavior of ρ_a and, with this value, the power law of $\rho_{\rm sat}$ in the supercritical region is analyzed. Since we do not know an accurate value of ρ_c, we use only the data which are the same for all-sample averages and surviving-sample averages. [Note that two samples are identical if no sample falls into absorbing states.] The predetermined value which yields the best results for the test is determined as true ρ_c.

The critical density is obtained as $\rho_c = 0.98285$, with the regression slope $\beta = 0.393(4)$, as shown in Fig. 4. The two dotted curves which veer up and veer down are for the trial densities $\rho_c^t = 0.98295$ (above) and 0.98275 (below), and the solid line is the power-law fit over the data using $\rho_c = 0.98285$. It is thus sufficient to estimate ρ_c up to four significant digits. The critical density obtained is larger than that by Lübeck and Heger, but the exponent β is close to the known values for the stochastic fixed-energy sandpile model, $\beta = 0.42$ [32], and for CTTP model by Lübeck and Heger, $\beta = 0.382$ [27]. The regression slope of $\rho_a(t)$ yields the exponent $\theta = 0.116(3)$, which is slightly smaller than that by Lübeck and Heger, the difference being apparently attributed to the larger value of ρ_c. With the estimates of β and θ, it follows $\nu_\parallel = \beta/\theta \simeq 3.39(4)$.

In order to crosscheck the value of $\nu_\parallel$, the off-critical scaling in Eq. (5) is examined for the data of $\rho_a(t)$. Plotted in Fig. 5 are the scaled densities $\rho_a(t)t^\theta$ against the scaled time $t|\rho - \rho_c|^{\nu_\parallel}$ using $\theta = 0.116$ and $\nu_\parallel = 3.39$. It is clear that data for various densities fall on the two separate curves, one for $\rho > \rho_c$ (above) and the other for $\rho < \rho_c$ (below), indicating that the scaling indeed holds. This

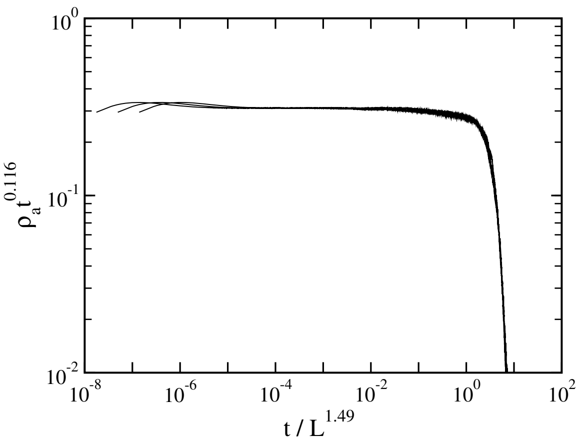

Fig. 6. The finite-size scaling function of the active site density $\rho_a t^\theta$ against the scaled time t/L^z for systems of sizes $L = 40000$, 80000, 160000 for the CTTP model in one dimension

confirms that the value of $\nu_\parallel$ is correct and, accordingly, the scaling relation in Eq. (1) holds.

The finite size scaling in Eq. (6) is also tested with the trial value of z. Plotted in Fig. 6 is the scaled density of active sites $\rho_a(t)t^\theta$ against the scaled time t/L^z for three selected sizes of systems, i.e., for $L = 4\times10^4$, 8×10^4, and 1.6×10^5. Data for different size systems exhibit the best collapse for $z = 1.49$, but the quality of scaling is not as good as for the off-critical scaling. If we, however, choose the size of system twice as large as the largest size selected, i.e., $L = 3.2 \times 10^5$, data near the inflection point deviate slightly. Similar behavior was also observed previously for the CTTP model on a checkerboard fractal substrate [33]. We believe this to be that the scaling region is relatively narrow for the CTTP model. Accepting $z = 1.49$, it is obtained that $\nu_\perp = \nu_\parallel/z = 2.28$. In Table 1, the estimates are summarized and compared with the estimates by Lübeck and Heger [27]. It should be noted that, with the estimates in the present work, we did not find any precursor of the violation of scaling relations and, thus, we believe that our estimates are correct and both relations in Eqs. (1) and (2)

Table 1. Summary of the critical exponents for the CLG model and the CTTP model both in one dimension, in comparison with the reported results. The values for the CLG model are conjecture to be exact.

Exponents	θ	β	$\nu_\parallel$	$\nu_\perp$	z
CLG model	$\frac{1}{4}$	1	4	2	2
CTTP model	0.116	0.393	3.39	2.28	1.49
Ref. [27]	0.141	0.382	2.452	1.760	1.393

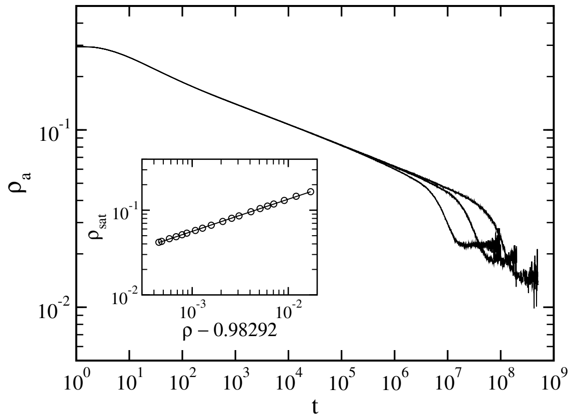

Fig. 7. The surviving sample averages of $\rho_a(t)$ at the trial value $\rho_c^t = 0.98292$ for $L = 25000$, 50000 for the CTTP model in one dimension

hold. It should also be noted that the scaling relation in Eq. (1) does not hold with the estimates by Lübeck and Heger.

In order to test the scaling in Eq. (7), the surviving-sample average data for ρ_a are also calculated for the CTTP model. Selecting the size of system as a multiple of 2 of the base size, i.e., L_0, $2L_0$, 2^2L_0, $\cdots$, with L_0 ranging from 10^3 to as large as 2×10^4, the steady-state values at ρ_a should be placed with an equal vertical spacing of $(\beta/\nu_\perp)\ln 2$ on a double logarithmic plot if the scaling $\rho_{\rm sat} \propto L^{-\beta/\nu_\perp}$ holds. For our estimate of $\rho_c = 0.98285$, the steady-state value does not exhibit such behaviors. We also calculate $\rho_{\rm sat}$ for different trial values of ρ_c. It is found that the steady-state densities are displayed with nearly equal spacing for $\rho_c^t = 0.98292$, as shown in Fig. 7 for $L = 2.5 \times 10^4$, 5×10^4, and 10^5. However, choosing the larger size $L = 2 \times 10^5$ or smaller size $L = 1.25 \times 10^4$, the steady-state value is no longer equally spaced and, with this value of ρ_c, the power-law fit of $\rho_{\rm sat}$ against $\rho - \rho_c$ becomes worse, though the exponent β is found to be similar. Therefore, it is not possible to estimate $\nu_\perp$ with the surviving-sample averages of $\rho_{\rm sat}$ for the CTTP model in one dimension. Similar behavior was also observed for the CTTP model on a checkerboard fractal substrate. Since the dynamics of the CTTP model on a checkerboard fractal is more likely one-dimensional as was claimed in Ref. [33], it is plausible that the critical behaviors of the CTTP model on a one-dimensional lattice and on a checkerboard fractal are similar. Indeed, some of the exponents on a checkerboard fractal are similar to those estimated in the present work.

5 Concluding Remarks

The two widely known scaling relations in absorbing phase transitions are examined carefully for the CLG model and the CTTP model both in one dimension. For the CLG model, it was found that both relations appeared to hold when the exponents obtained using the all-sample average data of ρ_a are considered. It was also found that the scaling in Eq. (7) was broken when the value of $\nu_\perp$ obtained from the surviving-sample averages was used, as was observed in the earlier works in two dimensions. Such a failure of the scaling appeared to be due to that the two time scales existing in the surviving-sample average ρ_a, i.e., one at the first inflection point where the finite-size effect first comes into the system and the other at the time when the saturation sets in, are different. The latter time exists only on the surviving-sample averages and appears to have influenced the value of $\nu_\perp$. In the thermodynamic limit of $L \to \infty$, since the second time scale associated with the saturation diminishes and a single value of $\nu_\perp$ would be obtained. It is believed that this value of $\nu_\perp$ is identical to that obtained from the finite-size scaling analysis with all-sample average data. Therefore, with the exponents valid in the thermodynamic limit, both scaling relations appear to hold.

For the CTTP model, we estimated the critical exponents β, θ, $\nu_\parallel$ and z from direct simulations and off-critical scaling and finite size scaling analyses. During the analyses, there was no precursor of violation of the scaling relations when all-sample average data were used. For the surviving sample averages for the CTTP model, it appeared that the scaling in Eq. (7) did not hold and, as a consequence, the value of $\nu_\perp$ was not obtained.

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References

1. Marro, J., Dickman, R.: Nonequilibrium Phase Transitions in Lattice Models. Cambridge University Press, Cambridge (1999)
2. Hinrichsen, H.: Adv. Phys. 49, 815 (2000); Hinrichsen, H.: Braz. J. Phys. 30, 69 (2000)
3. Ben-Avraham, D., Havlin, S.: Diffusion and Reaction in Fractals and Disordered Systems. Cambridge University Press, Cambridge (2000)
4. Ódor, G.: Rev. Mod. Phys. 76, 663 (2004)
5. Janssen, H.K.: Z. Phys. B: Condens. Matt. 42, 151 (1981)
6. Grassberger, P.: Z. Phys. B Condens. Matt. 47, 365 (1982)
7. Jensen, I., Dickman, R.: Phys. Rev. E 48, 1710 (1993); Jensen, I.: J. Phys. A 27, L61 (1994)
8. Muñoz, M.A., Grinstein, G., Dickamn, R., Livi, R.: Phys. Rev. Lett. 76, 451 (1996)
9. Takayasu, H., Tretyakov, A.Y.: Phys. Rev. Lett. 68, 3060 (1992)
10. Jensen, I.: Phys. Rev. E 50, 3623 (1994)
11. Kwon, S., Park, H.: Phys. Rev. E 52, 5955 (1995)
12. Cardy, J., Tauber, U.C.: Phys. Rev. Lett. 77, 4780 (1996)

13. For the bosonic mversion of the PCPD model, Howard, M.J., Täuber, J.C.: J. Phys. A 30, 7721 (1997); for the fermionic version, Carton, E., Henkel, M., Schollwöck, U.: Phys. Rev. E 63, 36101 (2001)
14. Kochelkoren, J., Chate, H.: Phys. Rev. Lett. 90, 125701 (2002)
15. Noh, J.D., Park, H.: Phys. Rev. E 69, 016122 (2004)
16. Hinrichsen, H.: Physica A 361, 457 (2006)
17. Park, K., Hinrichsen, H., Kim, I.-M.: Phys. Rev. E 66, 025101 (2002)
18. Ódor, G.: Phys. Rev. E 67, 056114 (2003)
19. Rossi, M., Pastor-Satorras, R., Vespignani, A.: Phys. Rev. Lett. 85, 1803 (2000)
20. Manna, S.S.: J. Phys. A 24, L363 (1991)
21. Dhar, D.: Physica 263A, 4 (1999)
22. Vespignani, A., Dickman, R., Muñoz, M.A., Zapperi, S.: Phys. Rev. E 62, 4564 (2000)
23. Stanley, H.E.: Introduction to Phase Transitions and Critical Phenomena. Oxford University Press, London (1971)
24. Amit, D.J.: Field Theory, the Renormalization Group, and Critical Phenomena. World Scientific, Singapore (1984)
25. Vespignani, A., Dickman, R., Muñoz, M.A., Zapperi, S.: Phys. Rev. Lett. 81, 5676 (1998); Dickman, R., Vespignani, A., Zapperi, S.: Phys. Rev. E 57, 5095 (1998)
26. Lübeck, S., Misra, A.: Eur. Phys. J. B 26, 75 (2002)
27. Lübeck, S., Heger, P.C.: Phys. Rev. E 68, 56102 (2003); Lübeck, S.: ibid 66, 046114 (2002)
28. Lee, S.-G., Lee, S.B.: Phys. Rev. E 77, 021113 (2008)
29. Lee, S.B., Lee, S.-G.: Phys. Rev. E 78, 040103(R) (2008)
30. de Oliveira, M.J.: Phys. Rev. E 71, 016112 (2005)
31. Lübeck, S.: Phys. Rev. E 64, 16123 (2001); Lübeck, S.: ibid 66, 46114 (2002)
32. Dickman, R., Alava, M., Muñoz, M.A., Peltola, J., Vespignani, A., Zapperi, S.: Phys. Rev. E 64, 056104 (2001)
33. Lee, S.-G., Lee, S.B.: Phys. Rev. E 77, 041122 (2008)