The Probability Distribution of Inter-car Spacings*

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Abstract. In this paper, the celluar automation model with Fukui-Ishibashi-type acceleration rule is used to study the inter-car spacing distribution for traffic flow. The method used in complex network analysis is applied to study the spacings distribution. By theoretical analysis, we obtain the result that the distribution of inter-car spacings follows power law when vehicle density is low and spacing is not large, while, when the vehicle density is high or the spacing is large, the distribution can be described by exponential distribution. Moreover, the numerical simulations support the theoretical result.

Keywords: FI model, Power Law, Exponential distribution, Complex network, Spacings distribution.

1 Introduction

Celluar automata(CA) has been considered as a model system in a wide variety of problems and phenomena in statistical physics and other fields[1]. Because it is conceptual simple, easy to use in simulation by computer and can model the complex behavior of the traffic flow, the CA became popular for the microscopic simulation of traffic flow. Two famous CA models are the Nagel-Schreckenberg (NS)model and the Fukui -Ishibashi(FI) model[2]. The CA model share several basic features: the lane is represented as one dimension lattices; the lattices sites are called "cell" which can be either empty or occupied by at most one vehicle at a given instant time, at every time step the speed of every vehicle will be updated following a well defined rule, the main difference between these models is the particular procedure implemented to change the speed of a vehicle. In the FI model, the vehicle can move by v_{max} (v_{max} is the maximum speed of the vehicle) cells at most in one time step if they are not blocked by cars in front. More precisely, if the empty cells $C_n(t)$ in front of a car is larger than v_{max} at time t, then the speed can be update to $v_{max} - 1$ by probability p and v_{max} by $1 - p$. The FI model differs from the NS model in that the increase in speed may not be gradual, and that stochastic delay only applies to high speed cars. In this paper we will consider the FI model due to its simplicity and leads to considerable improvement of the flow for higher velocities.

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Complex network has received much attention from the physics community in recent years[5][6][7]. Employing the theory and method of complex networks, one can hope that it provides some insight into the evolution mechanism of traffic flow. Some work such as [4], study the scale free property in traffic system by numerical simulation. In general, CA model is used to study the traffic system by computer simulation. It is difficult to study the traffic flow by theoretical analysis based on CA model, since in paper [3], author give some result for the traffic character by theoretical analysis and simulation. In this paper we study the inter-car spacings distribution using the methods used in the complex network study. Our result give a clear picture for the spacings distribution which is meaningful for the traffic flow theory.

The paper is organized as follows. In section 2 we give the model definition and theoretical analysis of spacings distribution. In section 3, the simulation results for $p = 0.5$ and all various density are given and compared with the theoretical results. At last, we summary with a discussion of our method.

2 Model and Analytical Solution of Inter-car Spacings Distribution

The modified Nagel-Schrechenberg model working with the Fukui-Ishibashi Acceleration rule is a probabilistic cell automation . In this model, the space, time and velocities are discrete. The one dimensional horizontal road with traffic flowing is divided into L cell. Each cell may either be empty, or may be occupied by a vehicle with speed $0, 1, 2, ...v_{max}$, v_{max} is the maximum speed of the vehicle. If there are N vehicles in the length L road, then the vehicle density is $\rho = N/L$.The speed of nth vehicle at time t is defined as $v_n(t)$,$n \in \{0, 1, 2, v_{max}\}$. If $C_n(t)$ represents the number of empty sites in front of nth vehicle at time t, then the spacings in front of nth vehicle can be calculated by

$$C_n(t + 1) = C_n(t) + v_{n+1}(t) - v_n(t) \tag{1}$$

Let $M = v_{max}$, the speed of vehicle v_n is updated through the following rules:

$$v_n(t+1) = \begin{cases} 0 & \text{if } C_n(t) = 0 \\ C_n(t) - 1 & \text{with probability } p\text{,if } 0 < C_n(t) < M \\ C_n(t) & \text{with probability } 1-p\text{,if } C_n(t) > M \\ M - 1 & \text{with probability } p \text{ ,if } C_n(t) \geq M \\ M & \text{with probability } 1-p\text{,if } C_n(t) \geq M \end{cases} \tag{2}$$

If $N_k(t)$ represents the number of inter-car spacings with length k at time t, the probability of finding such a spacing at time t is $P_k(t) = \frac{N_k(t)}{N}$. We assume that $t \to \infty$ is equivalent to $L \to \infty$ and $P_k = \lim_{t\to\infty} P_k(t)$. Next, we will derive the formula for P_k under different condition.

Let $\mathcal{F}_t = \sigma\{N_k(t), 0 \leq k \leq L - N)\}$, and from time step t to $t+1$ the change number of spacing k is $N_k(t + 1) - N_k(t)$, then we can calculate the condition expectation $E(N_k(t + 1) - N_k(t)|\mathcal{F}_t)$ as follows

$$E(N_k(t+1) - N_k(t)|\mathcal{F}_t) = \sum_{i=-N}^{N} iP(N_{k+1}(t) - N_k(t) = i|\mathcal{F}_t) \tag{3}$$

Let

$$\begin{aligned} Q_k^1(t) &= P(N_k(t+1) - N_k = 1|\mathcal{F}_t) \\ Q_k^2(t) &= P(N_k(t+1) - N_k = -1|\mathcal{F}_t) \end{aligned} \tag{4}$$

then $Q_k^1(t)$ and $Q_k^2(t)$ can be calculated by the following formulas:

$$\begin{aligned} Q_0^1(t) &= \sum_{i=1}^{M}(1-p)P_i(t)(P_0(t) + pP_1(t)) \\ Q_0^2(t) &= P_0(t)(1 - P_0(t) - pP_1(t)) \\ Q_1^1(t) &= P_0(t)[(1-p)P_1(t) + pP_2(t)] + \sum_{i=2}^{M} P_i(t)(1-p)[(1-p)P_1(t) + pP_2(t)] \\ &\quad + \sum_{i=2}^{M} P_i(t)p[P_0(t) + pP_1(t)] + P_{M+1}(t)(1-p)[P_0(t) + pP_1(t)] \\ Q_1^2(t) &= 1 - P_1(t)((1-p)((1-p)P_1(t) + pP_2(t)) - p(P_0(t) + pP_1(t)) \end{aligned} \tag{5}$$

when $2 \leq k \leq M$

$$\begin{aligned} Q_k^1(t) = P_0(t)&[(1-p)P_k(t) + pP_k + 1(t)] \\ &+ \sum_{i=1, i\neq k}^{M} P_i(t)(1-p)[(1-p)P_k(t) + pP_{k+1}(t)] \\ &+ \sum_{i=1, i\neq k}^{M} P_i(t)p[(1-p)P_{k-1} + pP_k] \\ &+ \sum_{i=1}^{k-1} P_{M+i}(t)(1-p)[(1-p)P_{k-i}(t) + pP_{k-i+1}(t)] \\ &+ \sum_{i=1}^{k-2} P_{M+i}(t)p[(1-p)P_{k-i-1}(t) + pP_{k-i}(t)] \\ &+ [P_{M+k}(t)(1-p) + P_{M+k-1}(t)p][P_0(t) + pP_1(t)] \\ Q_k^2(t) = 1 - &P_k(t)(1-p)[(1-p)P_k(t) \\ &+ p(P_{k+1}(t)] - P_k(t)p[(1-p)P_{k-1}(t) + pP_k(t)] \end{aligned} \tag{6}$$

For $k \geq M+1$

$$Q_k^1(t) = P_{k-1}(t)p(1-p)\sum_{i=M}^{L-N} P_i(t)$$

$$Q_k^2(t) = P_k(t)p(1-p)\sum_{j=M}^{L-N} P_j(t) \tag{7}$$

From (3), one can obtain the condition expectation as follows.

$$E(N_k(t+1) - N_k(t)|\mathcal{F}_t) = Q_k^1(t) - Q_k^2(t) + \delta(k,t) \tag{8}$$

Where the $\delta(k,t), (k = 0,1,2,\cdots L-N)$ is a small value, especially when k is large. The reason is that the probability $P(N_k(t+1) - N_k(t) = i|\mathcal{F}_t)$ and $P(N_k(t+1) - N_k(t) = -i|\mathcal{F}_t)$,$i \geq 2$ decrease rapidly with the i increasing. For example,$P(N_k(t+1) - N_k(t) = 2|\mathcal{F}_t) = (Q_k^1)^2$,because $0 \leq Q_k^1 < 1$ is a small value, then $(Q_k^1)^2$ is smaller .When $k > M$,for two cases we derive the formulas for P_k.

Case I: the vehicle density $\boldsymbol{\rho}$ is small and $\boldsymbol{k}$ is not very large
From t step to $t+1$ step, the $N_k(t)$ updating to $N_k(t+1)$ are decided by the probability $P_k(t) = \frac{kN_k(t)}{L-N}$. From (7)(8), we have

$$\begin{aligned} E(N_k(t+1) - N_k(t)|\mathcal{F}_t) = &\frac{(k-1)N_{k-1}(t)}{L(1-\rho)}p(1-p)\sum_{i=M}^{L-N}\frac{N_i(t)}{L(1-\rho)} \\ &-\frac{kN_k(t)}{L(1-\rho)}p(1-p)\sum_{j=M}^{L-N}\frac{jN_j(t)}{L(1-\rho)} + \delta_{k,t} \end{aligned} \tag{9}$$

Let $I_k(t) = E(N_k(t))$, it follows from (9) that

$$\begin{aligned} I_k(t+1) = &I_k(t)[1 - \frac{kp(1-p)}{L-N}\sum_{j=M}^{L-N}\frac{jN_j(t)}{L-N}] \\ &+\frac{(k-1)I_{k-1}(t)p(1-p)}{L-N}\sum_{j=M}^{L-N}\frac{jN_j(t)}{L-N} \end{aligned} \tag{10}$$

Because $\sum_{j=M}^{L-N}\frac{jN_j(t)}{L-N} = 1 - \frac{1}{L-N}\sum_{j=1}^{M-1}(jN_j(t)) \to C$ when $t \to \infty (L \to \infty)$ then we have the following equation

$$\begin{aligned} P_k &= \frac{(k-1)P_{k-1}p(1-p)C\frac{1}{\frac{1}{\rho}-1}}{1 + k\frac{Cp(1-p)}{\frac{1}{\rho}-1}} \\ &= \frac{(k-1)P_{k-1}p(1-p)C}{1/\rho - 1 + kCp(1-p)} \end{aligned} \tag{11}$$

Let $a = p(1-p)C$, from (11), we have

$$P_k = \frac{(k-1)P_{k-1}}{k + (1-\rho)/a\rho} \tag{12}$$

Solving this equation we obtain the result

$$P_k \approx \frac{1}{k^{1+\frac{1-\rho)}{a\rho}}} \tag{13}$$

Case II: The vehicle density ρ is large or $k \gg M$
From t step to $t+1$ step, the $N_k(t)$ updating to $N_k(t+1)$ are decided by the probability $P_k(t) = \frac{N_k(t)}{N}$, then from (7)(8), we have

$$E(N_k(t+1) - N_k(t)|\mathcal{F}_t) = \frac{N_{k-1}(t)}{N} p(1-p) \sum_{i=M}^{L-N} \frac{N_i(t)}{N} - \frac{N_k(t)}{N} p(1-p) \sum_{i=M}^{L-N} \frac{N_i(t)}{N} + \delta_{k,t} \tag{14}$$

Let $b_M(t) = \sum_{j=M}^{L-N} \frac{N_j(t)}{N}$, by (14) we have following equation

$$I_k(t+1) = I_k(t)[1 - \frac{p(1-p)b_M(t)}{N}] + \frac{I_{k-1}(t)p(1-p)b_M(t)}{N} \tag{15}$$

Because $b_M(t) \to b$ when $t \to \infty$,and $N = L\rho \to \infty$, we have the equation

$$P_k = \frac{bp(1-p)P_{k-1}}{1 + p(1-p)b} \tag{16}$$

Solving the equation, we obtain

$$P_k \approx (\frac{bp(1-p)}{1 + p(1-p)b})^k = e^{-kln(1+\frac{1}{bp(1-p)})} \tag{17}$$

Based on the above induction we can find that, when $k > M$,P_k can be represented as

$$P_k = \begin{cases} C_0 \frac{1}{k^\alpha} & \text{if } M < k < k^* \\ C_1 e^{-\lambda k} & \text{if } k > k^* \end{cases} \tag{18}$$

Where k^* relates to the density ρ. By the simulation in the next section, we can find that k^* decreases to M gradually while ρ is increasing. That is, at a point of ρ, P_k can be represented by exponential distribution completely.

3 Simulation

We now proceed to present computer simulation and compare the result with the theoretic result. At the beginning, the vehicle number N is chosen as 5000 and the road length L is adjusted so as to the vehicle density ρ can have desired value. The time length is $t^* = 50000$. In order to guarantee $L \to \infty$ increases with t and the density ρ unchanged, the simulations follow the rule that if a car leave the road then a new car will be added into. P_k are calculated by the formula as follows:

$$P_k = \frac{\sum_{t=1}^{t^*} N_k(t)}{Nt^*}, k > M. \tag{19}$$

Let $x_k = lnP_k$ and $y_k = k$, it is obvious that if (x_k, y_k) are in a line, then the distribution is $P_k = C_0 e^{-\lambda k}$. If $x_k = \ln P_k$ and $y_k = \ln k$, (x_k, y_k) are in a line, then the data fit the power law $P_k = C_1 \frac{1}{k^\alpha}$.

The simulation result are presented to the Fig 1 $\sim$ 5.

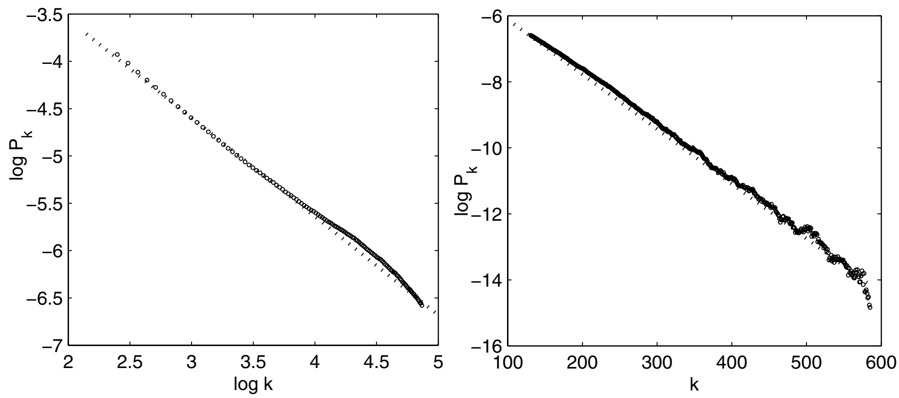

Fig. 1. Show that P_k follows power law when while$10 < k < 130$, while it is exponential distribution when $k \geq 130$. Where $v_{max} = 5, p = 0.5$.

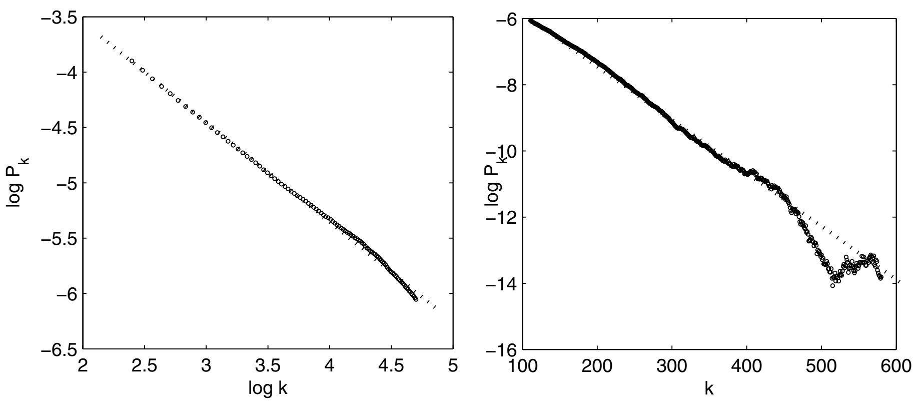

Fig. 2. Show that theP_k follows power law when $10 < k < 100$, while it is exponential distribution when $k \geq 100$. Where $v_{max} = 4, p = 0.5$.

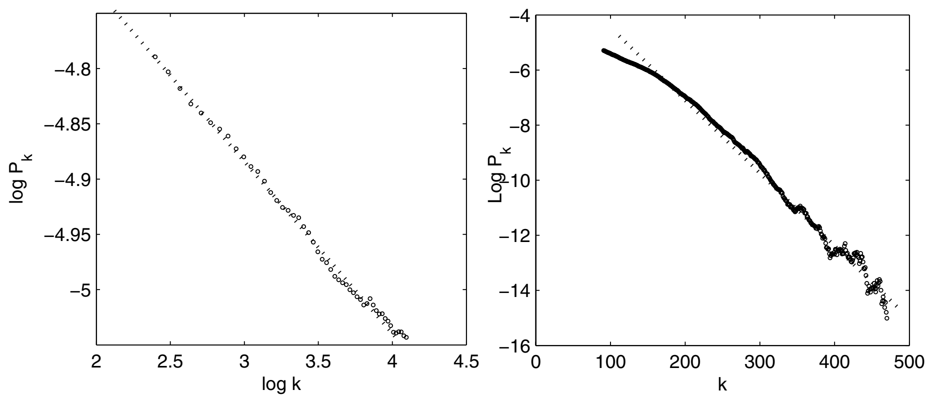

Fig. 3. Show that P_k follows power law when $0 < k < 60$, while it is exponential distribution when $k \geq 60$. Where $v_{max} = 3, p = 0.5$.

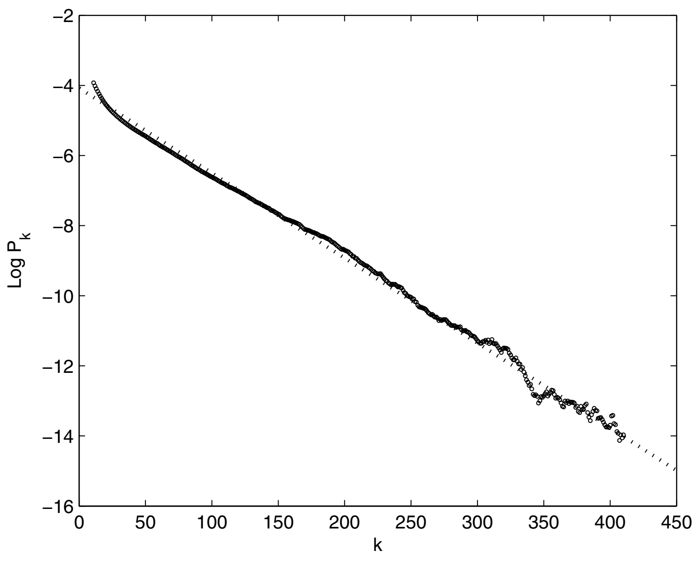

Fig. 4. Show that P_k follows exponential distribution when $k \geq 10$. Where $v_{max} = 2, p = 0.5$.

4 Notes and Comments

CA model have been widely used to model the traffic system, due to its simplicity for computer simulation. In our study, the methods for complex network analysis are applied to study the traffic flow characters. The inter-car spacings distribution is studied by theoretical analysis and numerical simulation. Our

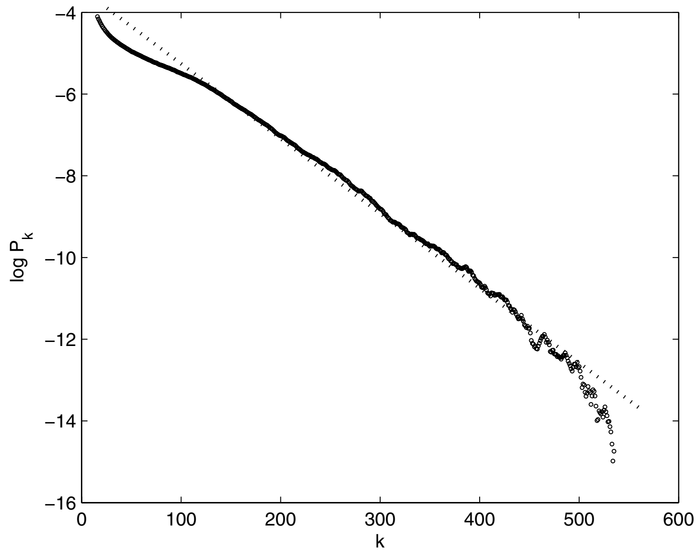

Fig. 5. Show that P_k follows exponential distribution when $k \geq 10$. Where $v_{max} = 1, p = 0.5$.

study methods may give some elicitation for studying the complexity of traffic flow.

Through the theoretical analysis and numerical simulation, the evolution of traffic flow has a clear picture. If the vehicle density is low, the distribution P_k shows the power law when $M < k < k^*$, k^* varying with density ρ. This tell us that traffic flow has scale free characteristic when the vehicle density is low. With the density increasing, such as $v_{max} = 1, 2$, P_k turn into complete exponential distribution. The simulation result coincide with the the theoretical result very well.

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