Modular Synchronization in Complex Network with a Gauge Kuramoto Model

C. Choi[1], E. Oh[1,2], B. Kahng[1], and D. Kim[1]

[1] Department of Physics and Astronomy and Center for Theoretical Physics, Seoul National University, Seoul 151-747, Korea
bkahng@snu.ac.kr
[2] Bioanalysis and Biotransformation Research Center, Korea Institute of Science and Technology, Seoul 136-791, Korea

Abstract. We modify the Kuramoto equation(KE) by introducing a gauge term which is a function of link betweenness centrality(BC). The gauge term induces the phase difference from 0 to π between two nodes that belong to different modules. Therefore, a synchronization occurs in each module individually even though the whole network is not synchronized globally. By measuring the phase similarity of all pairs of connected nodes, we can detect the modular structure of complex networks. This algorithm requires relatively little computational time $O(NL)$ for network with N nodes and L links.

Keywords: synchronization, module identification, modular complex network, Kuramoto model.

1 Introduction

Every system that has constituents and relationships between themselves can be represented by networks conceptually. Constituents of the system are nodes and their interactions are links which connect a node and another node in networks. Many physical and social systems have been studied via networks.

Among many kinds of dynamics on complex networks, synchronization is one of the most popular subjects. Synchronization is a process of adjusting some properties assigned to each node via interactions between the elements of complex network. This is so useful that it has been investigated in several kinds of fields - physics, biology, sociology, etc.

Module is a set of densely-connected nodes in complex network. Modular complex network has several modules and these modules are connected each other with relatively sparse links.

Synchronization in modular complex network shows some new features compared to synchronization in complex network having no module structure. Links are dense within modules and sparse between modules. So in synchronization process nodes in same module are synchronized first by intra-module links and the whole network becomes synchronized later by inter-module links or not. We reported on this phenomenon in our previous paper [1]. In this lecture note we follow the outline of the paper briefly and all figures in here are also taken from it.

J. Zhou (Ed.): Complex 2009, Part I, LNICST 4, pp. 429–434, 2009.

2 Module Identification

Before we introduce our method to indentify modules of complex network, it is also helpful to review some of the previous methods that are regardless of synchronization.

q-state Potts model [2] is very useful method in statistical mechanics. It concerns about spin-spin interactions. Spins assigned to each node in complex network have q states for spin value and they interact with the nearest neighbors nodes. When the energy of the whole system is minimized, it is likely to that nodes in same module have same value of spin.

Girvan and Newman algorithm(GN) [3] is one of the most famous module-detecting algorithms. Link BC is an important concept for GN algorithm. It is the number of paths between every pair of nodes that pass through the link. One calculates BC for all links and removes the link which has the maximum BC value. Repeating this step until the modularity Q of the network is maximized one can get module structures of the network. Due to the recalculation of BC at each step, the computational time of GN algorithm is relatively high and scales as $O(L^2 N)$, where L is the number of links and N is the number of nodes.

Clauset, Newman and Moore(CNM) [4] introduced a hierarchical agglomeration algorithm for detecting module structure. Its computational time scales to $O(L \log^2 N)$ for sparse networks.

3 Detecting Algorithms in Synchronization

Using synchronization to detect modular structure is another dynamic algorithm as well as q-state Potts model. Arenas *et al.* [5] showed oscillators of nodes in different modules are synchronized in different time scales and they are ordered hierarchically. But one must choose the characteristic time, t_c at which modular structure is determined.

And Boccaletti *et al.* [6] introduced another dynamic clustering algorithm called opinion changing rate (OCR) model. The dynamics of node are governed by

$$\frac{d\theta_i}{dt} = \omega_i - \frac{J}{\sum_{j \in \mathrm{nn}(i)} b_{ij}^{\alpha(t)}} \sum_{j \in \mathrm{nn}(i)} b_{ij}^{\alpha(t)} \sin(\theta_i - \theta_j) \beta e^{-\beta |\theta_j - \theta_i|}, \tag{1}$$

where ω is the natural frequency of node i, σ is the coupling strength and b_{ij} is BC of the link between node i and node j. By adjusting the parameter $\alpha(t)$ and β one can tune the interaction coupling between neighbouring nodes.

4 Kuramoto Model with Gauge Term

We introduce a modified KE [7] which we call gauge KE,

$$\frac{d\phi_i(t)}{dt} = \Omega_i - J \sum_{j=1}^{N} a_{ij} \sin(\phi_i(t) - \phi_j(t) - \eta g(b_{ij})). \tag{2}$$

Here ϕ_i is the phase of node i, Ω_i is the natural frequency of node i selected from the Gaussian distribution, $e^{-\Omega^2/2}/\sqrt{2\pi}$, J is the overall coupling constant and a_{ij} is the (i,j)-th component of the adjacency matrix, which is one when the node i and j are connected, and zero otherwise. η is a control parameter. When $\eta = 0$ the equation is equal to the original KE and we will set $\eta = 1$ for our algorithm. The gauge term $g(b_{ij})$ is defined as

$$g(b_{ij}) = \frac{b_{ij} - b_{\min}}{b_{\max} - b_{\min}}\pi, \tag{3}$$

where $b_{\min}$ ($b_{\max}$) is the minimum (maximum) value of link BC. Usually BC is relatively large for inter-module links and small for intra-module links. So inter-module links have large gauge term and their couplings become negative in Eq.(2). This means that two different modules which are connected with inter-module links have different average phases and velocities. This property enables us to employ gauge KE as modular structure detecting algorithm.

The followings are 4 steps of our algorithm.

i) With sufficiently large coupling constant J, we obtain the phases $\phi_i(t)$ of each oscillator in the steady state.

ii) We measure the phase similarity defined as $C_{ij} = \langle [1 + \cos(\phi_i(t) - \phi_j(t))]/2 \rangle$ for each links. The brackets are the average over different times, natural frequencies Ω_i and initial random phases $\phi_i(0)$.

iii) From the state in which every link is removed, we connect links one by one in descending order of C_{ij}.

iv) We repeat the step *iii*) until the modularity of the system becomes maximum. The modularity Q is defined as

$$Q = \sum_{\alpha} e_{\alpha\alpha} - a_{\alpha}^2, \tag{4}$$

where $a_\alpha = \sum_\beta e_{\alpha\beta}$, and $e_{\alpha\beta}$ is the fraction of edges that connect the nodes belonging to the modules α and β [8].

5 Simulation Results

We know the degree of synchronization of oscillators on the network quantitatively by measuring order parameter M defined as $\mathcal{M}_{\text{tot}} \equiv \langle | \sum_{j=1}^{N} e^{i\phi_j}/N | \rangle$, where $\langle \cdots \rangle$ means time average and ensemble average. We measure the order parameter in the steady state. If $\eta = 0$, the order parameter converges to 1 for sufficiently large J. As η is increased to 1, it has lower values than 1 as shown in Fig. 1(a).

We use *ad hoc* network for simulation. The network has four equal-sized modules. And the number of nodes is $N = 128$ and the number of links is $L = 1024$. We can adjust z_{out} which is the mean degree of inter-modular links.

We define the local order parameter as $\mathcal{M}_\alpha \equiv \langle | \sum_{j=1}^{N_\alpha} e^{i\phi_j}/N_\alpha | \rangle$, where α is the module index, N_α is the number of nodes within the module α and the sum

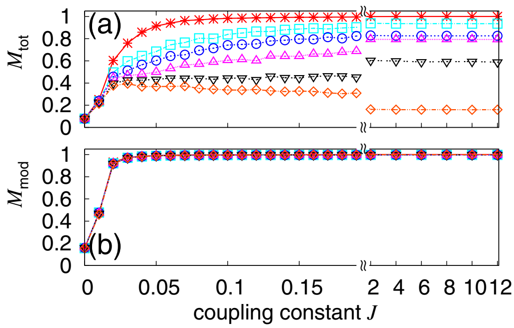

Fig. 1. The order parameter defined over the entire network (a) and within a module (b) versus the coupling constant J for the *ad hoc* network in case of $z_{out}/\langle k \rangle = 0.05$. Data are for $\eta = 0.0$, 0.6, 0.7, 0.8, 0.9 and 1.0 from the top in (a). The same symbols are used for (b), but data for different η collapse onto the single curve.

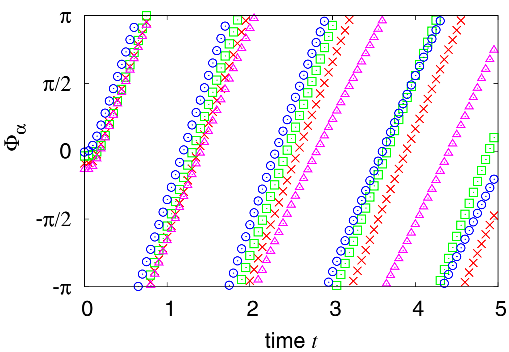

Fig. 2. The time evolution of average phases of the four modules, distinguished by different symbols, for the *ad hoc* network with $z_{out}/\langle k \rangle = 0.05$ when $\eta = 1.0$ and $J = 2.0$

is over nodes within the module. By comparing $\mathcal{M}_{tot}$ and $\mathcal{M}_\alpha$ (Fig. 1(b)) we can verify that the synchronization occurs within each module first.

The average phase of each module as a function of time is shown in Fig. 2. The modules are distinguishable because of the different average phases. The differnce of slopes indicates that the average phase velocities of each module are also different from each other.

To test performance of gauge KE model, we measure the mutual information defined as

$$I(A,B) = \frac{-2\sum_{i=1}^{M}\sum_{j=1}^{M'} \log(\frac{N_i^j}{N_i N^j})}{\sum_{i=1}^{M} N_i \log(\frac{N_i}{N}) + \sum_{j=1}^{M'} N^j \log(\frac{N^j}{N})} \tag{5}$$

where $M = 4$ is the number of preassigned modules and M' is the number of detected modules. N_i^j is the number of nodes belonging to the i-th preassigned and the j-th detected modules, $N_i = \sum_j N_i^j$ and $N^j = \sum_i N_i^j$. To test performances

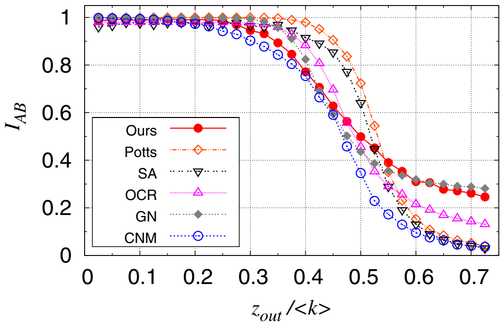

Fig. 3. The mutual information versus $z_{\text{out}}/\langle k \rangle$, the fraction of inter-modular edges per mean degree for the *ad hoc* network

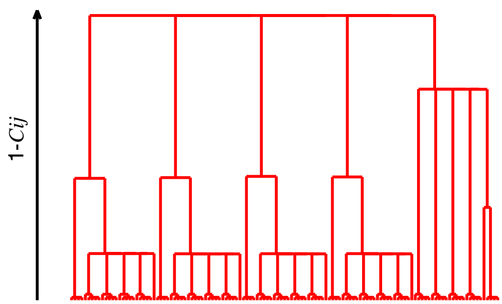

Fig. 4. The dendrogram based on the phase similarity between connected pairs of vertices for the hierarchical network with three levels

of several module-detecting algorithm we measure the mutual information on *ad hoc* networks [9] with several z_{out} values as shown in Fig. 3. The q state Potts model and the simulated annealing(SA) [10] are better than our algorithm in performence. However, if we consider the computational time, then ours($O(NL)$) may be useful for large scale networks in practical view. Our algorithm also does not need to tuning any parameters whereas OCR algorithm requires an extra task of parameter tuning.

Next, we applied our algorithm to the hierarchical network introduced by Ravasz and Barabási [11]. Fig. 4 shows the dendrogram constructed based on the phase similarity C_{ij}. The hub at each level is grouped with one of the four identical modules connected to it in that level.

6 Summary

In summary, we reviewed some module-detecting algorithms. And we introduced a gauge KE in which the gauge term is a function of link BC. The gauge term induces the phase difference between two nodes that belong to different modules.

Therefore, a synchronization occurs in each module individually even though the whole network is not synchronized globally. By measuring the phase similarity of all pairs of connected nodes, we detected the modular structure of *ad hoc* networks and this algorithm needs relatively low computational cost.

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