Optimality Conditions of a Three-Dimension Non-smooth Thermodynamic System of Sea Ice

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Abstract. This study is intended to provide the mathematical foundation for the numerical computation of the parameter identification problems of the three-dimensional two-layer thermodynamic system of sea ice. The non-smooth thermodynamic system with mixed boundary conditions is established, its properties are obtained and the first-order necessary conditions of the parameter identification problem of the non-smooth system are derived.

Keywords: coupled 3D non-smooth thermodynamic system, parameter identification, necessary condition for optimality.

1 Introduction

Sea ice plays an important role in the global climate system [1,2,3]. The thin sea ice, sometimes having a snow cover, forms a new interface between the lower ocean and the upper atmosphere, reducing the transfer of moisture, heat and momentum between the atmosphere and the ocean. The freezing and melting processes of sea ice are influenced by the temperature distribution, thus the numerical simulation for ice temperature distribution have attracted great attention [4-8]. However, the physical parameters such as the density, the specific heat, the thermal conductivity and the heat exchange coefficient, and so on, are crucial for exactly describing the sea ice temperature profile. Therefore, accurately estimating these physical parameters can improve the sea ice thermodynamic modelling.

Until now, these physical parameters in the sea ice thermodynamic system are mainly estimated by field data [4-8]. However, the field data are spare and unsatisfactory due to the difficulties associated with fieldwork, especially during the polar winter. Some parameters could not be detected continuously and automatically up to now, such as the ice salinity; some could not be detected directly, such as the ice thermal conductivity. Thus it could not help us to thoroughly understand the physical evolution of sea ice just by field data. The parameter identification method is effective to solve this problem. Parameter identification

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refers to the determination of the physical parameters that could be detected discontinuously or difficultly from the physical parameters which can be detected continuously in the system model such that the predicted response of the model is close, in some well-defined sense, to the process observations. In recent years, there are many researchers devoting to the parameter identification problems of thermodynamic systems. Some researchers [9-12] considered the determination of source terms, some researchers [13-15] considered the determination of thermal conductivities. In [16], the properties and the optimality conditions for a one-dimension non-smooth thermodynamic system of sea ice were provided. The coefficients describing the sea ice salinity and other two parameters in a one-dimension nonlinear and non-smooth thermodynamic system of sea ice were identified using the parameter identification method in [17]. In this paper, we will deal with a three-dimension two-layer thermodynamic system of sea ice, and identify the densities, the specific heats, the thermal conductivities and the heat exchange coefficients of the snow and the sea ice using the parameter identification method. The properties of the non-smooth thermodynamic system are discussed, and the first-order necessary conditions of the parameter identification problem of the non-smooth system are derived. Therefore, the parameter identification theories of the three-dimension non-smooth distributed parameter system are applied to the actual sea ice problems, and the mathematical foundation for the numerical computation of the parameter identification problems of the sea ice thermodynamic system is provided.

The rest of this paper is organized as follows. In Section 2, we describe the three-dimension two-layer thermodynamic system. Section 3 derives the properties of the system. In Section 4, we establish an identification model, prove the existence of optimal control, and derive the necessary conditions for optimality. Section 5 concludes this research.

2 The Coupled 3D Thermodynamic System of Sea Ice

In this section, according to the distribution characteristics of the sea ice temperature field, we will describe a three-dimension two-layer thermodynamic system coupled by the snow and the sea ice, (as in Fig.1 and Fig.2), which is denoted by SIS.

SIS is from longitude W_1 degrees west to longitude W_2 degrees west, latitude N_1 degrees north to latitude N_2 degrees north, and L meters depths, where W_1, W_2, N_1, N_2 and L are positive constants, and $W_1 < W_2$, $N_1 < N_2$. Take the position at longitude W_1 degrees west and latitude N_1 degrees north on the surface of SIS as the origin denoted by A_1. Let x_1 represent the distance coordinate in the latitude direction taken as positive north, x_2 the distance coordinate in the longitude direction taken as positive east, x_3 the depth coordinate of SIS taken as positive downward, and their units are meters; set $x = (x_1, x_2, x_3)$ be the spatial coordinate. For convenience of our analysis, set $I_n = \{1, 2, \cdots, n\}$ and $\overline{S}$ express the closure of the set S, where n is any positive integer. Let A_i, A_i'' $(i \in I_4)$ denote the eight vertexes of SIS. The plane $A_1'A_2'A_3'A_4'$ denotes the

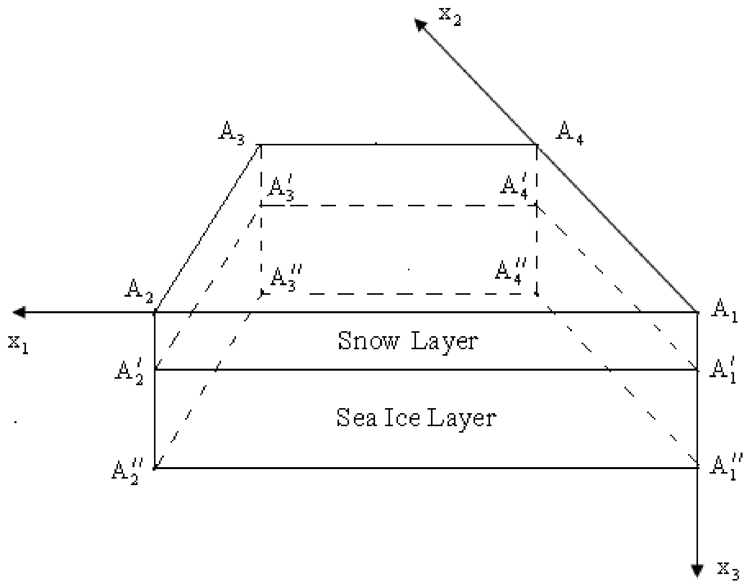

Fig. 1. The configuration of the 3D two-layer model

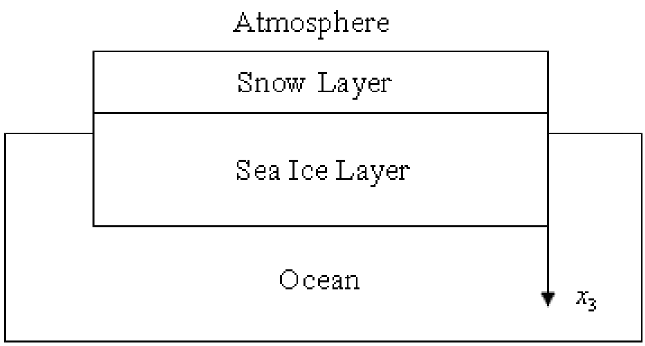

Fig. 2. The schematic diagram of the 3D two-layer model in profile

interface between the snow layer and the sea ice layer, where A_i' $(i \in I_4)$ denote the four vertexes. Let L_{sn}, L_{si} and L_{ia} be the depths of the snow layer, the sea ice layer and the sea ice layer in the atmosphere. Let t denote time, t_f the final time, $I = (0, t_f)$, and their units are seconds. Let $T(x,t)$ be the temperature of SIS at position x and time t, $T_0(x)$ the initial temperature, and their units are Kelvins. Let Γ_1 denote the plane $A_1A_2A_2''A_1''$, Γ_2 the plane $A_2A_3A_3''A_2''$, Γ_3 the plane $A_3A_4A_4''A_3''$, Γ_4 the plane $A_4A_1A_1''A_4''$, Γ_5 the plane $A_1''A_2''A_3''A_4''$, Γ_6 the plane $A_1A_2A_3A_4$, Γ_7 the plane $A_1'A_2'A_3'A_4'$, $\Gamma = \bigcup_{j=1}^{6} \Gamma_j$ the boundary of SIS. Let Ω_{sn} be the interior area of the snow layer, Ω_{si} the interior area of the sea ice layer, $\Gamma_{sn} = \partial\overline{\Omega}_{sn} \backslash ri\Gamma_7$ the boundary of the snow layer, $\Gamma_{si} = \partial\overline{\Omega}_{si} \backslash (\Gamma_7 \cup ri\Gamma_5)$

the boundary of the sea ice layer, Γ_{ia} the boundary of the sea ice layer in the atmosphere. $\Omega = \Omega_{sn} \cup \Omega_{si} \cup ri\Gamma_7$ is the interior area of SIS, $Q_{sn} = (\Omega_{sn} \cup ri\Gamma_7) \times I$, $Q_{si} = (\Omega_{si} \cup ri\Gamma_5) \times I$, $Q = \Omega \times I$. Take note that for any $(x,t) \in Q$, $T = T(x,t)$, $T_0 = T_0(x)$. Then according to the energy conservation law and the Fourier's law of heat conduction, the unsteady coupled three-dimensional two-layer thermodynamic system is described by the following heat equations

$$(\rho c)_{sn} \frac{\partial T}{\partial t} = \nabla \cdot (k_{sn} \nabla T) + g_{sn}(x,t), \qquad (x,t) \in Q_{sn}, \tag{1}$$

$$k_{sn} \nabla T = k_{si} \nabla T, \qquad (x,t) \in Q_{ni}, \tag{2}$$

$$(\rho c)_{si} \frac{\partial T}{\partial t} = \nabla \cdot (k_{si} \nabla T) + g_{si}(x,t), \qquad (x,t) \in Q_{si}. \tag{3}$$

The boundary conditions are

$$-k_{sn} \frac{\partial T}{\partial_{su}} = h_{sn}(T - T_{at}(x,t)), \qquad (x,t) \in \Gamma_6 \times I, \tag{4}$$

$$-k_{sn} \frac{\partial T}{\partial \boldsymbol{n}_{sn}} = h_{sn}(T - T_{at}(x,t)), \qquad (x,t) \in (\Gamma_{sn} \backslash \Gamma_6) \times I, \tag{5}$$

$$-k_{si} \frac{\partial T}{\partial \boldsymbol{n}_{si}} = h_{si}(T - T_{at}(x,t)), \qquad (x,t) \in \Gamma_{ia} \times I, \tag{6}$$

$$-k_{si} \frac{\partial T}{\partial \boldsymbol{n}_{si}} = h_{si}(T - T_{oc}(x,t)), \qquad (x,t) \in (\Gamma_{si} \backslash \Gamma_{ia}) \times I, \tag{7}$$

$$-k_{si} \frac{\partial T}{\partial \boldsymbol{n}_{io}} = h_{si}(T - T_{oc}(x,t)), \qquad (x,t) \in \Gamma_5 \times I. \tag{8}$$

The initial condition is

$$T(x,0) = T_0(x), \qquad x \in \Omega. \tag{9}$$

Where $\bigtriangledown$ is gradient operator; ρ, c, k, h and g are the density, the specific heat, the thermal conductivity, the heat exchange coefficient on the boundary and the source term, respectively; the subscripts sn, si and oc denote snow, sea ice and ocean, respectively; $\boldsymbol{n}_{su}$, $\boldsymbol{n}_{sn}$, $\boldsymbol{n}_{si}$ and $\boldsymbol{n}_{io}$ are the unit exterior normal vectors of the boundaries Γ_6, $\Gamma_{sn} \backslash \Gamma_6$, Γ_{si} and Γ_5, respectively; $T_{at}(x,t)$ and $T_{oc}(x,t)$ are the temperature functions of the atmosphere and the ocean which are adjacent to SIS.

Since the densities, the specific heats, the thermal conductivities, the heat exchange coefficients and the source terms are different in the three layers of SIS, the thermodynamic system is non-smooth.

According to the physical properties of SIS, we give the following assumptions.

(A1) *The temperature $T(x,t)$ of SIS is continuous on $\overline{Q}$.*
(A2) *The densities, the specific heats, the thermal conductivities and the heat exchange coefficients on the boundaries of snow, sea ice and ocean are bounded, so there exist $q_{Lj} > 0$, $q_{Uj} > 0$, and $q_{Lj} \leq q_{Uj}\,(j \in I_6)$, such that $q_{L1} \leq (\rho c)_{sn} \leq q_{U1}$,*

$q_{L2} \leq (\rho c)_{si} \leq q_{U2}$, $q_{L3} \leq k_{sn} \leq q_{U3}$, $q_{L4} \leq k_{si} \leq q_{U4}$, $q_{L5} \leq h_{sn} \leq q_{U5}$, $q_{L6} \leq h_{si} \leq q_{U6}$.

(A3) *$T_0(x) \in C^2(\Omega)$; $T_{at}(x,t) \in C^0((\Gamma_{sn} \cup \Gamma_{ia}) \times \bar{I})$, $T_{oc}(x,t) \in C^0((\Gamma \backslash \Gamma_{sn} \backslash \Gamma_{ia}) \times \bar{I})$; $g_{sn}(x,t) \in C^0(\overline{Q}_{sn})$, $g_{si}(x,t) \in C^0(\overline{Q}_{si})$; and for any $t \in \bar{I}$, $T_{at}(L_{sn} + L_{ia}, t) = T_{oc}(L_{sn} + L_{ia}, t)$, $g_{sn}(L_{sn}, t) = g_{si}(L_{sn}, t)$.*

For convenience of our analysis, next we will simplify the thermodynamic system. Let

$$\alpha(x,t) = \begin{cases} \alpha_1 = (\rho c)_{sn}^{-1}, & (x,t) \in \overline{\Omega}_{sn} \times I, \\ \alpha_2 = (\rho c)_{si}^{-1}, & (x,t) \in (\overline{\Omega}_{si} \backslash \Gamma_7) \times I. \end{cases}$$

$$\beta(x,t) = \begin{cases} \beta_1 = k_{sn}, & (x,t) \in \overline{\Omega}_{sn} \times I, \\ \beta_2 = k_{si}, & (x,t) \in (\overline{\Omega}_{si} \backslash \Gamma_7) \times I. \end{cases}$$

$$g(x,t) = \begin{cases} g_{sn}(x,t), & (x,t) \in \overline{\Omega}_{sn} \times I, \\ g_{si}(x,t), & (x,t) \in (\overline{\Omega}_{si} \backslash \Gamma_7) \times I. \end{cases}$$

$$\boldsymbol{n} = \begin{cases} \boldsymbol{n}_{su}, & (x,t) \in \Gamma_6 \times I, \\ \boldsymbol{n}_{sn}, & (x,t) \in (\Gamma_{sn} \backslash \Gamma_6) \times I, \\ \boldsymbol{n}_{si}, & (x,t) \in \Gamma_{si} \times I, \\ \boldsymbol{n}_{io}, & (x,t) \in \Gamma_5 \times I. \end{cases}$$

$$\gamma(x,t) = \begin{cases} \gamma_1 = h_{sn}, & (x,t) \in \Gamma_{sn} \times I, \\ \gamma_2 = h_{si}, & (x,t) \in \Gamma_{si} \times I. \end{cases}$$

$$T_c(x,t) = \begin{cases} T_{at}(x,t), & (x,t) \in (\Gamma_{sn} \cup \Gamma_{ia}) \times \bar{I}, \\ T_{oc}(x,t), & (x,t) \in ((\Gamma_{si} \backslash \Gamma_{ia}) \cup (\Gamma_{oc} \backslash \Gamma_5)) \times \bar{I}. \end{cases}$$

Then (1)-(9) can be simplified to the following form denoted by U3D2LTS.

$$\frac{\partial T}{\partial t} = \alpha(x,t) \nabla \cdot (\beta(x,t) \nabla T) + \alpha(x,t) g(x,t), \quad (x,t) \in Q_{sn} \cup Q_{si}, \tag{10}$$

$$k_{sn} \nabla T = k_{si} \nabla T, \quad (x,t) \in Q_{ni}, \tag{11}$$

$$-\beta(x,t) \frac{\partial T}{\partial \boldsymbol{n}} = \gamma(x)(T - T_c(x,t)), \quad (x,t) \in \Gamma \times \bar{I}, \tag{12}$$

$$T(x,0) = T_0(x), \quad x \in \overline{\Omega}. \tag{13}$$

Since $g_{sn}(x,t)$, $g_{si}(x,t)$, $T_{at}(x,t)$ and $T_{oc}(x,t)$ are given, the temperature T of U3D2LTS is dependent on the parameters ρ_{sn}, ρ_{si}, c_{sn}, c_{si}, k_{sn}, k_{si}, h_{sn} and h_{si}, that is $T = T(x,t;\rho_{sn},\rho_{si},c_{sn},c_{si},k_{sn},k_{si},h_{sn},h_{si})$. For convenience, let $q = (\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2) = (q_1,q_2,q_3,q_4,q_5,q_6)$. We often use the notation X_1 to denote $X_1(x)$ or X_2 to denote $X_2(x,t)$, where $x \in \overline{\Omega}$ or $(x,t) \in \overline{Q}$.

From the assumption (A2), set $q_L = \min\{q_{U1}^{-1}, q_{U2}^{-1}, q_{L3}, q_{L4}, q_{L5}, q_{L6}\}$, $q_U = \max\{q_{L1}^{-1}, q_{L2}^{-1}, q_{U3}, q_{U4}, q_{U5}, q_{U6}\}$, then $q_L \leq \alpha_j, \beta_j, \gamma_j \leq q_U$, $j \in I_2$. Let $Q_{ad} = \{q = (\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2) | 0 < q_L \leq \alpha_j, \beta_j, \gamma_j \leq q_U, j \in I_2\}$ be the parameter set, then Q_{ad} is a bounded closed convex subset of $\mathbb{R}^6$.

3 Properties of U3D2LTS

In this section, we will derive the properties of U3D2LTS.

Let $V = H^1(\Omega)$ be a Sobolev space, for any $\varphi \in V$, multiply the two sides of (10) by φ and integrate them over Ω, we have that

$$\int_\Omega \varphi \frac{\partial T}{\partial t} dx = \int_\Omega \alpha\varphi\nabla \cdot (\beta\nabla T)dx + \int_\Omega \alpha\varphi g dx, \tag{14}$$

for the first term on the right of (14),

$$\begin{aligned} \int_\Omega \alpha\varphi\nabla \cdot (\beta\nabla T)dx &= \int_\Omega \nabla \cdot (\alpha\varphi\beta\nabla T)dx - \int_\Omega \beta\nabla T \cdot \nabla(\alpha\varphi)dx \\ &= \int_\Gamma \alpha\gamma\varphi(T_c - T)d\Gamma - \int_\Omega \beta\nabla T \cdot \nabla(\alpha\varphi)dx. \end{aligned} \tag{15}$$

Substitute (15) into (14), and we get that

$$\begin{aligned} &\int_\Omega \varphi \frac{\partial T}{\partial t} dx + \int_\Omega \beta\nabla T \cdot \nabla(\alpha\varphi)dx + \int_\Gamma \alpha\gamma\varphi T d\Gamma \\ &= \int_\Omega \alpha\varphi g dx + \int_\Gamma \alpha\varphi\gamma T_c d\Gamma. \end{aligned} \tag{16}$$

For any $(x, t) \in Q$, let $T(x, t) = e^{rt}P(x, t)$, where r is a nonnegative constant, then (16) is transformed into

$$\begin{aligned} &\int_\Omega \varphi \frac{\partial P}{\partial t} dx + \int_\Omega \beta\nabla P \cdot \nabla(\alpha\varphi)dx + r\int_\Omega P\varphi dx + \int_\Gamma \alpha\gamma\varphi P d\Gamma \\ &= \int_\Omega \alpha\varphi g e^{-rt} dx + \int_\Gamma \alpha\gamma\varphi T_c e^{-rt} d\Gamma. \end{aligned}$$

Let $U_r(\alpha, \beta; u, v) = \int_\Omega \beta\nabla u \cdot \nabla(\alpha v)dx + r\int_\Omega uv dx$, $\forall\ u, v \in V$, then for any $\alpha, \beta \in Q_{ad}$, $U_r(\alpha, \beta; u, v)$ is a continuous linear functional on $V \times V$, so there exists $A_r(\alpha, \beta) \in \mathcal{L}(V, V')$, such that

$$U_r(\alpha, \beta; u, v) = \langle A_r(\alpha, \beta)u, v\rangle, \quad \forall\ \ u, v \in V.$$

Let $U_\Gamma(\alpha, \gamma; u, v) = \int_\Gamma \alpha\gamma uv d\Gamma$, $\forall\, u, v \in V$, then $U_\Gamma(\alpha, \gamma; u, v)$ is a bilinear functional on $V \times V$, the embedding operator $H^1(\Omega) \hookrightarrow L^2(\Gamma)$ is linear continuous, and there exists $A_\Gamma(\alpha, \gamma) \in \mathcal{L}(V, V')$, such that

$$U_\Gamma(\alpha, \gamma; u, v) = \langle A_\Gamma(\alpha, \gamma)u, v\rangle, \quad \forall\ \ u, v \in V.$$

Let $\mathcal{U}_r(q; u, v) = U_r(\alpha, \beta; u, v) + U_\Gamma(\alpha, \gamma; u, v)$, $\forall\ u, v \in V$, $\forall\ q \in Q_{ad}$, then

$$\mathcal{U}_r(q; u, v) = \int_\Omega \beta\nabla u \cdot \nabla(\alpha v)dx + r\int_\Omega uv dx + \int_\Gamma \alpha\gamma uv d\Gamma. \tag{17}$$

Property 1. *Suppose that assumptions (A1)-(A3) are valid, then for any $q \in Q_{ad}$, $u, v \in V$, $\mathcal{U}_r(q; u, v)$ defined by (17) is a bilinear functional on $V \times V$.*

Proof. For any $q \in Q_{ad}$, any $\lambda_1, \lambda_2 \in \mathbb{R}^1$, any $u, u_1, u_2, v, v_1, v_2 \in V$, we have that

$$\begin{aligned}
&\mathcal{U}_r(q;\lambda_1 u_1+\lambda_2 u_2,v)\\
&=U_r(\alpha,\beta;\lambda_1 u_1+\lambda_2 u_2,v)+U_\Gamma(\alpha,\gamma;\lambda_1 u_1+\lambda_2 u_2,v)\\
&=\lambda_1(U_r(\alpha,\beta;u_1,v)+U_\Gamma(\alpha,\gamma;u_1,v))+\lambda_2(U_r(\alpha,\beta;u_2,v)+U_\Gamma(\alpha,\gamma;u_2,v))\\
&=\lambda_1\mathcal{U}_r(q;u_1,v)+\lambda_2\mathcal{U}_r(q;u_2,v),
\end{aligned}$$

and $\mathcal{U}_r(q;u,\lambda_1 v_1+\lambda_2 v_2)$

$$\begin{aligned}
&=U_r(\alpha,\beta;u,\lambda_1 v_1+\lambda_2 v_2)+U_\Gamma(\alpha,\gamma;u,\lambda_1 v_1+\lambda_2 v_2)\\
&=\lambda_1(U_r(\alpha,\beta;u,v_1)+U_\Gamma(\alpha,\gamma;u,v_1))+\lambda_2(U_r(\alpha,\beta;u,v_2)+U_\Gamma(\alpha,\gamma;u,v_2))\\
&=\lambda_1\mathcal{U}_r(q;u,v_1)+\lambda_2\mathcal{U}_r(q;u,v_2),
\end{aligned}$$

which completes our proof. □

Property 2. *Suppose that assumptions (A1)-(A3) are valid, then for any $q \in Q_{ad}$, there exist $M_1 > 0$, $M_2 > 0$, such that*

$$\mathcal{U}_r(q;u,u) \ge M_1\|u\|_V^2, \quad \forall\, u\in V, \tag{18}$$

$$|\mathcal{U}_r(q;u,v)| \le M_2\|u\|_V\|v\|_V, \quad \forall\, u,v\in V. \tag{19}$$

Proof. For any $u \in V$, we obtain that

$$\begin{aligned}
U_r(q;u,u) &= \int_\Omega \beta\nabla u\cdot\nabla(\alpha u)dx + r\int_\Omega u^2dx\\
&= \int_{\Omega_{sn}} \alpha_1\beta_1(\nabla u)^2dx + \int_{\Omega_{si}} \alpha_2\beta_2(\nabla u)^2dx + r\int_\Omega u^2dx\\
&\ge q_L^2\|\nabla u\|^2_{L^2(\Omega)} + r\|u\|^2_{L^2(\Omega)},
\end{aligned}\tag{20}$$

and

$$\begin{aligned}
U_\Gamma(q;u,u) &= \int_\Gamma \alpha\gamma u^2 d\Gamma\\
&\ge q_L^2\|u\|^2_{L^2(\Gamma)}\\
&\ge q_L^2(\frac{1}{m}\|\nabla u\|^2_{L^2(\Omega)} + 3(1+m)\|u\|^2_{L^2(\Omega)}.
\end{aligned}\tag{21}$$

From the inequalities (20) and (21), we have that

$$\begin{aligned}
\mathcal{U}_r(q;u,u) &= U_r(\alpha,\beta;u,u)+U_\Gamma(\alpha,\gamma;u,u)\\
&\ge (r+3(1+m)q_L^2)\|u\|^2_{L^2(\Omega)} + (q_L^2+\frac{q_L^2}{m})\|\nabla u\|^2_{L^2(\Omega)},
\end{aligned}\tag{22}$$

take $m = 1$, then for any $r \ge 0$, $r + 3(1+m)q_L^2 > 0$, $q_L^2 + \frac{q_L^2}{m} > 0$, $M_1 = \min\{r+3(1+m)q_L^2, q_L^2+\frac{q_L^2}{m}\} > 0$, then $\mathcal{U}_r(q;u,u) \ge M_1\|u\|_V^2, \;\; \forall\, u \in V$.

Next we prove the second inequality. According to Poincaré inequality, we can obtain that $\|u\|^2_{L^2(\Omega)} \le C\|\nabla u\|^2_{L^2(\Omega)}$, where C is a positive constant, then

$$\begin{aligned}
&|\mathcal{U}_r(q;u,v)|\\
&=|U_r(\alpha,\beta;u,v)+U_\Gamma(\alpha,\gamma;u,v)|
\end{aligned}$$

$$
\begin{aligned}
&= |\int_{\Omega} \beta \nabla u \cdot \nabla(\alpha v) dx + r \int_{\Omega} uv dx + \int_{\Gamma} \alpha\gamma uv d\Gamma| \\
&\leq (r + q_U^2)\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + q_U^2\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} \\
&\leq B(\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}) \\
&\leq B(C+1)\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} \\
&\leq M_2\|u\|_V\|v\|_V,
\end{aligned}
$$

where $B = r + q_U^2$, $M_2 = 2B^2(C+1)^2$, which completes the proof. □

From (17), Property 1 and Property 2, we can get that $\mathcal{U}_r(q; u, v)$ is a bilinear continuous functional on $V \times V$, so there exists an $\mathcal{A}_r(q) \in \mathcal{L}(V, V')$, such that

$$\mathcal{U}_r(q; u, v) = \langle \mathcal{A}_r(q)u, v \rangle_{V',V}, \quad \forall \ u, v \in V, \tag{23}$$

that is $\mathcal{A}_r(q) = A_r(\alpha, \beta) + A_\Gamma(\alpha, \gamma)$.

Property 3. *Suppose that assumptions (A1)-(A3) hold, then for any $q \in Q_{ad}$, the following inequalities*

$$\langle \mathcal{A}_r(q)u, u \rangle_{V',V} \geq M_1\|u\|_V^2, \quad \forall \ u \in V, \tag{24}$$

$$|\langle \mathcal{A}_r(q)u, v \rangle_{V',V}| \leq M_2\|u\|_V\|v\|_V, \quad \forall \ u, v \in V, \tag{25}$$

are true, where M_1 and M_2 are defined in Property 2.

Let $\mathcal{F}_r(t, \alpha, \gamma; v) = \int_\Omega \alpha v g e^{-rt} dx + \int_\Gamma \alpha\gamma v T_c e^{-rt} d\Gamma$, $\forall \ v \in V$, then for any $t \in (0, t_f)$, any $\alpha, \gamma \in Q_{ad}$, $\mathcal{F}_r(t, \alpha, \gamma; v)$ is a linear continuous functional on V, so there exists $f_r(t, \alpha, \gamma) \in V'$, that is $f_r(t, \alpha, \gamma) \in \mathcal{L}(V, V')$, such that

$$\mathcal{F}_r(t, \alpha, \gamma; v) = \langle f_r(t, \alpha, \gamma), v \rangle, \quad \forall \ v \in V. \tag{26}$$

From (23) and (26), then U3D2LTS can be written as an evolution equation on V' denoted by EESS.

$$\frac{dP}{dt} + \mathcal{A}_r(q)P = f_r(t, \alpha, \gamma), \tag{27}$$

$$P(0) = P_0. \tag{28}$$

Where $P_0 = T_0$. Next we will present the existence and uniqueness of solution of EESS.

Property 4. *Suppose that assumptions (A1)-(A3) hold, then for any $q \in Q_{ad}$, EESS has a unique weak solution $P(t; q) \in L^2(0, t_f; V) \cap C(0, t_f; V)$.*

Proof. The proof is as Theorem 3.16 in Ref.[18]. □

Property 5. *Under the assumptions of Property 4, then U3D2LTS also has a unique weak solution $T(x, t; q) \in L^2(Q; V) \cap C(Q; V)$, for any $q \in Q_{ad}$.*

Define $S(Q_{ad}) = \{T(x,t;q) \in L^2(Q;V) \cap C(Q;V) | T(x,t;q)$ is the weak solution of U3D2LTS corresponding to $q \in Q_{ad}\}$ be the solution set of U3D2LTS.

4 Optimality Conditions of U3D2LTS

In this section, we will consider the parameter identification problem of U3D2LTS. Let $\widetilde{T}_{x,t}$ be the observed temperature data in $\overline{Q}$, $g_{sn}(x,t)$, $g_{si}(x,t)$, $T_0(x)$, $T_{at}(x,t)$ and $T_{io}(x,t)$ are all given. Define the performance criterion as follows

$$J(q) = \|T(x,t;q) - T_{mea}(x,t)\|^2_{L^2(Q)}, \tag{29}$$

where $T(x,t;q)$ is the temperature obtained from U3D2LTS, $T_{mea}(x,t)$ is a fitted continuous temperature function of $\widetilde{T}_{x,t}$ in the selected area $\overline{\Omega}$ and the time interval $[0,t_f]$, where $q \in Q_{ad}$. Our goal is to obtain the value of the parameter $q \in Q_{ad}$ to make $T(x,t;q)$ approach $T_{mea}(x,t)$, that is

$$\begin{aligned} PIP: \ min \quad & J(q) \\ s.t. \quad & T(x,t;q) \in S(Q_{ad}), q \in Q_{ad}. \end{aligned} \tag{30}$$

Where $J(q)$ is defined by (29), $S(Q_{ad})$ is the solution set of U3D2LTS.

4.1 Existence of Optimal Parameter of PIP

To get the existence of optimal solution of PIP, first we will consider the strong continuity of the mapping $\prod : q \to T(x,t;q)$.

Theorem 1. *Suppose that assumptions (A1)-(A3) hold, then the mapping* $\prod : q \to T(x,t;q)$ *is strongly continuous, where* $q \in Q_{ad}$, $T(x,t;q) \in S(Q_{ad})$.

Proof. For any fixed parameter $q_0 = (\alpha_1^0, \alpha_2^0, \beta_1^0, \beta_2^0, \gamma_1^0, \gamma_2^0) \in Q_{ad}$, let $\{q_n\} \subset Q_{ad}$ be a sequence, such that $\|q_n - q_0\|_{Q_{ad}} \to 0$, as $n \to \infty$, where $q_n = (\alpha_1^n, \alpha_2^n, \beta_1^n, \beta_2^n, \gamma_1^n, \gamma_2^n)$; let $T_n = T_n(x,t;q_n)$ and $T^0 = T^0(x,t;q_0)$ be the weak solutions of U3D2LTS corresponding to q_n and q_0 respectively; set $\omega_n = T_n - T^0$, $\alpha_0 = \{\alpha_1^0, \alpha_2^0\}$, $\alpha_n = \{\alpha_1^n, \alpha_2^n\}$, $\beta_0 = \{\beta_1^0, \beta_2^0\}$, $\beta_n = \{\beta_1^n, \beta_2^n\}$, $\gamma_0 = \{\gamma_1^0, \gamma_2^0\}$, $\gamma_n = \{\gamma_1^n, \gamma_2^n\}$, then we have

$$\begin{aligned} &\int_\Omega \omega_n \frac{\partial \omega_n}{\partial t} dx - \int_\Omega \omega_n \alpha_n \nabla \cdot (\beta_n \nabla \omega_n) dx \\ &= \int_\Omega \omega_n \alpha_n \nabla \cdot (\beta_n \nabla T^0) dx - \int_\Omega \omega_n \alpha_0 \nabla \cdot (\beta_0 \nabla T^0) dx + \int_\Omega \omega_n (\alpha_n - \alpha_0) g dx, \end{aligned} \tag{31}$$

$$-\beta_n \frac{\partial \omega_n}{\partial \boldsymbol{n}} - \gamma_n \omega_n = (\beta_n - \beta_0) \frac{\partial T^0}{\partial \boldsymbol{n}} + (\gamma_n - \gamma_0)(T^0 - T_c), \tag{32}$$

$$\omega_n(x,0) = 0. \tag{33}$$

For the equation (31), integrate over $[0,t]$, we have

$$\begin{aligned}&\int_0^t\int_\Omega \omega_n\frac{\partial\omega_n}{\partial s}dxds-\int_0^t\int_\Omega \alpha_n\omega_n\nabla\cdot(\beta_n\nabla\omega_n)dxds\\=&\int_0^t\int_\Omega \alpha_n\omega_n\nabla\cdot(\beta_n\nabla T^0)dxds-\int_0^t\int_\Omega \alpha_0\omega_n\nabla\cdot(\beta_0\nabla T^0)dxds\\&+\int_0^t\int_\Omega(\alpha_n-\alpha_0)\omega_n g dxds,\end{aligned}\tag{34}$$

then (34) can be simplified as

$$\begin{aligned}&\int_0^t\int_\Omega \omega_n\frac{\partial\omega_n}{\partial s}dxds+\int_0^t\int_\Omega \beta_n\nabla\omega_n\cdot\nabla(\alpha_n\omega_n)dxds+\int_0^t\int_\Gamma \alpha_n\gamma_n\omega_n^2d\Gamma ds\\=&\int_0^t\int_\Omega \omega_n((\alpha_n-\alpha_0)\nabla(\beta_n\nabla T^0)+\alpha_0\nabla((\beta_n-\beta_0)\nabla T^0))dxds\\&+\int_0^t\int_\Gamma \omega_n(\alpha_n(\beta_0-\beta_n)\frac{\partial T^0}{\partial \boldsymbol{n}}+\alpha_n(\gamma_0-\gamma_n)(T^0-T_c))dxds\\&+\int_0^t\int_\Omega \omega_n(\alpha_n-\alpha_0)gdxds.\end{aligned}\tag{35}$$

For the left term of (35), from (24), we get that

$$\begin{aligned}LEFT=&\frac{1}{2}\|\omega_n(t)\|^2_{L^2(\Omega)}+\int_0^t\langle A_0(\alpha_n,\beta_n)\omega_n,\omega_n\rangle_{V',V}ds\\&+\int_0^t\langle A_\Gamma(\alpha_n,\gamma_n)\omega_n,\omega_n\rangle_{V',V}ds\\=&\frac{1}{2}\|\omega_n(t)\|^2_{L^2(\Omega)}+\int_0^t\langle \mathcal{A}_0(q_n)\omega_n,\omega_n\rangle_{V',V}ds\\\geq&\frac{1}{2}\|\omega_n(t)\|^2_{L^2(\Omega)}+M_1\int_0^t\|\omega_n(s)\|^2_V ds,\end{aligned}\tag{36}$$

where $\mathcal{A}_0(q_n)=A_0(\alpha_n,\beta_n)+A_\Gamma(\alpha_n,\gamma_n)$.

For the right term of (35),

$$\begin{aligned}RIGHT\leq&\frac{M_1}{4}\int_0^t\|\omega_n(s)\|^2_{L^2(\Omega)}ds\\&+\frac{3}{M_1}(\int_0^t\|(\alpha_n-\alpha_0)\nabla\cdot(\beta_n\nabla T^0)+\alpha_0\nabla((\beta_n-\beta_0)\nabla T^0)\|^2_{L^2(\Omega)}ds\\&+\int_0^t\|\alpha_n(\beta_0-\beta_n)\frac{\partial T^0}{\partial\boldsymbol{n}}+\alpha_n(\gamma_0-\gamma_n)(T^0-T_c)\|^2_{L^2(\Gamma)}ds\\&+\int_0^t\|(\alpha_n-\alpha_0)g\|^2_{L^2(\Omega)}ds).\end{aligned}\tag{37}$$

Substitute (36) and (37) into (35), we can obtain that

$$
\begin{aligned}
&\|\omega_n(t)\|^2_{L^2(\Omega)} + \frac{3M_1}{2}\int_0^t \|\omega_n(s)\|^2_V ds \\
\le &\frac{6}{M_1}\Big(\int_0^t \|(\alpha_n - \alpha_0)\nabla\cdot(\beta_n\nabla T^0) + \alpha_0\nabla((\beta_n-\beta_0)\nabla T^0)\|^2_{L^2(\Omega)} ds \\
&+ \int_0^t \|\alpha_n(\beta_0-\beta_n)\frac{\partial T^0}{\partial \boldsymbol{n}} + \alpha_n(\gamma_0-\gamma_n)(T^0 - T_c)\|^2_{L^2(\Gamma)} ds \\
&+ \int_0^t \|(\alpha_n-\alpha_0)g\|^2_{L^2(\Omega)} ds\Big) \\
&+ \int_0^t \Big(\|\omega_n(s)\|^2_{L^2(\Omega)} + \frac{3M_1}{2}\int_0^s \|\omega_n(\theta)\|^2_V d\theta\Big) ds.
\end{aligned}
$$

Set $Y_n(t) = \|\omega_n(t)\|^2_{L^2(\Omega)} + \frac{3M_1}{2}\int_0^t \|\omega_n(s)\|^2_V ds$, using Gronwall's lemma, we have that

$$
\begin{aligned}
Y_n(t) \le e^{t_f}\frac{6}{M_1}\Big(&\int_0^t \|(\alpha_n - \alpha_0)\nabla\cdot(\beta_n\nabla T^0) + \alpha_0\nabla((\beta_n-\beta_0)\nabla T^0)\|^2_{L^2(\Omega)} ds \\
&+ \int_0^t \|\alpha_n(\beta_0-\beta_n)\frac{\partial T^0}{\partial \boldsymbol{n}} + \alpha_n(\gamma_0-\gamma_n)(T^0 - T_c)\|^2_{L^2(\Gamma)} ds \\
&+ \int_0^t \|(\alpha_n-\alpha_0)g\|^2_{L^2(\Omega)} ds\Big).
\end{aligned}
$$

Since $\|q_n - q_0\|_{Q_{ad}} \to 0$, as $n \to \infty$, so we can get that $\Pi : q \to T(x,t;q)$ is strongly continuous, and obtain the desired result. □

Theorem 2. *Suppose the assumptions (A1)-(A3) hold, then there exists at least one optimal parameter* $q^* = (\alpha_1^*, \alpha_2^*, \beta_1^*, \beta_2^*, \gamma_1^*, \gamma_2^*) \in Q_{ad}$ *satisfying (31).*

Proof. For any $q \in Q_{ad}$, let $X(T;q) = T(x,t;q) - T_{mea}(x,t)$, then $J(q) = \|X(T;q)\|^2_{L^2(Q)} \ge 0$. Obviously, $X(T;q)$ is continuous on V. Hence, by Theorem 1, the mapping $\Pi : q \to T(x,t;q)$ is continuous for all $q \in Q_{ad}$, so $q \to J(q)$ is continuous on Q_{ad}. Since Q_{ad} is a nonempty bounded closed set, there exists $q^* \in Q_{ad}$, such that for all $q \in Q_{ad}$, $J(q^*) \le J(q)$, i.e. $q^* \in Q_{ad}$ is an optimal parameter, thus we obtain the desired result. □

4.2 Necessary Conditions for Optimality

Here we will derive the necessary conditions for optimality. For this purpose, we will give a theorem as follows.

Theorem 3. *Assume assumption (A1)-(A3) hold, let $P(q)$ be the solution of EESS corresponding to $q \in Q_{ad}$, then at each point $q^0 \in Q_{ad}$ the function $q \to P(q)$ has a weak Gâteaux differential in the direction $q - q^0$, denoted by $P'(q^0; q - q^0)$, and it is the solution of the Cauchy problem*

$$\varphi_t + \mathcal{A}_r(t, q^0)\varphi = -\mathcal{A}_r'(t, q^0; q - q^0)P(q_0) + f_r'(t, q^0; q - q^0), \tag{38}$$

$$\varphi(0) = 0, t \in I, \tag{39}$$

where $P(q^0)$ is the solution of EESS corresponding to $q = q^0$, $\mathcal{A}'_r(q^0; q - q^0)$ and $f'_r(t, q^0; q - q^0)$ denote the weak Gâteaux differential of $\mathcal{A}_r(q)$ and $f_r(t, q)$ in the direction $q - q^0$.

Proof. Since Q_{ad} is a closed convex set, set $q_\delta = q^0 + \delta(q - q^0) \in Q_{ad}$ for $0 \leq \delta \leq 1$, $\varphi^\delta = \frac{P(q_\delta) - P(q^0)}{\delta}$. Then from (23), we obtain

$$\varphi_t^\delta + \mathcal{A}_r(q_\delta)\varphi^\delta = \frac{f_r(t, q_\delta) - f_r(t, q^0)}{\delta} + \frac{\mathcal{A}_r(q^0) - \mathcal{A}_r(q^\delta)}{\delta} P(q^0).$$

Let $\delta \to 0$, then we obtain the desired result. □

With the help of the above Theorem 3, we get the following necessary conditions for optimality.

Theorem 4. *Suppose that assumptions (A1)-(A3) are valid, let $q^* \in Q_{ad}$ is an optimal solution of PIP, and $T(q^*) \in S(Q_{ad})$ satisfies the following equations,*

$$\frac{dT(q^*)}{dt} + \mathcal{A}_0(q^*)T(q^*) = f_0(t, q^*), \tag{40}$$

$$T(0, q^*) = T_0, \tag{41}$$

then the adjoint state $\Psi(q^*)$ of $T(q^*)$ is determined by the adjoint equations

$$-\frac{d\Psi(q^*)}{dt} + \mathcal{A}_0^*(q^*)\Psi(q^*) = 2(T(q^*) - T_{mea}), \tag{42}$$

$$\Psi(t_f, q^*) = 0, \tag{43}$$

and the inequality

$$\int_0^{t_f} \langle -\mathcal{A}'_0(q^*; q - q^*)T(q^*), \Psi(q^*)\rangle_{V',V} dt + \int_0^{t_f} \langle f'_0(t, q^*; q - q^*), \Psi(q^*)\rangle_{V',V} dt \geq 0. \tag{44}$$

is true, where $\mathcal{A}_0^*(q^*)$ is the conjugate operator of $\mathcal{A}_0(q^*)$, $\mathcal{A}'_0(q^*; q - q^*)$ and $f'_0(t, q^*; q - q^*)$ are the weak Gâteaux differentials of $\mathcal{A}_0(q^*)$ and $f_0(t, q^*)$ in the direction $q - q^*$, $T_{mea} = T_{mea}(x, t)$ is the fitted temperature function of the observed data, $\mathcal{A}_0(q^*) = A_0(\alpha^*, \beta^*) + A_\Gamma(\alpha^*, \gamma^*)$.

Proof. Since $\Pi : q \to T(q)$ has a weak Gâteaux differential, it follows that $J(q)$ as defined by (25) also has a Gâteaux differential. Then in order that $J(q)$ attains its minimum at $q^* \in Q_{ad}$, it is necessary that

$$J'(q^*)(q - q^*) = \lim_{\sigma \to 0} \frac{J(q^* + \sigma(q - q^*)) - J(q^*)}{\sigma} \geq 0, \tag{45}$$

for all $q \in Q_{ad}$. Using the result of Theorem 3, it follows from the above that

$$J'(q^*)(q - q^*) = 2\int_0^{t_f} \langle T'(q^*; q - q^*), T(q^*) - T_{mea}\rangle_{L^2(\Omega)} dt \geq 0. \tag{46}$$

Let $\Psi(q^*)$ is the adjoint function of $T(q^*)$, it is the solution of the following equation

$$-\frac{d\Psi(q^*)}{dt} + \mathcal{A}_0^*(q^*)\Psi(q^*) = 2(T(q^*) - T_{mea}), \tag{47}$$

$$\Psi(t_f, q^*) = 0. \tag{48}$$

Then, since $T(q^*)-T_{mea} \in L^2(\Omega)$, reversing the flow of time $t \to t_f - t$, it follows from Property 5 that the adjoint system also has a unique solution $\Psi(q^*)$.

Next we will prove the inequality (44).

Multiplying both sides of (47) by $T(q) - T(q^*)$ and integrating over $[0, t_f]$, we have

$$\int_0^{t_f} \langle \Psi(q^*), \frac{d(T(q) - T(q^*))}{dt} + \mathcal{A}_0(q^*)(T(q) - T(q^*))\rangle_{V,V'} dt$$
$$= 2\int_0^{t_f} \langle T(q^*) - T_{mea}, T(q) - T(q^*)\rangle_{L^2(\Omega)} dt. \tag{49}$$

Let $q_\theta = q^* + \theta(q - q^*)$, $0 \leq \theta \leq 1$, and multiply (49) by $\frac{1}{\theta}$,

$$\int_0^{t_f} \langle \Psi(q^*), \frac{d}{dt}(\frac{T(q) - T(q^*)}{\theta}) + \mathcal{A}_0(q^*)\frac{T(q) - T(q^*)}{\theta}\rangle_{V,V'} dt$$
$$= 2\int_0^{t_f} \langle T(q^*) - T_{mea}, \frac{T(q) - T(q^*)}{\theta}\rangle_{L^2(\Omega)} dt.$$

Let $\theta \to 0$, we obtain

$$\int_0^{t_f} \langle \Psi(q^*), \frac{dT'(q^*; q - q^*)}{dt} + \mathcal{A}_0(q^*)T'(q^*; q - q^*)\rangle_{V,V'} dt$$
$$= 2\int_0^{t_f} \langle T(q^*) - T_{mea}, T'(q^*; q - q^*)\rangle_{L^2(\Omega)} dt.$$

Since $T'(q^*; q - q^*)$ is the solution of (38) by Theorem 3,

$$\int_0^{t_f} \langle -\mathcal{A}_0'(q^*; q - q^*)T(q^*) + f_0'(q^*; q - q^*), \Psi(q^*)\rangle_{V',V} dt$$
$$= 2\int_0^{t_f} \langle T'(q^*; q - q^*), T(q^*) - T_{mea}\rangle_{L^2(\Omega)} dt.$$

Then from (46), we obtain the desired result. □

5 Conclusions

In this paper, we have considered an unsteady three-dimension two-layer thermodynamic system coupled by the snow, the sea ice, presented the properties and derived the optimality conditions of the system. Thus we provide the mathematical foundation for the numerical computation of the parameter identification problems of the three-dimensional two-layer thermodynamic system of sea ice. The optimization algorithm and numerical results will be presented in a forthcoming paper.

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