Less Restrictive Synchronization Criteria in Complex Networks with Coupling Delays

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Abstract. This paper considers the synchronization in complex networks with coupling delays, whose topologies could be be symmetric and asymmetric. Differing from most works on the synchronization in complex networks with coupling delays, this paper only uses a positively-defined function, which is definitely not a Krasovskii-Lyapunov function, to analyze the synchronization criteria. Further, we can derive novel but less restrictive synchronization criteria than those resulting from the Krasovskii-Lyapunov theory. Theoretical analysis and numerical simulations fully verify the main results.

Keywords: complex networks, synchronization, matrix measure.

1 Introduction

Recently, the dynamics of complex networks has been extensively investigated, with special emphasis on the interplay between the overall topology and the local dynamics of coupled nodes. As a typical kind of dynamics, the synchronization in complex networks has been a research topic [1-17]. In 1998, Pecora and Carroll proposed the *master stability function* (MSF) based method to study the synchronization in networks [5]. The Lyapunov's direct method can be also used to study the synchronization in networks by constructing a Lyapunov function, which decreases along trajectories and gives analytical criteria for local and global synchronization [6-9,11,12].

Due to the finite speeds of transmission and spreading as well as traffic congestions, a signal or influence traveling through a complex network is often associated with time delays, and this is very common in biological and physical networks. Complex networks with coupling delays have recently attracted attention in many fields. From works [12,15-17], the Krasovskii-Lyapunov theory [18] is a useful and powerful tool to discuss the synchronization in networks with coupling delays. According to this kind of theory, some sufficient delay-independent and delay-independent conditions are given to ensure synchronization in networks with coupling delays. However, synchronization criteria resulting from the Krasovskii-Lyapunov theory may be too strict since they require that the

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derivative of a positively-defined Krasovskii-Lyapunov function is non-positive for all time [12,15-17].

Without using the Krasovskii-Lyapunov theory, this paper tries to derive less restrictive criteria for the synchronization in complex networks with coupling delays. Differing from most works on the synchronization in complex networks with coupling delays, this paper only uses a positively-defined function, which is definitely not a Krasovskii-Lyapunov function, to analyze the synchronization. We can derive novel but less restrictive synchronization criteria than those resulting from the Krasovskii-Lyapunov theory. The idea in this paper can be applied to complex networks with symmetric and asymmetric topologies.

The rest of this paper is organized as follows. A complex network model with coupling delays is presented, and some preliminaries are introduced in Section 2. In Section 3, we give one novel synchronization criterion for complex networks with symmetric topology. The synchronization criterion in complex networks with asymmetric topology is considered in Section 4. Numerical simulations are illustrated to show the effectiveness of the proposed synchronization criteria in Section 5. The last Section draws our conclusion.

2 A Complex Network Model and Necessary Preliminaries

Consider a complex network consisting of N nodes in the following form

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j \neq i} g_{ij} \Gamma (x_j(t-\tau) - x_i(t-\tau)) \tag{1}$$

for $1 \leq i \leq N$, where $x_i(t) = (x_{i1}(t), \ldots, x_{in}(t))^T$ is the state vector of node i, $\tau > 0$ is the time delay, the initial states for states $x_i(t)$ are $x_{i0} = \psi_{i0}(t), t \in [-\tau, 0]$, $\Gamma = \mathrm{diag}\{r_1, \cdots, r_n\}$ with r_i being 0 or 1 is the inner coupling matrix, and $f : R^n \to R^n$ is a smooth vector-valued function. Matrix $G = (g_{ij})_{N \times N}$ is the outer coupling matrix representing the topology of networks, and its elements are chosen as follows: if nodes i and j are connected, $g_{ij} = g_{ji} \neq 0$; otherwise $g_{ij} = g_{ji} = 0$, and the diagonal elements are defined by $g_{ii} = -\sum\limits_{j=1, j \neq i}^{N} g_{ij}$.

Network (1) is said to be in a synchronized manifold Ξ: $\{x_1(t) = \cdots = x_N(t) = s(t)\}$ if $\lim\limits_{t \to \infty} (x_i(t) - s(t)) = 0$ for $1 \leq i \leq N$, where $s(t)$ is a solution of an isolated node, denoted by $\dot{s}(t) = f(s(t))$. In this paper, suppose that $s(t)$ is an orbitally stable solution of the isolated node, and the Jacobian matrix $J(t) = Df(s(t))$ satisfies that $\frac{dJ(t)}{dt} = \frac{Df(s(t))}{dt}$ is bounded for all time.

In this paper our main results are based on the concept of matrix measure and one lemma with respect to the stability of time-delayed equations.

Definition 1: The matrix measure of a complex square matrix $C = (c_{ij})$ is defined as follows [20]:

$$\mu.(C) = \lim_{\varepsilon \to 0^+} \frac{||I_n + \varepsilon C|| - 1}{\varepsilon} \tag{2}$$

in which $|| \cdot ||$ is a matrix norm, and I_n is the identity matrix.

When $||C||_1 = \max_j \sum_{i=1}^{n} |c_{ij}|$, $||C||_2 = [\lambda\max(C^TC)]^{1/2}$ and $||C||_\infty = \max_i \sum_{j=1}^{n} |c_{ij}|$, we obtain $\mu_1(C) = \max_j\{\mathrm{Re}(c_{jj}) + \sum_{i=1,i\neq j}^{n} |c_{ij}|\}$, $\mu_2(C) = \frac{1}{2}\lambda_{\max}(C^* + C)$, and $\mu_\infty = \max_i\{\mathrm{Re}(c_{ii}) + \sum_{j=1,j\neq i}^{n} |c_{ij}|\}$ respectively, where C^* is the complex conjugate transpose of a complex matrix.

Lemma 1 [21,22]**:** Consider the following time-delayed equations

$$\dot{x}(t) = Ax(t) + Bx(t-\tau), x(t) = \phi(t),\ t \in [-h,\ 0] \tag{3}$$

where $A, B \in R^{n\times n}$, $\phi(t)$ is a continuous vector-valued initial function. Eq.(3) is asymptotically stable i.o.d (independent of delay) if and only if, for any given positive definite hermitian matrix $Q(z)$, $\forall |z| = 1$, the solution of $P(z)$ of the complex Lyapunov matrix equation

$$A^*(z)P(z) + P(z)A(z) = -Q(z),\ |z| = 1 \tag{4}$$

is also a positive definite hermitian matrix, where $A(z) = A + zB$, $|z| = 1$, $z = \exp(jw)$, $w \in [0,\ 2\pi]$, $j = \sqrt{-1}$.

Lemma 1 can be viewed as the asymptotical stability condition of a generalized linear system described by [22]

$$\dot{y}(t) = A(z)y(t),\ |z| = 1 \tag{5}$$

3 Synchronization in Complex Networks with Symmetric Topology

We first give a fundamental lemma for the network (1) with symmetric topology.

Lemma 2: For network (1) with symmetric topology, assume the outer coupling matrix G is a *nonnegative diffusively coupled matrix*, and can be irreducible and diagonalized. The manifold $\varXi$ is asymptotically stable, if the following $N-1$ systems are asymptotically stable:

$$\dot{w}_i(t) = J(t)w_i(t) + \lambda_i \varGamma w_i(t-\tau), \quad 2 \le i \le N \tag{6}$$

where λ_i are nonzero eigenvalues of G.

Proof: Since G is a *nonnegative diffusively coupled matrix*, G has zero-sum rows and nonnegative off-diagonal elements. In addition, 0 is one eigenvalue of multiplicity 1, and there exists a nonsingular matrix $\varPhi = (\phi_1, \cdots, \phi_N)$ such that $G^T\phi_k = \lambda_k\phi_k$ for $1 \le k \le N$, where $0 = \lambda_1 > \lambda_2 \ge \cdots \ge \lambda_N$ [9]. After a similar procedure [6,7,9,13-15], the manifold $\varXi$ is asymptotically stable if the $N-1$ linear systems (6) are asymptotically stable. □

From works [12,15-17], the Krasovskii-Lyapunov theory is very useful and powerful to analyze the stability of Eq. (6). Generally speaking, a Krasovskii-Lyapunov function can be chosen as

$$V^i(t) = w_i(t)^T P w_i(t) + \mu \int_{t-\tau}^{t} w_i(\alpha)^T Q w_i(\alpha) d\alpha \tag{7}$$

where $P = P^T > 0$, $Q = Q^T > 0$, and $\mu > 0$ is an arbitrary positive parameter. The main purpose of the Krasovskii-Lyapunov theory is to find the condition for the negativeness of $\frac{dV^i(t)}{dt}$ when the error $w_i(t)$ is not zero. In this paper, differing from most results with respect to the Krasovskii-Lyapunov theory, we only use a positively-defined function

$$V_i(t) = w_i(t)^T P w_i(t) \tag{8}$$

to analyze the stability of Eq. (6). Our aim in this paper is to make $\lim_{t\to\infty} V_i(t) = 0$, which also leads to $\lim_{t\to\infty} w_i(t) = 0$.

In order to do so, we first introduce a segmentation strategy for Eq. (6). For the time interval $[t_0, \infty)$, we segment it into $[t_0, \infty) = \bigcup_{j\geq 0} [t_j, t_{j+1})$, where $\tau_0 > 0$ is sufficiently small, j is an integer, $t_{j+1} = t_j + \tau_0$, and τ is a multiple of τ_0. If τ_0 is sufficiently small, the isolated dynamics $s(t)$ can be approximated by $s(t_j)$ within the interval $[t_j, t_{j+1})$, which further results in the approximation of $J(t)$ by $J(t_j)$. Therefore, within the interval $[t_j, t_{j+1})$, Eq. (6) can be approximated by

$$\dot{w}_i(t) = J(t_j)w_i + \lambda_i \Gamma w_i(t-\tau), \quad 2 \leq i \leq N \tag{9}$$

For the approximation system (9), we have the following result.

Theorem 1: Eq. (9) is asymptotically stable for the sample time τ_0, if there exists a symmetric positive definite matrix $P = M^T M \in R^{n\times n}$, $||M|| \neq 0$, such that

$$\int_{t_0}^{\infty} [\mu_\theta(MJ(t_j)M^{-1} + M^{-T}J(t_j)^T M^T) + (\frac{k^2}{\varsigma} + \varsigma) + 2|\lambda_i|]dt = -\infty \tag{10}$$

where μ_θ is one of μ_1, μ_2, μ_∞, ς and k are two sufficiently small positive constants.

Proof: Along with the solution of system (9), we get $\dot{V}_i(t) = w_i(t)^T[PJ(t_j) + J(t_j)^T P]w_i(t) + 2\lambda_i w_i^T(t)P\Gamma w_i(t-\tau)$. The second term satisfies $2\lambda_i w_i(t)^T P\Gamma w_i(t-\tau) \leq 2|\lambda_i|\cdot||Mw_i(t)||\cdot||M\Gamma w_i(t-\tau)|| \leq |\lambda_i|\cdot[V_i(t)+V_i(t-\tau)]$. Hence we obtain $\dot{V}_i(t) \leq w_i(t)^T M^T[MJ(t_j)M^{-1} + M^{-T}J(t_j)^T M^T]Mw_i(t) + |\lambda_i|\cdot[V_i(t) + V_i(t-\tau)]$. Inspired by the concept of matrix measure (2), we get $\dot{V}_i(t) \leq (\mu_\theta(MJ(t_j)M^{-1} + M^{-T}J(t_j)^T M^T) + |\lambda_i|)V_i(t) + |\lambda_i|V_i(t-\tau)$. Hence

$$V_i(t) \leq \exp(\textstyle\int_{t_0}^t (\mu_\theta(MJ(t_j)M^{-1} + M^{-T}J(t_j)^T M^T) + |\lambda_i|)d\alpha)V_i(0)$$
$$+ \textstyle\int_{t_0}^t \exp(\int_\vartheta^t(\mu_\theta(MJ(t_j)M^{-1} + M^{-T}J(t_j)^T M^T) + |\lambda_i|)d\alpha)|\lambda_i|V_i(\vartheta-\tau)d\vartheta$$

From the comparison theorem [19], the solution $V_i(t)$ satisfies

$$V_i(t) \leq \Gamma_i(t) \tag{11}$$

where $\Gamma_i(t)$ is the maximal solution of $\Gamma_i(t) = \exp(\int_{t_0}^t(\mu_\theta(MJ(t_j)M^{-1} + M^{-T}J(t_j)^T M^T)+|\lambda_i|)d\alpha)V_i(0)+\int_{t_0}^t \exp(\int_\vartheta^t(\mu_\theta(MJ(t_j)M^{-1}+M^{-T}J(t_j)^T M^T)+|\lambda_i|)d\alpha)|\lambda_i|\Gamma_i(\vartheta-\tau)d\vartheta$, or equivalently,

$$\frac{d\Gamma_i(t)}{dt} = (\mu_\theta(MJ(t_j)M^{-1} + M^{-T}J(t_j)^T M^T) + |\lambda_i|)\Gamma_i(t) + |\lambda_i|\Gamma_i(t-\tau) \quad (12)$$

with the same initial condition $V_i(0)$. From Lemma 1, within the interval $[t_j, t_{j+1})$, the stability of Eq. (12) is equivalent to the stability of

$$\frac{d\Gamma_i'(t)}{dt} = [\mu_\theta(MJ(t_j)M^{-1} + M^{-T}J(t_j)^T M^T) + |\lambda_i| + \vartheta|\lambda_i|]\Gamma_i'(t) \quad (13)$$

where $\vartheta = \exp(j\theta),\ \theta \in [0, 2\pi],\ j = \sqrt{-1}$. Since the relationship that $||\exp[(E+\vartheta F)t]|| \leq \exp[\mu(E+\vartheta F)t] \leq \exp[\mu(E) + ||F||)t]$ for $E, F \in R^{n\times n}$ and $\forall\, |\vartheta| = 1$ (please refer to Ref. [25] and Lemma 2 in Ref. [26]), we get

$$\begin{aligned}
||\Gamma_i'(t)|| = &\ ||\exp(\textstyle\int_{t_j}^{t}[\mu_\theta(MJ(t_j)M^{-1} + M^{-T}J(t_j)^T M^T) + |\lambda_i| + \vartheta|\lambda_i|]ds)\cdots\\
&\times\exp(\textstyle\int_{t_0}^{t_1}[\mu_\theta(MJ(t_0)M^{-1} + M^{-T}J(t_0)^T M^T) + |\lambda_i| + \vartheta|\lambda_i||]ds)V_i(0)||\\
&\leq \exp(\textstyle\int_{t_{j-1}}^{t}[\mu_\theta(MJ(t_j)M^{-1} + M^{-T}J(t_j)^T M^T) + 2|\lambda_i|]ds)\cdots\\
&\quad\times\exp(\textstyle\int_{t_0}^{t_1}[\mu_\theta(MJ(t_0)M^{-1} + M^{-T}J(t_0)^T M^T) + 2|\lambda_i|]ds)V_i(0)\\
&= \exp(\textstyle\int_{t_0}^{t}[\mu_\theta(MJ(t_j)M^{-1} + M^{-T}J(t_j)^T M^T) + 2|\lambda_i|]ds)V_i(0)
\end{aligned} \quad (14)$$

when $t \in [t_j,\ t_{j+1})$. From condition (10), $\int_{t_0}^{\infty}[\mu_\theta(MJ(t_j)M^{-1}+M^{-T}J(t_j)^T M^T)+2|\lambda_i|]dt = -\infty$ holds. Hence we get $\lim_{t\to\infty}\Gamma_i'(t) = 0$, which means $\lim_{t\to\infty}\Gamma_i(t) = 0$ and $\lim_{t\to\infty}V_i(t) = 0$. This implies that the approximation system (9) can be asymptotically stable. □

Note that there exists the term of $\frac{k^2}{\varsigma} + \varsigma$ in condition (10), and this can be approximatively zero if we choose two sufficiently small constants k and ς. In the following we show that this term is very useful for dealing with the approximation error between Eq. (6) and Eq. (9). From Theorem 1, we know that condition (10) only ensures the stability of Eq. (9), but it cannot ensure the stability of Eq. (6). Now we consider the stability condition for Eq. (6).

Theorem 2: Eq. (6) is asymptotically stable, if there exists a symmetric positive definite matrix $P = M^T M \in R^{n\times n}$, $||M|| \neq 0$, such that

$$\int_{t_0}^{\infty}[\mu_\theta(MJ(t)M^{-1} + M^{-T}J(t)^T M^T) + (\frac{k^2}{\varsigma} + \varsigma) + 2|\lambda_i|]dt = -\infty \quad (15)$$

Proof: Obviously, the following relationships hold:

$$\begin{aligned}
&\textstyle\int_{t_0}^{t}[\mu_\theta(MJ(t)M^{-1} + M^{-T}J(t)^T M^T) + (\frac{k^2}{\varsigma} + \varsigma) + 2|\lambda_i|]dt\\
&= \lim_{\tau_0\to 0}\{\textstyle\sum_{j\geq 0}^{n-1}\int_{t_j}^{t_{j+1}}[\mu_\theta(MJ(t_j)M^{-1} + M^{-T}J(t_j)^T M^T) + (\frac{k^2}{\varsigma} + \varsigma) + 2|\lambda_i|]dt\\
&\qquad + \textstyle\int_{t_n}^{t}[\mu_\theta(MJ(t_n)M^{-1} + M^{-T}J(t_n)^T M^T) + (\frac{k^2}{\varsigma} + \varsigma) + 2|\lambda_i|]dt\}\\
&= \lim_{\tau_0\to 0}\textstyle\int_{t_0}^{t}[\mu_\theta(MJ(t_j)M^{-1} + M^{-T}J(t_j)^T M^T) + (\frac{k^2}{\varsigma} + \varsigma) + 2|\lambda_i|]d\vartheta
\end{aligned}$$

when $t \in [t_n, t_{n+1})$, and

$$\begin{aligned}
&\textstyle\int_{t_0}^{\infty}[\mu_\theta(MJ(t)M^{-1} + M^{-T}J(t)^T M^T) + (\frac{k^2}{\varsigma} + \varsigma) + 2|\lambda_i|]dt\\
&= \lim_{\tau_0\to 0}\textstyle\int_{t_0}^{\infty}[\mu_\theta(MJ(t_j)M^{-1} + M^{-T}J(t_j)^T M^T) + (\frac{k^2}{\varsigma} + \varsigma) + 2|\lambda_i|]dt
\end{aligned}$$

This implies that, for arbitrary small positive constant ε, there exists a constant $\delta_1 > 0$ such that

$$\begin{aligned}&|\int_{t_0}^{\infty}[\mu_\theta(MJ(t)M^{-1} + M^{-T}J(t)^TM^T) + (\tfrac{k^2}{\varsigma} + \varsigma) + 2|\lambda_i|]dt \\ &\quad - \int_{t_0}^{\infty}[\mu_\theta(MJ(t_j)M^{-1} + M^{-T}J(t_j)^TM^T) + (\tfrac{k^2}{\varsigma} + \varsigma) + 2|\lambda_i|]dt| < \varepsilon\end{aligned} \tag{16}$$

if $0 < \tau_0 < \delta_1$. From Eq. (14), we have

$$\begin{aligned}&\lim_{t\to\infty} ||\Gamma_i'(t)|| \leq \exp(\int_{t_0}^{\infty}[\mu_\theta(MJ(t_j)M^{-1} + M^{-T}J(t_j)^TM^T) + (\tfrac{k^2}{\varsigma} + \varsigma) + 2|\lambda_i|]dt(V_i(0) \\ &\leq \exp(\int_{t_0}^{\infty}[\mu_\theta(MJ(t)M^{-1} + M^{-T}J(t)^TM^T) + (\tfrac{k^2}{\varsigma} + \varsigma) + 2|\lambda_i|]dt)(\exp(\varepsilon)V_i(0)\end{aligned} \tag{17}$$

It means that condition (15) can be also one sufficient condition for the asymptotical stability of Eq. (9) if the sample time τ_0 satisfies $0 < \tau_0 < \delta_1$.

Now we prove the stability of Eq. (6) if condition (15) is satisfied. For Eq. (6), if the sample time τ_0 is sufficiently small, we obtain

$$\dot{w}_i(t) = J(t_j)w_i(t) + \lambda_i\Gamma w_i(t-\tau) + O(t, t_j, \tau_0)w_i(t) \tag{18}$$

where $O(t, t_j, \tau_0) = J(t) - J(t_j)$. Since $O(t, t_j, \tau_0) = \frac{dJ(t_j)}{dt}(t - t_j)$ and the assumption that $\frac{dJ(t_j)}{dt}$ is bounded for all time, we obtain $\lim_{\tau_0\to 0} ||O(t, t_j, \tau_0)|| = 0$. Therefore, for the constant k, there exists a constant δ_2 satisfying $\delta_1 > \delta_2 > 0$ such that $-kI_n < O(t, t_j, \tau_0) < kI_n$ for $0 < \tau_0 < \delta_2$. From the function $V_i(t)$ given by Eq. (8), we get $\dot{V}_i(t) \leq w_i(t)^TM^T[MJ(t_j)M^{-1} + M^{-T}J(t_j)^TM^T + (\frac{k^2}{\varsigma} + \varsigma)]Mw_i(t) + |\lambda_i| \cdot [V_i(t) + V_i(t-\tau)]$ since $2w_i^T(t)O^T(t, t_j, \tau_0)M^TMw_i(t) \leq (\frac{k^2}{\varsigma} + \varsigma)w_i(t)^TM^TMw_i(t)$. Similar to the proof procedure in Theorem 1, we conclude that $\lim_{t\to\infty} V_i(t) = 0$ if condition (10) is satisfied for $0 < \tau_0 < \delta_2$. This means that Eq. (18), namely Eq. (6), is asymptotically stable for $0 < \tau_0 < \delta_2$ provided that condition (10) holds. Further, from Ineqs. (16,17), we conclude that condition (15) is also a sufficient condition for the stability of Eq. (6). □

We have several remarks.

Remark 1: From Ref. [15], the stability of Eq. (6) can be analyzed by the Krasovskii-Lyapunov function (7), and a general condition is $PJ(t) + J^T(t)P + Q + \lambda_N^2c^2PAQ^{-1}A^TP < 0$ for all time t. Let $P = M^TM$ with $||M|| \neq 0$, and we get $MJ(t)M^{-1} + M^{-T}J^T(t)M^T < -M^{-T}QM^{-1} - \lambda_N^2c^2MAQ^{-1}A^TM^T < 0$ for all time. Obviously, this is too strict for all time, and this can not be applied to the case where $MJ(t)M^{-1} + M^{-T}J^T(t)M^T$ is larger than zero during certain time intervals. In this paper condition (15) does not require the condition that $MJ(t)M^{-1} + M^{-T}J^T(t)M^T < 0$ for all time. In this sense condition (15) is less restrictive than Theorem 2 in Ref. [15]. For the case of $s(t) = (1/N)\sum_{k=1}^{N} x_k(t)$, we can also give one less restrictive condition for synchronization in network (1) with coupling delays than Theorem 1 in Ref. [17].

Remark 2: Based on the above idea, we can consider the case where the inner coupling $\Gamma(t) = \mathrm{diag}\{r_1(t), \cdots, r_N(t)\}$ is continuously time-varying. If $\Gamma(t)$ is independent of the node dynamics $x_i(t)$, the stability of the synchronized state is equivalent to the stability of the linear systems $\dot{w}_i(t) = J(t)w_i(t)+\lambda_i\Gamma(t)w_i(t-\tau)$ for $2 \le i \le N$. Similar to Theorems 1 and 2, one sufficient stability condition is given as follows

$$\int_{t_0}^{\infty}[\mu_\theta(MJ(t)M^{-1}+M^{-T}J(t)^TM^T)+(\frac{k^2}{\varsigma}+\varsigma)+|\lambda_i|(1+\frac{||M\Gamma(t)||^2}{\lambda_{\min}(M^TM)})]dt=-\infty \tag{19}$$

where $P = M^TM \in R^{n\times n}$, $||M|| \ne 0$, is a n-dimensional symmetric positive definite matrix, and $\lambda_{\min}(M^TM)$ is the minimum eigenvalue of matrix M^TM.

Remark 3: We can also consider the case where a coupling delay occurs when the signals from each of the nodes are transmitted to interconnected nodes. In this case the dynamics of the network is given by $\dot{x}_i(t) = f(x_i(t)) + c\sum_{j\ne i} g_{ij}\Gamma(x_j(t-\tau) - x_i(t))$ where c is the coupling strength. Let $g_i = \sum_{j\ne i} g_{ij}$. Under the condition of $g_1 = g_2 = \cdots = g_N = g$, the synchronized state is given by $\dot{s}(t) = f(s(t)) + cg\Gamma(s(t-\tau) - s(t))$. From Ref. [16], the stability of the synchronized state $x_1(t) = \cdots = x_N(t) = s(t)$ can be transformed into the stability of the following linear systems $\frac{d}{dt}(\varphi(t)) = (J(t)-cg\Gamma)\varphi(t)+c(\lambda_i+g)\Gamma\varphi(t-\tau)$ for $2 \le i \le N$. Similar to Theorems 1 and 2, we can obtain the following sufficient synchronization condition

$$\begin{aligned}\int_{t_0}^{\infty}[\mu_\theta(M(J(t)-cg\Gamma)M^{-1}+M^{-T}(J(t)-cg\Gamma)^TM^T)+(\tfrac{k^2}{\varsigma}+\varsigma)\\ +2|c(\lambda_i+g)|]dt=-\infty\end{aligned} \tag{20}$$

where $P = M^TM$, $||M|| \ne 0$, is a n-dimensional symmetric positive definite matrix. Compared with Theorem 1 in Ref. [16], condition (20) is also less restrictive.

Remark 4: Now we extend the procedure in Theorems 1 and 2 to the stability of the n-dimensional time-varying linear systems

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-\tau) \tag{21}$$

where $A(t)$ and $B(t)$ are continuously time-varying, and are independent of the dynamics $x(t)$. Similar to Theorems 1 and 2, we can obtain one asymptotical stability condition

$$\int_{t_0}^{\infty}[\mu_\theta(M(A(t)M^{-1}+M^{-T}A(t)M^T)+(\tfrac{k^2}{\varsigma}+\varsigma)+(1+\tfrac{||MB(t)||^2}{\lambda_{\min}(M^TM)})]dt=-\infty \tag{22}$$

where $P = M^TM$, $||M|| \ne 0$, is a n-dimensional symmetric positive definite matrix. Similar to the analysis in Remark 1, condition (22) is less restrictive than conditions from the Krasovskii-Lyapunov theory.

4 Synchronization in Complex Networks with Asymmetric Topology

If the topology in network (1) is symmetric, criteria (10,15) are not be applicable since eigenvalues of G may have the non-zero imaginary part. Hence we further analyze the synchronization criteria for complex networks with asymmetric topology. Note that the procedure developed in this section can be also applied to networks with symmetric topology.

Network (1) can be rewritten in an equivalent form

$$\dot{X}(t) = F(X(t)) + (G \otimes \Gamma)X(t-\tau) \tag{23}$$

where '$\otimes$' is the Kronecker product, $F(X(t)) = (f(x_1(t))^T, \cdots, f(x_N(t))^T)^T$, and $X(t) = (x_1^T(t), \cdots, x_N^T(t))^T$. By choosing a suitable continuously time-varying matrix $K(t) \in R^{n\times n}$, Eq. (23) is equivalent to the following system

$$\dot{X}(t) = F'(X(t)) - (I_N \otimes K(t))X(t) + (G \otimes \Gamma)X(t-\tau) \tag{24}$$

where $F'(X(t)) = ((f(x_1(t)) + K(t)x_1(t))^T, \cdots, (f(x_N(t)) + K(t)x_N(t))^T)^T$.

Let $\eta_j(t) = x_{j+1}(t) - x_1(t)$ for $1 \le j \le N-1$, and $\eta(t) = (\eta_1^T(t), \cdots, \eta_{N-1}^T(t))^T$. Then we get

$$\dot{\eta}(t) = \bar{F}(X(t)) - (I_{N-1} \otimes K(t))\eta(t) + (S_G \otimes \Gamma)\eta(t-\tau) \tag{25}$$

where $\bar{F}(X(t)) = ((f(x_2(t)) + K(t)x_2(t) - f(x_1(t)) - K(t)x_1(t))^T, \cdots, (f(x_N(t)) + K(t)x_N(t) - f(x_1(t)) - K(t)x_1(t))^T)^T$, and S_G is described by

$$S_G = \begin{bmatrix} -g_{12} - \sum\limits_{j\neq 2} g_{2j} & g_{23} - g_{13} & \cdots & g_{2N} - g_{1N} \\ g_{32} - g_{12} & -g_{13} - \sum\limits_{j\neq 3} g_{3j} & \cdots & g_{3N} - g_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ g_{N2} - g_{12} & g_{N3} - g_{13} & \cdots & -g_{1N} - \sum\limits_{j\neq N} g_{Nj} \end{bmatrix} \tag{26}$$

The above procedure can be seen in Refs. [10,14]. The procedure given by Eqs. (25,26) is very useful for dealing with the synchronization in networks without coupling delays, and the derived synchronization criteria are less restrictive than many exiting synchronization criteria [14]. In this paper we also utilize the procedure to consider the synchronization in networks with coupling delays.

Suppose that the feedback gain $K(t)$ is not affected by the node dynamics $x_i(t)$. Applying the segmentation strategy developed in the previous section, Eq. (25) can be approximated by the following system

$$\dot{\eta}(t) = \bar{F}(X(t)) - (I_{N-1} \otimes K(t_j))\eta(t) + (S_G \otimes \Gamma)\eta(t-\tau) \tag{27}$$

within the interval $[t_j, t_{j+1})$. Further, stability conditions for Eqs. (25,27) are stated as follows:

Theorem 3: Let $K(t)$ be a suitable feedback gain such that $f(x(t)) + K(t)x(t)$ is V-uniformly decreasing for a symmetric positive definite matrix $V \in R^{n\times n}$. Eq. (27) is asymptotically stable if there exists a positive definite matrix $U = \mathrm{diag}(u_1, \cdots, u_{N-1})$ such that

$$\begin{aligned} \int_0^{+\infty} [\mu_\theta(-M(I_{N-1} \otimes K(t_j))M^{-1} - M^{-T}(I_{N-1} \otimes K(t_j))^T M^T) \\ +(\tfrac{k^2}{\varsigma} + \varsigma) + (1 + \tfrac{||M(S_G \otimes \Gamma_2)||}{\lambda_{\min}(M^T M)}) - \tfrac{c_0 u_0}{\lambda_{\max}(M^T M)}]dt = -\infty \end{aligned} \tag{28}$$

where $U \otimes V = M^T M$, $M \in R^{n(N-1)\times n(N-1)}$ is nonsingular, $u_0 = \min\{u_1, \cdots, u_{N-1}\}$, c_0 is a positive constant, and $\lambda_{\max}$ stands for the maximal eigenvalue of $M^T M$.

Proof: We choose a positively-defined function $V_0(t) = \eta^T(t)(U \otimes V)\eta(t)$. Hence its derivative along with the trajectory of Eq. (27) is $\dot{V}_0(t) = 2\eta^T(t)(U \otimes V)\bar{F}(X(t)) + 2\eta^T(t)(U \otimes V)(S_G \otimes \Gamma)\eta(t-\tau) + \eta^T(t)((U \otimes V)(-(I_{N-1} \otimes K(t_j)) - (I_{N-1} \otimes K(t_j))^T(U \otimes V))\eta(t)$. From the V-uniformly decreasing property of $f+K$ [11,12], the first term is of the form $2\eta^T(t)(U \otimes V)\bar{F}(X(t)) \leq -c_0 \sum_{j=2}^{N} u_{j-1}||x_j - x_1||^2 \leq -\frac{c_0 u_0}{\lambda_{\max}(M^T M)} V_0(t)$. The second term satisfies $2\eta^T(t)(U \otimes V)(S_G \otimes \Gamma)\eta(t-\tau) \leq V_0(t) + \frac{||M(S_G \otimes \Gamma)||}{\lambda_{\min}(M^T M)} V_0(t-\tau)$. The third term satisfies $\eta^T(t)((U \otimes V)(-I_{N-1} \otimes K(t_j)) - (I_{N-1} \otimes K(t_j))^T(U \otimes V))\eta(t) \leq \mu_\theta(-M(I_{N-1} \otimes K(t_j))M^{-1} - M^{-T}(I_{N-1} \otimes K(t_j))^T M^T)V_0(t)$. Similar to the proof of Theorem 1, condition (28) is a sufficient stability condition for the approximation system (27). □

Theorem 4: Assume that $K(t)$ and V satisfy Theorem 3. Eq. (25) is asymptotically stable if there exists a positive definite matrix $U = \mathrm{diag}(u_1, \cdots, u_{N-1})$ such that

$$\begin{aligned}\int_0^{+\infty} [\mu_\theta(-M(I_{N-1} \otimes K(t))M^{-1} - M^{-T}(I_{N-1} \otimes K(t))^T M^T) \\ +(\frac{k^2}{\varsigma} + \varsigma) + (1 + \frac{||M(S_G \otimes \Gamma_2)||}{\lambda_{\min}(M^T M)}) - \frac{c_0 u_0}{\lambda_{\max}(M^T M)}]dt = -\infty\end{aligned} \tag{29}$$

where $U \otimes V = M^T M$, $||M|| \neq 0$, $u_0 = \min\{u_1, \cdots, u_{N-1}\}$, c_0 is a positive constant, and $\lambda_{\max}$ stands for the maximal eigenvalue of $M^T M$.

Proof: This can be easily by the procedure in Theorems 1, 2 and 3. □

Remark 5: Theorems 3 and 4 do not require the linearization strategy (please see Eq.(6)). Moreover, conditions (28,29) can be regarded as global synchronization criteria. If the feedback gain matrix $K(t)$ is chosen as a constant matrix K_0, similar to the proof procedure in Theorem 3, we get $\dot{V}_0(t) \leq (\mu_\theta(-M(I_{N-1} \otimes K_0))M^{-1} - M^{-T}(I_{N-1} \otimes K_0)^T M^T) + 1 - \frac{c_0 u_0}{\lambda_{\max}(M^T M)})V_0(t) + \frac{||M(S_G \otimes \Gamma)||}{\lambda_{\min}(M^T M)} V_0(t-\tau)$. From Lemma 1 and Ref. [23], one sufficient stability condition for the case of the time-invariant feedback gain K_0 is

$$\begin{aligned}(\mu_\theta(-M(I_{N-1} \otimes K_0))M^{-1} - M^{-T}(I_{N-1} \otimes K_0)^T M^T) + \\ 1 - \frac{c_0 u_0}{\lambda_{\max}(M^T M)}) + \frac{||M(S_G \otimes \Gamma)||}{\lambda_{\min}(M^T M)} < 0\end{aligned} \tag{30}$$

Remark 6: The idea in Theorems 3 and 4 can be also extended to the synchronization in networks, whose topology $G(t)$ is time-varying. Let $G(t) = (g_{ij}(t))$ have the same definition as matrix G in the network (1) at the t instant. Suppose that the topology $G(t)$ is continuously time-varying, and it is not affected by the node dynamics $x_i(t)$. Inspired by Theorems 3 and 4, we also obtain one sufficient synchronization condition

$$\begin{aligned}\int_0^{+\infty}[\mu_\theta(-M(I_{N-1}\otimes K(t))M^{-1}-M^{-T}(I_{N-1}\otimes K(t))^TM^T)\\ +(\tfrac{k^2}{\varsigma}+\varsigma)+(1+\tfrac{||M(S_G(t)\otimes\Gamma_2)||}{\lambda_{\min}(M^TM)})-\tfrac{c_0u_0}{\lambda_{\max}(M^TM)}]dt=-\infty\end{aligned}\tag{31}$$

where $S_G(t)$ has the same structure as Eq. (26) at the t instant.

Remark 7: We further consider the synchronization in the network given by

$$\dot{x}_i(t)=f(x_i(t))+\sum_{j\neq i}g_{ij}(t)\Gamma_1(x_j(t)-x_i(t))+\sum_{j\neq i}g_{\tau,ij}(t)\Gamma_2(x_j(t-\tau)-x_i(t-\tau))\tag{32}$$

where $\Gamma_i\,(i=1,2)$ are the inner coupling matrices. Let $G(t)=(g_{ij}(t))$ and $G_\tau(t)=(g_{\tau,ij}(t))$ have same definition as matrix G in the network (1) at the t instant. Similar to Theorems 3 and 4, one sufficient synchronization condition for network (32) is

$$\begin{aligned}\int_0^{+\infty}[\mu_\theta(M(S_G(t)\otimes\Gamma_1-I_{N-1}\otimes K(t))M^{-1}\\ +M^{-T}(S_G(t)\otimes\Gamma_1-I_{N-1}\otimes K(t))^TM^T)\\ +(\tfrac{k^2}{\varsigma}+\varsigma)+(1+\tfrac{||M(S_{G_\tau}(t)\otimes\Gamma_2)||}{\lambda_{\min}(M^TM)})-\tfrac{c_0u_0}{\lambda_{\max}(M^TM)}]dt=-\infty\end{aligned}\tag{33}$$

Further, Theorem 4 can be also generalized to network (32) with continuous time-varying inner coupling matrices $\Gamma_1(t)$ and $\Gamma_2(t)$. From Eq. (2) in the work [12], one necessary condition for the synchronization in network (32) is $(U\otimes V)(G(t)\otimes\Gamma_1(t)-I_n\otimes K)\leq 0$ for all time. Similar to the above discussion, this condition is too strict, and condition (33) for the coupling matrices $\Gamma_1(t)$ and $\Gamma_2(t)$ is less restrictive.

5 Numerical Simulations

In this section we verify the effectiveness of the proposed synchronization criteria by using a three-dimensional system as a node in network (1). Each individual node is described by $\dot{x}_1(t)=(-1+1.5\sin(t))x_1(t)$, $\dot{x}_2(t)=-3x_2(t)$, $\dot{x}_3(t)=-3x_3(t)$. Further, its Jacobian is $J(t)=\mathrm{diag}\{-1+1.5\sin(t),\,-3,\,-3\}$. To begin with, we prove the stability of the above system at its zero solution. From the proofs of Theorems 1 and 2, the stability condition for the isolated node is $\int_{t_0}^{\infty}[\mu_1(J(t)+J(t)^T)+(\frac{k^2}{\varsigma}+\varsigma)]dt=-\infty$ for $M=I_3$, and arbitrary small positive constants ς and k. If we choose $\frac{k^2}{\varsigma}+\varsigma=1$, $\int_{t_0}^{\infty}[\mu_1(J(t)+J(t)^T)+1]dt=\int_{t_0}^{\infty}[-2+3\sin(t)+1]dt=\int_{t_0}^{\infty}[-1]dt+\int_{t_0}^{\infty}[3\sin(t)]dt=-\infty$. Therefore the isolated node can be asymptotically stable at its zero solution.

In this section the star-type coupled network is chosen to be the simulated network. In this network, only one node is a center node with degree $N-1$, and all the other nodes with degree 1 are connected to this center node. Suppose that all nodes are connected by their first states, namely $\Gamma=\mathrm{diag}\{1,0,0\}$. In this case the coupling matrix is

$$G = c \begin{bmatrix} -1 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -1 & 1 \\ 1 & \cdots & 1 & -(N-1) \end{bmatrix}$$

where c is a positive constant coupling. Let the number of nodes $N = 10$ and $c = 0.08$. Obviously, two different nonzero eigenvalues of G are $\lambda_1 = -0.08$ and $\lambda_2 = -0.8$. From Theorems 1 and 2, the synchronization condition for arbitrary delay time τ is $\int_{t_0}^{\infty} [\mu_1(J(t) + J(t)^T) + (\frac{k^2}{\varsigma} + \varsigma) + |\lambda_i|]dt = -\infty$, which can be easily verified by the the above analysis. Simulations results with respect to three states $x_1(t)$ (the dashed lines), $x_2(t)$ (the solid lines), and $x_3(t)$ (the dashdot lines) are plotted in Figure 1 for the delayed time $\tau = 2$. From this figure, the network can be asymptotically stabilized at the zero solution of the isolated node. Since $\mu_1(J(t) + J(t)^T) = -2 + 3\sin(t)$ are larger than zero during certain time intervals, the Krasovskii-Lyapunov theory can not be successfully used to analyze the stability of the network.

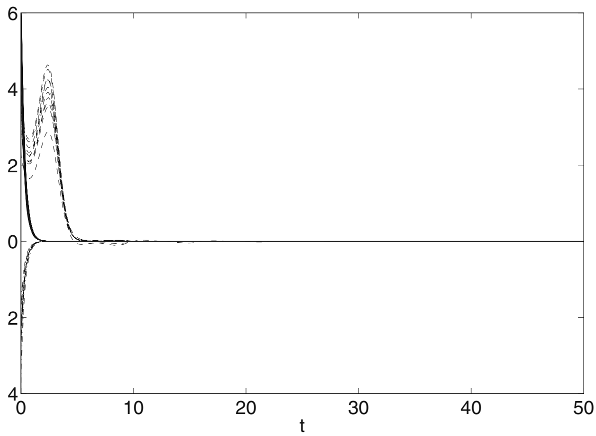

Fig. 1. The history curves of states $x_i(t)$ $(i = 1, 2, 3)$

6 Conclusion

In this paper we propose some novel synchronization criteria in complex networks with coupling delays, in which the topologies in networks can be symmetric and asymmetric. Compared with synchronization criteria resulting from the Krasovskii-Lyapunov theory, the proposed synchronization criteria are less restrictive.

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