Phase Transition of Active Rotators in Complex Networks

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Abstract. We study the nonequilibrium phenomena of a coupled active rotator model in complex networks. From a numerical Langevin simulation, we find the peculiar phase transition not only on globally connected network but also on other complex networks and reveal the corresponding phase diagram. In this model, two phases — stationary and quasi-periodic moving phases — are observed, in which microscopic dynamics are thoroughly investigated. We extend our study to the non-identical oscillators and the more heterogeneous degree distribution of complex networks.

Keywords: active rotator model, phase transitions, complex networks.

1 Introduction

Various coupled oscillatory systems in nature have been known to exhibit many interesting behaviors including synchronization. Collective synchronization has attracted much interest due to the beauty of simultaneousness and the spontaneous emergence in such phenomena as the synchronous flashing of fireflies, the chorusing of crickets, and the clapping of hands after an astonishing orchestral performance [1]. In order to understand such synchronized behaviors, nonlinear coupled oscillators have been studied extensively with various models. Among them, the Kuramoto model is one of the most studied models due to its simplicity and analytical tractability [2,3]. The Kuramoto model has been extended with many variations for applications in diverse systems [3]. One natural extension is to add external fields, which implies the external current applied to a neuron to describe an excitable systems. This is also known to be an *active rotator* model when each oscillator has the constant natural frequency [3,4].

Most studies of the active rotator model have assumed that all oscillators are connected to each other, i.e., globally connected network, or sometimes 2 and 3-dimensional regular lattice is used [4,5,6]. However, such a type of interaction has a limitation when applied to most real systems. Therefore we need to consider such nontrivial connectivity and extend the study of synchronization to complex networks. Thus, in the present paper, we report our study of active rotator model in complex networks.

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2 Model System

The dynamics of N coupled limit-cycle oscillators having the phase $\{\phi_i(t)|i = 1, 2, \ldots, N\}$ is described by the set of equations

$$\frac{d\phi_i}{dt} = \omega_i - b\sin\phi_i - \frac{K}{\langle k\rangle}\sum_{j=1}^{N} a_{ij}\sin(\phi_i - \phi_j) + \eta_i(t). \tag{1}$$

The first term ω_i represents the natural frequency of the ith oscillator, which is assumed the random normal distribution having the correlation $\langle\omega_i\omega_j\rangle = \sigma^2\delta_{ij}$ with the variance σ^2 and the mean $\langle\omega_i\rangle = \omega_0$. The second and third terms indicate the pinning force and the coupling between the oscillators respectively; the coupling strength K is set to be a positive one ($K > 0$), so the interacting oscillators favor their phase difference minimized. The adjacency matrix element $a_{ij} = 1(0)$ if oscillators i and j are connected (disconnected), and $\langle k\rangle$ denotes the mean degree given by $\sum_i k_i/N$, where the degree $k_i = \sum_j a_{ij}$. In the last term of Eq. (1), $\eta_i(t)$ is the Gaussian white noise with properties $\langle\eta_i(t)\rangle = 0, \langle\eta_i(t)\eta_j(t')\rangle = 2D\delta(t - t')\delta_{ij}$.

When all oscillators are connected to each other, i.e., $a_{ij} = 1$ for all $i \neq j$, and $b = 0$, $D = 0$, the model corresponds to the original Kuramoto model [2]. If all oscillators are identical and $b = 0$, it describes the thermodynamic system of classical XY spins, where D plays role of the temperature of the spin systems [6]. When all oscillators have the same frequency, we call the system as active rotators.

Collective phase synchronization is conveniently described by the order parameter defined by

$$r(t)e^{i\theta(t)} \equiv \frac{1}{N}\sum_{j=1}^{N} e^{i\phi_j(t)}, \tag{2}$$

where $r > 0$ implies emergence of the phase synchronization. Then we take the time average of $r(t)$ such as $r \equiv \overline{r(t)} = (2/T)\sum_{t=T/2+1}^{T} r(t)$, where the over line represents the time averaging and we set T to enough large number after confirming the state passes over the transient period. In the case of the original Kuramoto model, the time averaged r delivers most information since $r(t)$ saturates to a value r. However, active rotators do not always go to the stationary phase but show periodic behavior. Therefore, Shinomoto *et al.* [4] introduced another order parameter σ and a kind of fluctuation measure $\widetilde{\chi}$ defined by

$$\sigma e^{i\varphi} \equiv \overline{r(t)e^{i\theta(t)}} = \frac{2}{T}\sum_{t=T/2+1}^{T} r(t)e^{i\theta(t)}, \tag{3}$$

$$\widetilde{\chi} \equiv N \cdot \overline{|r(t)e^{i\theta(t)} - \sigma e^{i\varphi}|^2}. \tag{4}$$

One can easily show that $\widetilde{\chi}$ is equivalent to $N \cdot [\overline{r^2(t)} - \sigma^2]$, which measures the difference between r and σ.

In order to investigate phase transition, we have performed a numerical simulation of Eq. (1). We use the second-order Runge-Kutta method [7] with discrete time step $\Delta t = 0.01$. For given b and D, we get the total simulation time $T = 2 \times 10^4$ steps so the first 10^4 steps are discarded as a transient period to achieve steady state and 10^4 steps are used to compute the order parameters.

3 Phase Transition in Active Rotator Model

First of all, we fix the natural frequency $\omega_i = 1$ for all i in order to study the active rotator model on Erdös-Rényi random networks [8]. To generate the random network, we visit each node and connect to other nodes with the probability $p = \langle k \rangle/(N-1)$, where we fix $\langle k \rangle = 5$ for convenience. Then we perform a numerical simulation on the Eq. (1), and investigate phase transition.

Figure 1 shows the phase diagram with $K = 5$. When $D = 0$, the active rotators show a transition at $b = 1$, which corresponds to the natural frequency $\omega = 1$. For $b > 1$, the system becomes a steady state and rotators are fixed to specific angle, otherwise rotators are synchronized and move periodically. And when $b = 0$, Eq. (1) becomes simply coupled identical oscillators without any external fields. As D increases, the order parameter r becomes smaller since the noise disturbs the oscillators to be synchronized. Finally, the system becomes a desynchronized state at $D = 2.5$, which corresponds to the half of coupling strength $K/2$. This transition point $D = K/2$ well agrees with the result of globally coupling case and overall features of phase diagram are not much different from the mean-field expectation.

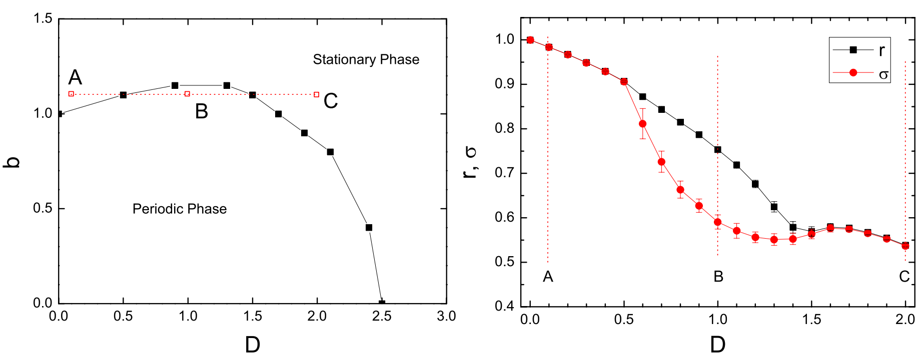

Fig. 1. Phase diagram for the random network with fixed $\omega_i = 1$ for every node i and $K = 5$ (left). When b and D is small enough, the active rotator behaves periodic motion. However, the active rotator goes to the stationary phase if the external field strength b or noise strength D gets strong. On the right: Order parameter behavior for $b = 1.1$. When the noise amplitude is small, oscillators fixed to the external field potential. However, if noise becomes a proper level, oscillators show the periodic motion. For the strong noise, each oscillator scatters and overall behavior shows stationary state.

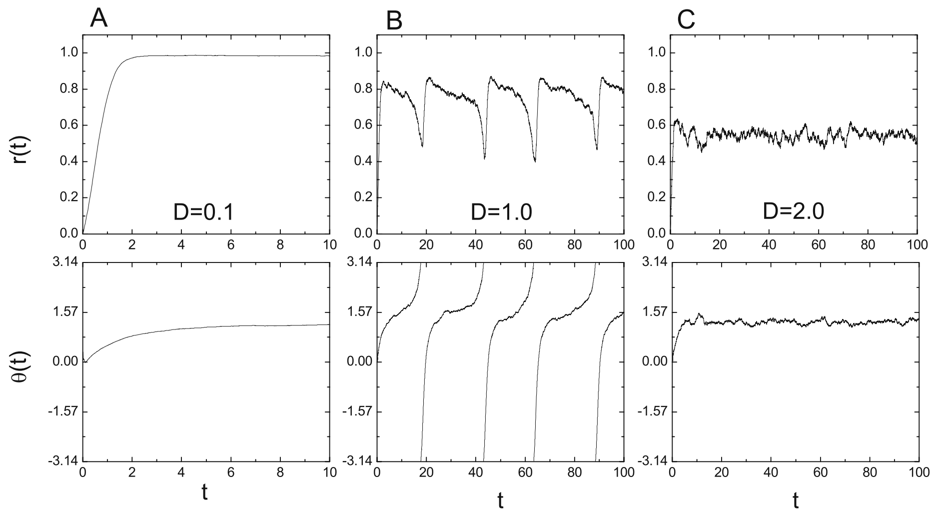

Fig. 2. Three phase states for $b = 1.1$. From the phase diagram, we observe the behaviors of order parameter at the three different points. The label A, B, and C correspond with the three points in Fig. 1.

Noise doesn't always make synchronization bad. For instance, when $b = 1.1$, since the rotator natural frequency ω is smaller than the external fields strength, rotators cannot overcome the external field and stay at a fixed point if the coupling and noise are very weak. However, if the coupling is strong enough and moderate strength of noises are added, the rotators shows a periodic motion. First several oscillators excited by noises try to overcome the external potential and these oscillators pull other oscillators to upward. Average phase gradually moves up to the peak of the potential and then slides down fast. This transition is shown in the left panel of Fig. 1. After $D = 0.5$ the system shows a periodic phase, then it becomes a stationary state again above $D = 1.5$. These three different phases are shown in Fig. 2. The label A, B, and C corresponds to that of Fig. 1. As shown in the middle panels in Fig. 2, the average angle θ rotates quasi-periodically, which means oscillators are rotating together. Therefore, the order parameter σ becomes small since σ is calculated from averaging over complex order parameters including phase information also. Each different phases cancel each other. In this reason, we can find the transition points by observing the difference between order parameters r and σ or the divergent behaviors of the susceptibility $\widetilde{\chi}$.

We extend this study to the non-identical oscillators, i.e. oscillators having the natural frequency distribution [5], and the complex networks having more heterogeneous degree distribution such as *scale-free* networks [9]. As a primary result, we observe that the area of phase diagram is enlarged as networks' degree distribution becomes more heterogeneous. And if oscillators have the natural frequency distribution, the dynamics become more complicated since oscillators make several clusters which have a similar effective frequency.

4 Summary and Remarks

We study the phase transition of the active rotator model on random networks by performing a numerical Langevin simulation. For a specific external field strength, we observe the peculiar phase transition as increasing the noise strength. Even the external field strength is stronger than driving natural frequency, noise-induced coupled oscillators show a periodic rotations, which is also observed in that of globally coupled case. In this model we observe and visualize two different phases, stationary and quasi-periodic phases. Even though the connectivity of random network is local and sparser than all-to-all globally connected network, the overall behavior of oscillators are similar with that of on global network since mean-field approximation works in the case of random networks. We extend this work to the scale-free networks, which have more heterogeneous degree distribution, and the nonidentical oscillators having natural frequency distribution. These variations would show richer dynamics with various applications. This work was supported by KOSEF through the grant No. R17-2007-073-01001-0 and R01-2007-000-20084-0 (H.H).

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