Average Consensus in Delayed Networks of Dynamic Agents with Impulsive Effects

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Abstract. In this paper, the issues of average consensus in undirected delayed networks of dynamic agents with impulsive effects are investigated. The primary contribution of this paper is to propose the consensus schemes in undirected delayed networks of dynamic agents having impulsive effects as well as fixed, switching topology. Based on impulsive stability theory on delayed dynamical systems, we derive some simple sufficient conditions under which all the nodes in the network achieve average consensus globally exponentially. It is shown that average consensus in the networks is heavily dependent on impulsive effects of communication topology of the networks. Subsequently, two numerical examples illustrate and visualize the effectiveness and feasibility of our theoretical results.

Keywords: average consensus, undirected network, multi-agent systems, time-delays, impulsive effects.

1 Introduction

During the last few decades, distributed coordination in dynamic networks of multi-agents has attracted a great deal of attention in many fields such as biology, ecology, robotics, physics, etc., [1,2,3,4,5,6,7,8,9]. This is partly due to potential applications in many areas including cooperative control of unmanned air vehicles (UAVs), formation control, flocking, distributed sensor networks, attitude alignment of cluster of satellites, and congestion control in communication networks [8,10,11,12,13]. A critical problem in distributed coordinated control of multiple agents is to design appropriate protocols and algorithms such that all agents can reach an agreement regarding a certain quantity of interest that depends on the states of all agents. This problem is usually called the consensus problem.

Consensus problems have a long history in the field of computer science, and it has also been considered as the foundation of the distributed computing [14]. The study of consensus problems in groups of experts originated in management science and statistics in 1960s [15]. In the past decade, many researchers have investigated the consensus problems from various perspectives [6,8,10,16,17,18,19,20]: Vicsek et al. proposed a

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simple model for phase transition of a group of self-driven particles and numerically demonstrated complex dynamics of the model [8]; Jadbabaie et al. attempted to provide a formal analysis of emergence of alignment in the simplified model of flocking proposed by Vicsek [16]; Saber et al. addressed the consensus problems under a variety of assumptions on the network topology (fixed or switching), presence or lack of communication delays, and directed or undirected network information flow [6]; Ren et al. extended the results of Jadbabaie et al. and gave some more relaxable conditions [19]; Moreau focused on the consensus problems under dynamically changing interaction topologies[18].

In many evolutionary systems there are two common phenomena: delay effects and impulsive effects [21,22,23,24]. Time delays often occurs in such systems as transportation and communication systems, chemical and metallurgical processes, environmental models and power networks. In many scenarios, networked systems can possess a dynamic topology that is time-varying due to node and link failures/creations, packet-loss [25], asynchronous consensus [26], state-dependence [27], formation reconfiguration, evolution, and flocking [7]. There has been increasing interest in the study of consensus problem in dynamic networks of multi-agents with time delays in the last several years. Among the existing research works, Earl and Strogatz proposed a stability criterion for a network of specific oscillators with time-delayed coupling, a necessary and sufficient consensus condition was established [28]. Olfati-Saber discussed average consensus problems in undirected networks with a common constant communication delay and fixed topology [6]. Moreau studied the case when the common constant delay affects only those variables that are actually being communicated between distinct agents in the network [18]. Sun discussed the average consensus problem in undirected networks of dynamic agents with fixed and switching topologies as well as multiple time-varying communication delays [35]. Lin studied the average-consensus problem in directed networks of agents with both switching topology and time-delays [30]. On the other hand, many evolutionary processes, particularly some biological systems such as biological neural networks and bursting rhythm models in pathology, as well as optimal control models in economics, frequency-modulated signal processing systems, and flying object motions, are characterized by abrupt changes of states at certain time instants. This is the familiar impulsive phenomena [31,32]. However, to our best knowledge, up to now just few works involved the consensus problems in delayed dynamic networks with impulsive effects. Therefore, as an interesting and challenging topic, this motivates the present investigation of consensus issue in dynamic networks of multi-agents associated with time delays and impulsive effects.

In this paper, we investigate average consensus problem in undirected delayed networks of dynamic agents with impulsive effects. The primary contribution of this paper is to propose the consensus schemes in undirected delayed networks of dynamic agents having impulsive effects as well as fixed, switching topology. Based on impulsive stability theory on delayed dynamical systems, we derive some simple sufficient conditions under which all the nodes in the network achieve average consensus globally exponentially. It is shown that average consensus in the networks is heavily dependent on impulsive effects of communication topology of the networks. Subsequently, two numerical examples illustrate and visualize the effectiveness and feasibility of our theoretical results.

An outline of this paper is as follows. In Section 2, some mathematical preliminaries are first prepared. Section 3 introduces the problem formulations with respect to average consensus problem in dynamic networks of multi-agents with time delays and impulsive effects. Section 4 deals with the average consensus problem in undirected delayed networks of dynamic agents having impulsive effects as well as fixed, switching topology. Some numerical simulations are presented in Section 5. Finally, some conclusions are drawn in Section 6.

2 Mathematical Preliminaries

Throughout this paper, the following notations and definitions will be used.

Let R denotes the set of real number, and $R^n = \underbrace{R \times R \times \cdots \times R}_{n}$, $R^{n\times n}$ is $n \times n$ the set of real matrices, $\mathrm{diag}(\gamma_1, \cdots, \gamma_n) \in R^{n\times n}$ is the diagonal matrix with diagonal entries $\gamma_i\,(i = 1, \cdots, n)$. For $x = (x_1, \cdots, x_n)^T \in R^n$, $A = (a_{ij})_{n\times n} \in R^{n\times n}$, x^T denotes its transpose, we denote $|x| = (|x_1|, \cdots, |x_n|)^T$. The norm of the vector x is defined as $\|x\| = (x^\top x)^{1/2}$. For $\psi : R \to R$, denote: $\psi(t^+) = \lim_{s\to 0^+} \psi(t + s)$, $\psi(t^-) = \lim_{s\to 0^-} \psi(t + s)$, $[\psi(t)]_\tau = \sup_{-\tau\le s\le 0}\{\psi(t + s)\}$, $[\psi(t)]_{\tau^-} = \sup_{-\tau\le s<0}\{\psi(t + s)\}$. $PC([t_0 - \tau, t_0], R^n])$ denotes the set of all functions of the bounded variation and right-hand continuous on any compact subinterval of $[t_0 - \tau, t_0]$. Denote $\|\phi(t)\|_\tau = \sup_{-\tau\le s\le 0} \|\phi(t + s)\|$.

For the later use, the following inequality and the famous Halanay differential inequality on delayed impulsive dynamical systems are listed in the following:

Lemma 1. *(Park [36]). For any positive scalar ϵ and vectors x and y, the following inequality holds:*

$$x^T y + y^T x \le \epsilon x^T x + \epsilon^{-1} y^T y. \tag{1}$$

Lemma 2. *(Yang and Xu [34]). Suppose $p > q \ge 0$ and $u(t)$ satisfies scalar impulsive differential inequality:*

$$\begin{cases} D^+u(t) \le -pu(t) + q[u(t)]_\tau, & t \ne t_k, \quad t \ge t_0 \\ u(t_k^+) \le b_k u(t_k^-) + d_k[u(t_k)]_{\tau^-}, & k \in N \\ u(t) = \phi(t), & t \in [t_0 - \tau, t_0] \end{cases} \tag{2}$$

where $u(t)$ is continuous at $t \ne t_k$, $t \ge t_0$, $u(t_k) = u(t_k^+)$, and $u(t_k^-)$ exists, $\phi \in PC$ with $n = 1$. Then

$$u(t) \le \Big(\prod_{t_0<t_k\le t} \theta_k\Big) e^{-\lambda(t-t_0)} \|\phi(t_0)\|_\tau, \quad t \ge t_0 \tag{3}$$

where $\delta_k := max\{1, |b_k| + |d_k| e^{\lambda\tau}\}$ and $\lambda > 0$ is a solution of the inequality $\lambda - p + qe^{\lambda\tau} \le 0$.

3 Problem Formulations

3.1 Graph Theory

Let $G = (\mathscr{V}, \mathscr{E}, \mathscr{A})$ be a weighted undirected graph of order $n\,(n \geq 2)$ with the set of nodes $\mathscr{V} = \{v_1, \cdots, v_n\}$, set of edges $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$, and a symmetric weighted adjacency matrix $\mathscr{A} = [a_{ij}]$ with nonnegative adjacency elements a_{ij}. The node indexes belong to a finite index set $\mathscr{I} = \{1, 2, \cdots, n\}$. An edge of G is denoted by $e_{ij} = (v_i, v_j)$. The adjacency elements associated with the edges of the graph are positive, i.e., $e_{ij} \in \mathscr{E}$ if and only if $a_{ij} > 0$. Moreover, we assume $a_{ii} = 0$ for all $i \in \mathscr{I}$. The set of *neighbors* of the node v_i is denoted by $N_i = \{v_j \in \mathscr{V} : (v_i, v_j) \in \mathscr{E}\}$. An undirected graph is called *connected* if any two distinct nodes of the graph can be connected via a path that follows the edges of the graph.

3.2 Consensus Problems

Let $x_i \in R$ denote the value of the node v_i. We refer to $G_x = (G, x)$ with $x = (x_1, \cdots, x_n)^T$ as a *network* (or *algebraic graph*) with the value $x \in R^n$ and the *topology* (or *information flow*) G. The value of a node might represent physical quantities such as attitude, position, temperature, voltage, and so on. We say both the nodes v_i and v_j *agree* in a network if and only if $x_i = x_j$. We say the nodes of a network have reached a *consensus* if and only if $x_i = x_j$ for all $i, j \in \mathscr{I}, i \neq j$. Whenever the nodes of a network are all in agreement, the common value of all nodes is called the *group decision value*.

Suppose each node of a graph is a *dynamic agent* with dynamics

$$\dot{x}_i = f(x_i, u_i), \qquad i \in \mathscr{I} \tag{4}$$

A *dynamic graph* (or *dynamic network*) is a dynamical system with a state (G, x) in which the value x evolves according to the *network dynamics* $\dot{x} = F(x, u)$. Here, $F(x, u)$ is the column-wise concatenation of the elements $f(x_i, u_i)$ for $1, \cdots, n$. In a dynamic network with switching topology, the information flow G is a discrete-state of the system that changes in time.

Let $\mathscr{X} : R^n \to R$ be a function of n variables $x_1, \cdots, x_n$ and $a = x(0)$ denote the initial state of the system. The $\mathscr{X}$*-consensus problem* in a dynamic graph is a distributed way to calculate $\mathscr{X}(a)$ by applying inputs u_i that only depend on the states of node v_i and its neighbors. We say a state feedback

$$u_i = k_i(x_{j_1}, \cdots, x_{j_{m_i}}) \tag{A}$$

is a *protocol* with topology G if the cluster $J_i = \{v_{j_1}, \cdots, v_{j_{m_i}}\}$ of nodes with indexes $j_1, \cdots, j_{m_i} \in \mathscr{I}$ satisfies the property $J_i \subseteq \{v_i\} \cup N_i$. In addition, if $|J_i| < n$ for all $i \in \mathscr{I}$, (A) is called a *distributed protocol*.

We say protocol (A) asymptotically solves the $\mathscr{X}$-consensus problem if and only if there exists an asymptotically stable equilibrium x^* of $\dot{x} = F(x, k(x))$ satisfying $x_i^* = \mathscr{X}(x(0))$ for all $i \in \mathscr{I}$. We are interested in distributed solutions of the $\mathscr{X}$–consensus problem in which no node is connected to all other nodes. The special case with $\mathscr{X}(x) = Ave(x) = \dfrac{1}{n}(\sum_{i=1}^{n} x_i)$, is called *average-consensus*.

Solving the average-consensus problem is an example of *distributed computation* of a linear function $\mathscr{X}(a) = Ave(a)$ using a network of dynamic systems (or integrators). This is a more challenging task than reaching a consensus with initial state a. Since an extra condition $x_i^* = \mathscr{X}(a)$, $\forall i \in \mathscr{I}$ has to be satisfied which relates the limiting state x^* of the system to the initial state a.

3.3 Model Formulations

In this paper, we are interested in discussing average consensus problem in undirected delayed networks of dynamic agents having impulsive effects as well as fixed, switching topology, where the information (from v_j to v_i) passes through edge (v_i, v_j) with the coupling time-delays $0 < \tau(t) \leq \tau$. To solve such a problem, we use the following consensus protocol:

$$u_i(t) = \sum_{v_j \in N_i} a_{ij}\Big(x_j(t-\tau(t)) - x_i(t-\tau(t))\Big)Dw_j(t), \tag{5}$$

where D denotes the distributional derivative, $w_i : J = [t_0, +\infty) \to R$ are functions of the bounded variations which are right-continuous on any compact subinterval of J. We remark that the model formulation given above implies that Dw_i represents the impulsive effect of switching topology in the dynamical network.

Under the consensus protocol (5), the system (4) can be described by the following measure differential equations:

$$Dx_i(t) = \sum_{v_j \in N_i} a_{ij}\Big(x_j(t-\tau(t)) - x_i(t-\tau(t))\Big)Dw_j(t). \tag{6}$$

Without loss of generality, we assume that

$$w_j(t) = t + \sum_{m=1}^{\infty} \mu_m H_m(t), \tag{7}$$

with discontinuity points

$$t_1 < t_2 < \cdots < t_m < \cdots, \qquad \lim_{m\to\infty} t_m = \infty,$$

where μ_m are constants, represents the strength of impulsive effects of connection between the *j*th node and the *i*th node at time t_m, and $H_m(t)$ are Heaviside functions defined by

$$H_m(t) = \begin{cases} 0, & t < t_m, \\ 1, & t \geq t_m. \end{cases} \tag{8}$$

It is easy to see that

$$Dw_j = 1 + \sum_{m=1}^{\infty} \mu_m \delta(t - t_m), \tag{9}$$

where $\delta(t)$ is the Dirac impulsive function. Clearly, if $\mu_k = 0$, then the model (6) becomes continuous consensus scheme with time delays [35].

$$\dot{x}_i(t) = \sum_{v_j \in N_i} a_{ij}\Big(x_j(t-\tau(t)) - x_i(t-\tau(t))\Big). \tag{10}$$

For the impulsive functional differential equation (6), its initial conditions are given by $x_i(t) = \phi_i(t) \in PC([t_0 - \tau, t_0], R^n)$. We always assume that Eq. (6) has a unique solution with respect to initial conditions [33,34].

Rewrite (6) in matrix form as

$$Dx(t) = -Lx(t - \tau(t))Dw(t), \tag{11}$$

where $L = [l_{ij}]$ is called *graph Laplacian* (Laplacian matrix or Laplacian) induced by the information flow G and is defined by

$$l_{ij} = \begin{cases} \sum\limits_{k=1,k\neq i}^{n} a_{ik}, & j = i \\ -a_{ij}, & j \neq i \end{cases} \tag{12}$$

It is easy to see that the matrix L is given by

$$L = \mathcal{L}(G) = D - A \in R^{n\times n}, \tag{13}$$

where $A = [a_{ij}]_{n\times n}$, and $D = diag(d_1, \cdots, d_n) \in R^{n\times n}$, which is called as a degree matrix of the topology G, whose diagonal elements $d_i = \sum\limits_{j\in N_i} a_{ij}$ for $i = 1, 2, \cdots, n$.

As indicated in [6], the topology G with the Laplacian matrix L is a connected undirected graph, then all eigenvalues but one simple eigenvalue at zero of L have positive real-parts. Furthermore, it always has a zero eigenvalue corresponding to a right eigenvector $\mathbf{1} = (1, , \cdots, 1)^T$. This means that $rank(L) \leq n - 1$.

Notice that $\mathbf{1}^T L = 0$. Thus, $\alpha = Ave(x)$ is an invariant quantity. The invariance of $Ave(x)$ allows decomposition of x according to the following equation:

$$x = \alpha\mathbf{1} + \delta, \tag{14}$$

where $\alpha = Ave(x)$ and $\delta = (\delta_1, \cdots, \delta_n)^T \in R^n$ satisfies $\mathbf{1}^T\delta = 0$. Here, we refer to δ as the (group) *disagreement vector*. The vector δ is orthogonal to $\mathbf{1}$ and belongs to an $(n-1)$-dimensional subspace. Moreover, δ evolves according to the (group) *disagreement dynamics* given by

$$D\delta(t) = -L\delta(t - \tau(t))Dw(t). \tag{15}$$

In what follows, we will consider the average consensus problem of the in two cases: 1) networks with fixed topology; 2) networks with switching topology. We will prove that under appropriate conditions the system achieves average consensus globally exponentially.

4 Average Consensus in Delayed Networks with Impulsive Effects

4.1 Networks with Fixed Topology

Based on impulsive stability theory on delayed dynamical systems, the following sufficient condition for average consensus of the system (11) is established.

Theorem 1. *Let the eigenvalues of the Laplacian matrix L can be ordered as*

$$0 = \lambda_1(L) < \lambda_2(L) \leq \cdots \leq \lambda_n(L).$$

Assume that the topology of G with L is connected and the following conditions are satisfied for $m \in Z^+ = \{1, 2, \cdots, \infty\}$:

(A_1) Denote $p = 2\lambda_2(L) - \lambda_n^2(L)\tau$, $q = \lambda_n^2(L)\tau$, such that $\tau < \dfrac{\lambda_2(L)}{\lambda_n^2(L)}$.

(A_2) Let $\lambda > 0$ and $\varepsilon > 0$ satisfy $\lambda - p + qe^{2\lambda\tau} \leq 0$, and

$$\theta_m = 1 + \varepsilon + (1 + \varepsilon^{-1})\mu_m^2\lambda_n^2 e^{2\lambda\tau}, \quad \theta = \sup_{m \in z^+}\{\frac{\ln \theta_m}{t_m - t_{m-1}}\}.$$

such that $\theta < \lambda$.
Then the dynamical network (11) achieve average consensus globally exponentially. □

Proof. Now we rewrite Eq. (15) as

$$D\delta(t) = -L\delta(t - \tau(t))Dw(t) = \Big[- L\delta(t) + L \int_{t-\tau(t)}^{t} \dot{\delta}(s)ds\Big]Dw(t). \tag{16}$$

Let us construct the Lyapunov functional as the following:

$$V(t) = \frac{1}{2}\delta^T(t)\delta(t). \tag{17}$$

For $t \neq t_m$, we have

$$\begin{aligned} D^+V(t) &= -\delta^T(t)L\delta(t) - \delta^T(t)L \int_{t-\tau(t)}^{t} L\delta(s - \tau(s))ds \\ &\leq -\lambda_2(L)\delta^T(t)\delta(t) + \frac{1}{2}\lambda_n^2(L) \int_{t-\tau(t)}^{t} \Big[\delta^T(t)\delta(t) + \delta^T(s - \tau(s))\delta(s - \tau(s))\Big]ds \\ &\leq -\lambda_2(L)\delta^T(t)\delta(t) + \lambda_n^2(L)\tau\Big[V(t) + \sup_{t-2\tau \leq s \leq t} V(s)\Big] \\ &\leq -pV(t) + q \sup_{t-2\tau \leq s \leq t} V(s). \end{aligned} \tag{18}$$

On the other hand, by using the properties of Dirac measure, we have

$$\delta(t_m) = \delta(t_m^-) - L\delta(t_m - \tau(t))\mu_m.$$

Then, by Lemma 1, we can get

$$\begin{aligned} V(t_m) &= \frac{1}{2}|\delta(t_m)|^T|\delta(t_m)| \\ &\leq \frac{1}{2}[|\delta(t_m^-)|^T|\delta(t_m^-)| + |\mu_m||\delta(t_m^-)|^T L|\delta(t_m - \tau(t))| \\ &\quad +|\mu_m||\delta(t_m - \tau(t))|^T L|\delta(t_m^-)| + \mu_m^2|\delta(t_m - \tau(t))|^T L^2|\delta(t_m - \tau(t))|] \\ &\leq \frac{1}{2}[(1 + \varepsilon)|\delta(t_m^-)|^T|\delta(t_m^-)| + (1 + \varepsilon^{-1})\mu_m^2\lambda_n^2|\delta(t_m - \tau(t))|^T|\delta(t_m - \tau(t))|] \\ &\leq (1 + \varepsilon)V(t_m^-) + (1 + \varepsilon^{-1})\mu_m^2\lambda_n^2 \sup_{-2\tau \leq s < 0} V(t_m + s). \end{aligned} \tag{19}$$

It follows from Lemma 2 that, for $t_{m-1} \leq t < t_m$, $m \in Z^+$, we have

$$\begin{aligned} V(t) &\leq \theta_1 \cdots \theta_{m-1} e^{\lambda(t-t_0)} \sup_{-2\tau \leq s \leq 0} V(t_0 + s) \\ &\leq e^{-(\lambda-\theta)(t-t_0)} \sup_{-2\tau \leq s \leq 0} V(t_0 + s). \end{aligned} \tag{20}$$

Therefore, for all $t \geq t_0$,

$$V(t) \leq e^{-(\lambda-\theta)(t-t_0)} \sup_{-2\tau \leq s \leq 0} V(t_0 + s). \tag{21}$$

This completes the proof of Theorem 1. □

4.2 Networks with Switching Topology

Since the nodes of the network are moving, it is not hard to imagine that some of the existing communication links can fail simply due to the existence of an obstacle between two agents. The opposite situation can arise where new links between nearby agents are created because the agents come to an effective range of detection with respect to each other. In terms of the network topology G, this means that certain number of edges are added or removed from the graph. Here, we are interested in investigating such a problem: for a *network with switching topology*, whether it is still possible to reach a consensus or not. In this case, the following hybrid system is considered:

$$Dx(t) = -L_k x(t - \tau(t)) Dw(t), \quad k = s(t) \in \mathscr{I}_0, \tag{22}$$

where $L_k = L(G_k)$ is the Laplacian of graph G_k, and $s(t) : [0, +\infty) \to \mathscr{I}_0 \subseteq \{1, \cdots, \frac{n(n-1)}{2}\}$ ($\frac{n(n-1)}{2}$ denotes the total number of all possible undirected graphs) is a switching signal that determines the communication topology G. If $s(t)$ is a constant function, then the corresponding topology is fixed.

Under arbitrary switching signal, $\alpha = Ave(x)$ is also an invariant quantity. This allows the decomposition of any solution $x(t)$ of the system (22) in the form Eq. (14). Therefore, the disagreement switching system induced by the system (22) takes the following form:

$$D\delta(t) = -L_k \delta(t - \tau(t)) Dw(t), \quad k = s(t) \in \mathscr{I}_0. \tag{23}$$

Theorem 2. *Let the eigenvalues of the Laplacian matricex L_k be ordered as*

$$0 = \lambda_1(L_k) < \lambda_2(L_k) \leq \cdots \leq \lambda_n(L_k).$$

and denote

$$\overline{\lambda}_2 = \min \lambda_2(L_k), \qquad \overline{\lambda}_n = \max \lambda_n(L_k).$$

Assume that the topology of G_k with L_k is connected, and the following conditions are satisfied for $m \in Z^+ = \{1, 2, \cdots, \infty\}$:

(A_1) *Denote $p = 2\overline{\lambda}_2 - \overline{\lambda}_n^2\tau$, $q = \overline{\lambda}_n^2\tau$, such that $\tau < \frac{\overline{\lambda}_2}{\overline{\lambda}_n^2}$.*

(A_2) *Let $\lambda > 0$ and $\varepsilon > 0$ satisfy $\lambda - p + qe^{2\lambda\tau} \leq 0$, and*

$$\theta_m = 1 + \varepsilon + (1 + \varepsilon^{-1})\mu_m^2\overline{\lambda}_n^2 e^{2\lambda\tau}, \quad \theta = \sup_{m \in z^+}\{\frac{\ln\theta_m}{t_m - t_{m-1}}\}.$$

such that $\theta < \lambda$.

Then the dynamical network (22) achieve average consensus globally exponentially. □

Proof. The proof of Theorem 2 is similar to that of Theorem 1 by the same Lyapunov function, and hence it is omitted. The proof of Theorem 2 is completed. □

5 Numerical Simulations

In this section, numerical simulations will be given to illustrate the theoretical results obtained in the previous section.

Example 1. Consider an undirected network with fixed topology G_a in Fig. 1. It is easy to see that G_a is a connected graph. For the case with fixed topology, we have

$$\mathbf{L}_a = \begin{pmatrix} 2 & -1 & 0 & \cdots & -1 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 2 \end{pmatrix}$$

and

$$\lambda_2 = 0.3820, \qquad \lambda_{10} = 4, \qquad \frac{\lambda_2}{\lambda_{10}^2} = 0.0239.$$

By taking $\tau = 0.0200$, we obtain $p = 2\lambda_2 - \lambda_{10}^2\tau = 0.4440$, $q = \lambda_{10}^2\tau = 0.3200$.

For simplicity, we consider the equidistant impulsive interval $t_m - t_{m-1} = 0.5$, and $\mu_m = 0.0050$. By picking $\varepsilon = 0.01$, it is easy to find that $\theta_m = 1.0506$, and

$$\frac{\ln\theta_m}{t_m - t_{m-1}} = 0.0987 < \lambda = 0.1224,$$

where $\lambda = 0.1224$ is an unique solution of equation: $\lambda - p + qe^{2\lambda\tau} = 0$. It follows from Theorem 1 that the dynamical network (11) with the graph G_a achieve average

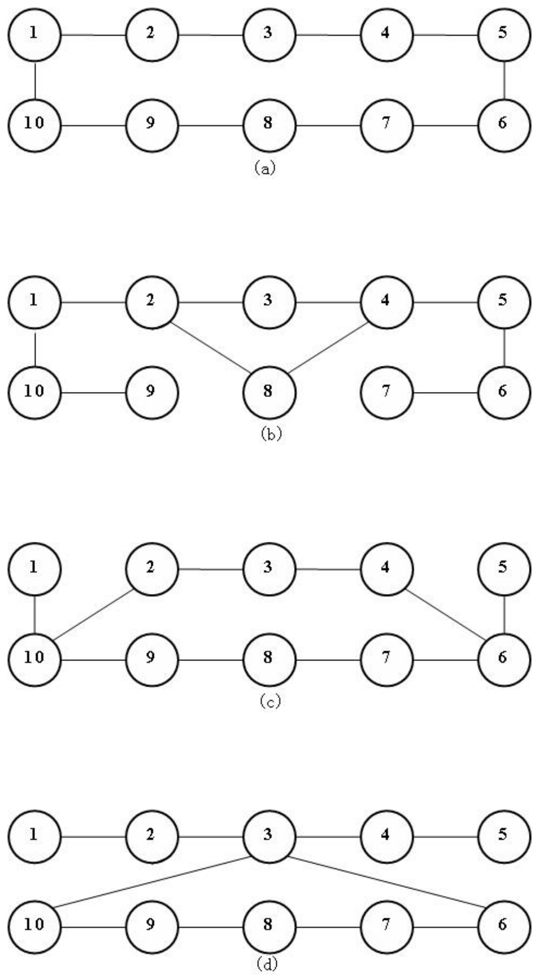

Fig. 1. Four examples of undirected graphs

consensus globally exponentially. Fig. 2 is the simulations results corresponding to this situations.

Example 2. Consider an undirected network with switching topology $\{G_a, G_b, G_c, G_d\}$ in Fig. 1. In Fig. 3, a finite state machine is shown with four states $\{G_a, G_b, G_c, G_d\}$, which represent the discrete-states of a network with switching topology and time-delays as a hybrid system. The hybrid system starts at the discrete state G_a, and switches every 0.2s to the next state according to the state machine as shown in Fig. 3. In this case, some of the existing communication links fail and some of them are created due to the moving of the agents. It is easy to see that the topologies G_b, G_c and G_d are all connected,

$$\mathbf{L_b} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

$$\mathbf{L_c} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 \end{pmatrix},$$

and

$$\mathbf{L_d} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

with

$$\lambda_2(L_b) = 0.1522, \quad \lambda_{10}(L_b) = 4.8260, \quad \lambda_2(L_c) = 0.3249,$$
$$\lambda_{10}(L_c) = 4.4812, \quad \lambda_2(L_d) = 0.3187, \quad \lambda_{10}(L_d) = 5.3234,$$

and so

$$\overline{\lambda}_2 = \min \lambda_2(L_k) = 0.1522, \quad \overline{\lambda}_{10} = \max \lambda_{10}(L_k) = 5.3234, \quad \frac{\overline{\lambda}_2}{\overline{\lambda}_{10}^2} = 0.0054.$$

By taking $\tau = 0.0050$, we obtain $p = 2\overline{\lambda}_2 - \overline{\lambda}_{10}^2\tau = 0.1627$, $q = \overline{\lambda}_{10}^2\tau = 0.1417$.

We also consider the equidistant impulsive interval $t_m - t_{m-1} = 0.8$, and $\mu_m = 0.0015$. By picking $\varepsilon = 0.01$, it is easy to find that $\theta_m = 1.0164$, and

$$\frac{\ln \theta_m}{t_m - t_{m-1}} = 0.0203 < \lambda = 0.0210,$$

where $\lambda = 0.0210$ is an unique solution of equation: $\lambda - p + qe^{2\lambda\tau} = 0$. From Theorem 2, we know that the dynamical network (22) with switching topology $\{G_a, G_b, G_c, G_d\}$ given in Fig. 3. achieve average consensus globally exponentially, whose the simulations is shown in Fig. 4.

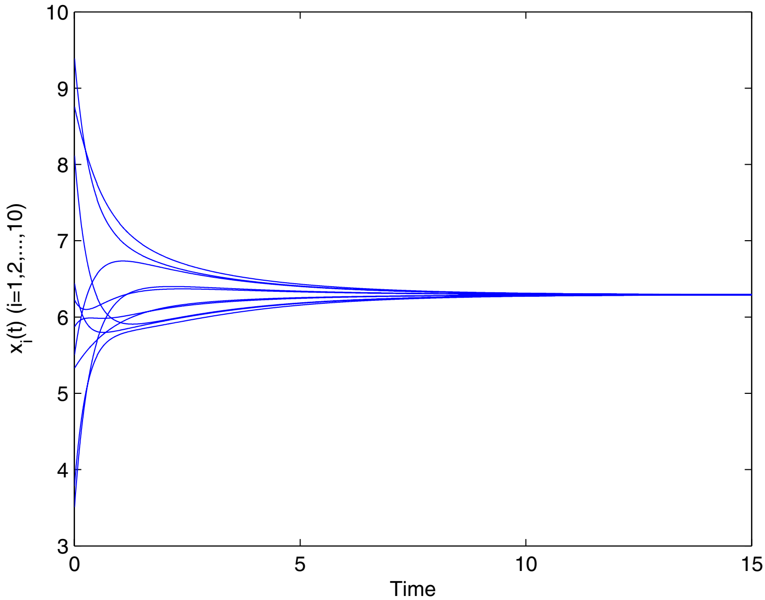

Fig. 2. Consensus problem with fixed topology and communication time-delays on graph G_a given in Fig. 1

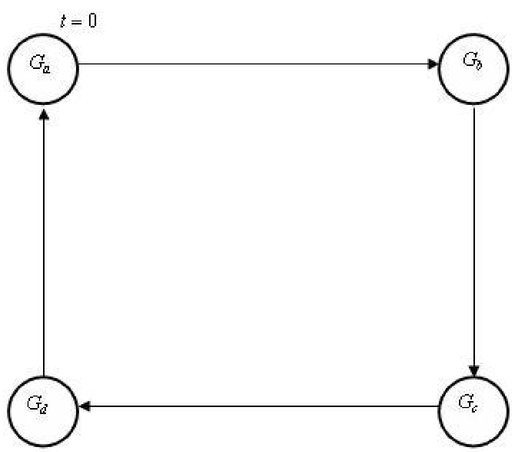

Fig. 3. Finite automation with four states representing the discrete-states of a network with switching topology and time-delays

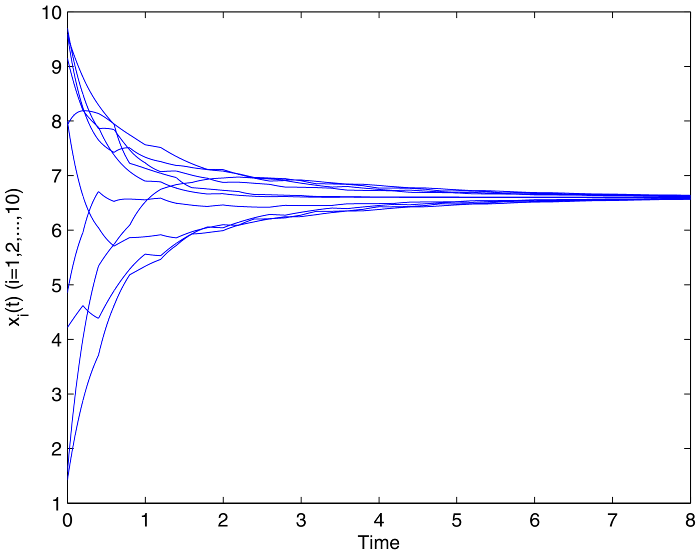

Fig. 4. Consensus problem with switching topology and communication time-delays given in Fig. 1

6 Conclusions

In this paper, we have investigated the consensus problems in undirected delayed networks of dynamic agents having impulsive effects as well as fixed, switching topology. Based on impulsive stability theory on delayed dynamical systems, we derive some simple sufficient conditions under which all the nodes in the network achieve average consensus globally exponentially. It is shown that average consensus in the networks is heavily dependent on impulsive effects of communication topology of the networks. To this end, two numerical examples have been presented to demonstrate the effectiveness of our theoretical results.

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