Asymptotic Behavior of Ruin Probability in Insurance Risk Model with Large Claims

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Abstract. For the renewal risk model with subexponential claim sizes, we established for the finite time ruin probability a lower asymptotic estimate as initial surplus increases, subject to the demand that it should hold uniformly over all time horizons in an infinite interval. In the case of Poisson model, we also obtained the upper asymptotic formula so that an equivalent formula was derived. These extended a recent work partly on the topic from the case of Pareto-type claim sizes to the case of subexponential claim sizes and, simplified the proof of lower bound in Leipus and Siaulys ([9]).

Keywords: asymptotic formula, finite time ruin probability, strongly subexponential distributions, the compound Poisson/renewal model, uniform convergence.

1 Introduction

Consider the renewal risk model, in which the claim sizes Z_i, $i = 1, 2, \ldots$, form a sequence of independent, identically distributed (i.i.d.), nonnegative random variables with common distribution B, while the inter-occurrence times θ_i, $i = 1, 2, \ldots$, form another sequence of i.i.d. positive random variables with common finite mean $1/\lambda$. Two sequences $\{Z_i, i = 1, 2, \ldots\}$ and $\{\theta_i, i = 1, 2, \ldots\}$ are assumed to be mutually independent. The locations of claims $\tau_k = \sum_{i=1}^{k} \theta_i$, $k = 1, 2, \ldots$, constitute a renewal counting process

$$N(t) = \max\{k = 1, 2, \ldots : \tau_k \in (0, t]\}, \qquad t \geq 0, \tag{1}$$

with a mean function $\lambda(t) = \mathrm{E}N(t) \sim \lambda t$ as $t \to \infty$. The meaning of $\sim$ will be given in the following. The surplus process is then defined as

$$R(t) = x + ct - \sum_{i=1}^{N(t)} Z_i, \qquad t \geq 0, \tag{2}$$

where $R(0) = x \geq 0$ denotes the initial surplus, $c > 0$ denotes the constant premium rate, and a summation over an empty set of index is 0 by convention.

Ruin probability is one of the most important concept in modern actuarial risk theory.

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Definition 1. *Finite time ruin probability within time t is defined as*

$$\psi(x;t) = \Pr\left(\inf_{0\le s\le t} R(s) < 0 \;\middle|\; R(0) = x \right), \qquad t \ge 0. \tag{3}$$

If $t = \infty$,

$$\psi(x;\infty) = \lim_{t\to\infty} \psi(x;t) = \Pr\left(\inf_{0\le s<\infty} R(s) < 0 \;\middle|\; R(0) = x \right) \tag{4}$$

is called ultimate ruin probability.

In order for the ultimate ruin not to be certain, it is natural to assume the safety loading condition

$$\mu = \frac{c}{\lambda} - \mathrm{E}Z_1 > 0. \tag{5}$$

We refer readers to Asmussen ([1]) and ([2]) for a nice reviews on the study of the finite time ruin probability and to Tang ([12]) for a list of references devoted to this study. Our goal in the current paper is to derive an asymptotic estimate as the initial surplus x increases for the finite time ruin probability $\psi(x;t)$, subject to the requirement that the asymptotic result should hold uniformly over all time horizons t in an infinite interval.

Hereafter, all limit relationships are for $x \to \infty$ unless stated otherwise. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write$a(x) \lesssim b(x)$ if $\limsup a(x)/b(x) \le 1$, write $a(x) \gtrsim b(x)$ if $\liminf a(x)/b(x) \ge 1$, and write $a(x) \sim b(x)$ if both. As done in the main result of this paper, we shall assign a certain uniformity property to some asymptotic relations under discussion. Let us take an example to clarify the meaning of uniformity. For two positive bivariate functions $a(\cdot;\cdot)$ and $b(\cdot;\cdot)$, we say that the asymptotic relation $a(x;t) \sim b(x;t)$ holds uniformly over all t in a nonempty set Δ if

$$\lim_{x\to\infty} \sup_{t\in\Delta} \left| \frac{a(x;t)}{b(x;t)} - 1 \right| = 0. \tag{6}$$

That is, for each fixed $\varepsilon > 0$, there exists some $x_0 > 0$ irrespective to t such that the two-sided inequality

$$(1-\varepsilon)b(x;t) \le a(x;t) \le (1+\varepsilon)b(x;t) \tag{7}$$

holds for all $x \ge x_0$ and $t \in \Delta$. This is further equivalent to that both $a(x) \lesssim b(x)$ and $a(x) \gtrsim b(x)$ hold uniformly over all $t \in \Delta$. Admittedly, results that hold with such a uniformity property are of higher theoretical and practical interest.

Heavy-tailed risk has played an important role in insurance and finance because it can describe large claims; see Embrechts et al. ([3]). We shall mainly discuss heavy-tailed claims in this paper. The most important class of heavy-tailed distributions is the subexponential class.

Definition 2. *a distribution F on $[0,\infty)$ is said to be subexponential, written as $F \in \mathcal{S}$, if its right tail $\overline{F} = 1 - F$ satisfies $\overline{F}(x) > 0$ for all x and the relation*

$$\overline{F^{*2}}(x) \sim 2\overline{F}(x) \tag{8}$$

holds, where F^{*2} denotes the convolution of F with itself.

More generally, a distribution F on $(-\infty, \infty)$ is still said to be subexponential if the distribution $F^+(x) = F(x)1_{(0\le x<\infty)}$ is subexponential, where 1_A denotes the indicator function of A. It is well known that every subexponential distribution F is long tailed, written as $F \in \mathcal{L}$, in the sense that the relation

$$\overline{F}(x+y) \sim \overline{F}(x) \tag{9}$$

holds for each fixed real number y; see, for example, Embrechts et al. ([3], Lemma 1.3.5).

Very often the class $\mathcal{S}$ appears to be too wide to possess desirable probabilistic properties. For this reason, researchers in applied probability have introduced many subclasses of $\mathcal{S}$ to meet certain special requirements. In this regard, Korshunov ([8]) introduced the class of strongly subexponential distributions. For a distribution F on $(-\infty, \infty)$ with $0 < m = \int_0^\infty \overline{F}(u)\mathrm{d}u < \infty$ and for each fixed $l \in (0, \infty]$, we write

$$\overline{F_l}(x) = \begin{cases} \min\left\{1, \int_x^{x+l} \overline{F}(u)\mathrm{d}u\right\}, & x \ge 0, \\ 1, & x < 0. \end{cases} \tag{10}$$

Clearly, for each $l \in (0, \infty]$ the function F_l defines a standard distribution on $[0, \infty)$. In the terminology of Korshunov ([8]), the distribution F is said to be strongly subexponential, denoted by $F \in \mathcal{S}_*$, if the relation

$$\lim_{x\to\infty} \frac{\overline{F_l^{*2}}(x)}{\overline{F_l}(x)} = 2 \tag{11}$$

holds uniformly over all $l \in [1, \infty]$. It is easy to check that relation (11) with an arbitrarily fixed number $l \in [1, \infty)$ implies $F \in \mathcal{S}$; see Kaas and Tang ([6]). Hence, $\mathcal{S}_*$ is a subclass of $\mathcal{S}$. From the discussions of Korshunov ([8]), we see that the class $\mathcal{S}_*$ covers almost all useful subexponential distributions with $m < \infty$. Specifically, the class $\mathcal{S}_*$ contains all Pareto-like distributions with $m < \infty$, all lognormal-like distributions, and all heavy-tailed Weibull-like distributions. We should point out that, Pareto-like function class with index $-\alpha$ is usually denoted by $\mathcal{R}_{-\alpha}$: if

$$\overline{F}(x) = x^{-\alpha} L(x), \quad x > 0,$$

where $L(x)$ is a slowly varying function as $x \to \infty$ and index $-\alpha < 0$. $\mathcal{R}_{-\alpha}$ is also called regularly varying function class.

2 Main Result and Insurance Significance

The main results of this paper are as follows:

Theorem 1. *Consider the renewal model with the safety loading condition (5), which is introduced at the very beginning of this paper. If $B \in \mathcal{L}$, then for every positive function $f(\cdot)$ with $f(x) \to \infty$, it holds uniformly over all $t \in [f(x), \infty]$ that*

$$\psi(x; t) \gtrsim \frac{1}{\mu} \int_x^{x+\mu\lambda t} \overline{B}(u)\mathrm{d}u. \tag{12}$$

When $t=\infty$, formula (12) is reduced to

$$\psi(x;\infty) \gtrsim \frac{1}{\mu}\int_x^\infty \overline{B}(u)\mathrm{d}u. \tag{13}$$

Specially, in the case of Poisson risk model, we can get the following asymptotic formula:

Theorem 2. *Consider the Compound Poisson model with the safety loading condition (5). If $B\in\mathcal{L}$, then for every positive function $f(\cdot)$ with $f(x)\to\infty$, it holds uniformly over all $t\in[f(x),\infty]$ that*

$$\psi(x;t) \lesssim \frac{1}{\mu}\int_x^{x+\mu\lambda t} \overline{B}(u)\mathrm{d}u. \tag{14}$$

When $t=\infty$, formula (16) is reduced to

$$\psi(x;\infty) \lesssim \frac{1}{\mu}\int_x^\infty \overline{B}(u)\mathrm{d}u, \tag{15}$$

which is well known, first established by Veraverbeke ([13]) and Embrechts and Veraverbeke ([4]). which is well known, first established by Veraverbeke ([13]) and Embrechts and Veraverbeke ([4]).

An corollary of These two Theorems can be obtained directly.

Corollary 1. *Consider the Compound Poisson model with the safety loading condition (5). If $B\in\mathcal{L}$, then for every positive function $f(\cdot)$ with $f(x)\to\infty$, it holds uniformly over all $t\in[f(x),\infty]$ that*

$$\psi(x;t) \sim \frac{1}{\mu}\int_x^{x+\mu\lambda t} \overline{B}(u)\mathrm{d}u. \tag{16}$$

When $t=\infty$, formula (16) is reduced to

$$\psi(x;\infty) \sim \frac{1}{\mu}\int_x^\infty \overline{B}(u)\mathrm{d}u, \tag{17}$$

Tang ([12]) established (16) in the form of equivalence in the renewal model under the assumption, among others, that the distribution B is consistently varying tailed in the sense that

$$\lim_{l\nearrow 1}\limsup_{x\to\infty}\frac{\overline{B}(lx)}{\overline{B}(x)} = 1. \tag{18}$$

Hence, his result works essentially only for the case of Pareto-like claim sizes. Recently, under the three assumptions as following:

(1) There exists a nonnegative function q: $R_+\to R_+$ such that

$$Q(u)=\int_0^u q(v)dv, \quad u\in R_+ \ \ and \ \ \limsup_{u\to\infty}\frac{uq(u)}{Q(u)} =: r \ \ is \ \ finite;$$

(2) The hazard rate $q(u)$ satisfies $\liminf_{u\to\infty} uq(u) \geq \max\left\{1, \frac{1}{1-r}\right\}$;
(3) The random variable θ is such that $P(0 \leq \theta < \epsilon)$ and $P(\theta = 0) = 1$ for every positive $\epsilon > 0$,

Leipus and Siaulys ([9]) obtained that, in the renewal risk model with the safety loading condition (5), if $B \in \mathcal{S}_*$, then for every positive function $f(\cdot)$ with $f(x) \to \infty$, (16) holds uniformly over all $t \in [f(x), \gamma x]$.

By analyzing this result carefully, we could see that, firstly, the assumptions it demands seem to be too strong. Hence, it is very difficult to be suitable for more general case. Secondly, the proof of their results is too complicated. It isn't pretty mathematically. Finally, that fact that class $\mathcal{L}$ is much bigger than class S_* illustrates that, Theorem 1 has much wider usage.

In modelling extremal events, heavy-tailed risk has played an important role in insurance and finance because they can describe large claims efficiently; see Embrechts et al. ([3]) and Goldie & Klüppelberg ([5]) for a nice review. We give here several important classes of heavy-tailed distributions for further references.

The insurance significance of Theorem 1 is as the following: first, it provides a lower bound of ruin probability of an insurance company. It is useful in risk management of the insurance company. Second, asymptotic formula when $x \to \infty$ often means that large claim is concerns. In other words, even very large initial capital is paid out by claims. This is just the case of extremal event! In fact, this is one of the reasons that we study heavy-tailed claims.

Uniformity is an important concept in mathematics. The main advantage of uniformity is, changing order of limit and integral is permissable. Thus, some results, say, about finite time ruin probability, can be extended easily to the ultimate time ruin probability.

The following three lemmas play the important roles in obtaining these two Theorems.

Lemma 1. *Let $\{X_i, i = 1, 2, \dots\}$ be a sequence of i.i.d. random variables with common distribution F and finite mean $\mathrm{E}X_1 = -\mu < 0$. If $F \in \mathcal{L}$, then it holds uniformly over all $n = 1, 2, \dots$ that*

$$\Pr\left(\max_{1\leq k\leq n}\sum_{i=1}^{k} X_i > x\right) \gtrsim \frac{1}{\mu}\int_x^{x+\mu n} \overline{F}(u)\mathrm{d}u. \tag{19}$$

This lemma is the extension to the Theorem of Korshunov ([8]); See also Tang (2004a). Lemma 2 reflects the basic property of homogenous Poisson process and it can be found, for example, in Theorem 2.3.1 of Ross ([10]):

Lemma 2. *Let $\{N(t), t \geq 0\}$ be a homogenous Poisson process with arrival times τ_k, $k = 1, 2, \dots$. Given $N(t) = n$ for arbitrarily fixed $t > 0$ and $n = 1, 2, \dots$, the random vector $(\tau_1, \cdots, \tau_n)$ is equal in distribution to the random vector $(tU_{(1,n)}, \cdots, tU_{(n,n)})$ with $U_{(1,n)}, \dots, U_{(n,n)}$ being the order statistics of n independent and uniformly distributed random variables $U_1, \dots, U_n$ in $(0, 1)$.*

Lemma 3 is from Lemma 2.1 of Klüppelberg and Mikosch ([7]):

Lemma 3. *Let $\{N(t), t \geq 0\}$ be a homogenous Poisson process with intensity $\lambda > 0$. Then, it holds for every $\varepsilon > 0$ and $\delta > 0$ that*

$$\lim_{t\to\infty} \sum_{n>(1+\delta)\lambda t} (1+\varepsilon)^n \Pr(N(t) = n) = 0. \tag{20}$$

3 The proof of the Main Results

3.1 Proof of Theorem 1

Since, by definition, $B \in \mathcal{L}$ implies $B_I \in \mathcal{S}$, by virtue of relation (15), it suffices to prove the uniformity of (16) over all $t \in [f(x), \infty)$. For arbitrarily fixed $\delta > 0$, we write

$$M_-(\delta) = \min_{0\leq k<\infty} \left(\frac{(1+\delta)k}{\lambda} - \tau_k \right), \tag{21}$$

which is nonpositive and finite almost surely. From the equivalent definition of finite time ruin probability

$$\psi(x;t) = \Pr\left(\max_{0\leq k\leq N(t)} \left(\sum_{i=1}^{k} Z_i - c\tau_k \right) > x \right), \qquad t > 0, \tag{22}$$

for each fixed $L > 0$, we have

$$\begin{aligned} &\psi(x;t) \\ &= \Pr\left(\max_{0\leq k\leq N(t)} \left(\sum_{i=1}^{k} \left(Z_i - \frac{c(1+\delta)}{\lambda} \right) + c\left(\frac{(1+\delta)k}{\lambda} - \tau_k \right) \right) > x \right) \\ &\geq \Pr\left(\max_{0\leq k\leq N(t)} \sum_{i=1}^{k} \left(Z_i - \frac{c(1+\delta)}{\lambda} \right) > x + cL,\ M_-(\delta) > -L \right) \\ &= \sum_{n=1}^{\infty} \Pr\left(\max_{0\leq k\leq n} \sum_{i=1}^{k} \left(Z_i - \frac{c(1+\delta)}{\lambda} \right) > x + cL \right) \Pr(N(t) = n, M_-(\delta) > -L) \end{aligned} \tag{23}$$

We write

$$\mu_2(\delta) = \frac{c(1+\delta)}{\lambda} - \mathrm{E}Z_1 > 0. \tag{24}$$

Applying Lemma 1, it holds uniformly over all $n = 1, 2, \ldots$ that

$$\Pr\left(\max_{0\leq k\leq n} \sum_{i=1}^{k} \left(Z_i - \frac{c(1+\delta)}{\lambda} \right) > x + cL \right) \gtrsim \frac{1}{\mu_2(\delta)} \int_x^{x+\mu_2(\delta)n} \overline{B}(u + cL)\,\mathrm{d}u. \tag{25}$$

Substituting this into (23) and considering an arbitrarily fixed number $0 < l < 1$, we have that, uniformly over all $t \in [f(x), \infty)$,

$$\begin{aligned}
&\psi(x;t) \\
&\gtrsim \frac{1}{\mu_2(\delta)} \sum_{n=1}^{\infty} \int_x^{x+\mu_2(\delta)n} \overline{B}(u+cL)\,\mathrm{d}u \cdot \Pr(N(t)=n,\ M_-(\delta) > -L) \\
&\geq \frac{1}{\mu_2(\delta)} \sum_{n\geq(1-l)\lambda t} \int_x^{x+\mu n} \overline{B}(u+cL)\,\mathrm{d}u \cdot \Pr(N(t)=n,\ M_-(\delta) > -L) \\
&\geq \frac{1}{\mu_2(\delta)} \int_x^{x+(1-l)\mu\lambda t} \overline{B}(u+cL)\,\mathrm{d}u \cdot \Pr\left(\frac{N(t)}{\lambda t} \geq 1-l,\ M_-(\delta) > -L\right). \qquad (26)
\end{aligned}$$

We apply an elementary inequality, $\Pr(AB) \geq \Pr(A) + \Pr(B) - 1$, to obtain that

$$\Pr\left(\frac{N(t)}{\lambda t} \geq 1-l,\ M_-(\delta) > -L\right) \geq \Pr\left(\frac{N(t)}{\lambda t} \geq 1-l\right) + \Pr(M_-(\delta) > -L) - 1. \qquad (27)$$

As $t \to \infty$, it is well known that $N(t)/\lambda t \to 1$ holds almost surely; see, for example, Section 2.5 of Embrechts et al. ([3]). Hence for each $\varepsilon > 0$, we may find some $x_0 > 0$ and $L_0 > 0$ such that the inequality

$$\Pr\left(\frac{N(t)}{\lambda t} \geq 1-l, M_-(\delta) > -L_0\right) \geq 1-\varepsilon \qquad (28)$$

holds for all $t \in [f(x_0), \infty)$. Substitution of this into (26) with $L = L_0$ gives that, uniformly over all $t \in [f(x), \infty)$,

$$\begin{aligned}
\psi(x;t) &\gtrsim \frac{1-\varepsilon}{\mu_2(\delta)} \int_x^{x+(1-l)\mu\lambda t} \overline{B}(u+cL_0)\,\mathrm{d}u \\
&\sim \frac{1-\varepsilon}{\mu_2(\delta)} \left(\int_x^{x+\mu\lambda t} - \int_{x+(1-l)\mu\lambda t}^{x+\mu\lambda t}\right) \overline{B}(u)\,\mathrm{d}u \\
&\geq \frac{1-\varepsilon}{\mu_2(\delta)} \int_x^{x+\mu\lambda t} \overline{B}(u)\,\mathrm{d}u \left(1 - \frac{\int_{x+(1-l)\mu\lambda t}^{x+\mu\lambda t} \overline{B}(u)\,\mathrm{d}u}{\int_x^{x+(1-l)\mu\lambda t} \overline{B}(u)\,\mathrm{d}u}\right) \\
&\geq \frac{1-\varepsilon}{\mu_2(\delta)} \int_x^{x+\mu\lambda t} \overline{B}(u)\,\mathrm{d}u \left(1 - \frac{l\mu\lambda t\overline{B}(x+(1-l)\mu\lambda t)}{(1-l)\mu\lambda t\overline{B}(x+(1-l)\mu\lambda t)}\right) \\
&= \frac{1-\varepsilon}{\mu_2(\delta)} \frac{1-2l}{1-l} \int_x^{x+\mu\lambda t} \overline{B}(u)\,\mathrm{d}u.
\end{aligned}$$

Since the constants $\delta > 0$, $0 < l < 1$, and $\varepsilon > 0$ can be arbitrarily small, we finally obtain the desired relation (16) with the indicated uniformity property. □

3.2 Proof of Theorem 2

We complete the proof by proving that, when claim-arrival follows Poisson process, the right hand of (12) is also the upper bound, uniformly over all $n = 1, 2, \ldots$.

For arbitrarily fixed $0 < \delta < 1$ such that $(1-\delta)v > \mathrm{E}Z_1$ and for each $n = 1, 2, \ldots$, we have

$$\Pr\left(\max_{1\le k\le n}\left(\sum_{i=1}^{k} Z_i - vnU_{(k,n)}\right) > x\right) \le \Pr\left(\xi_n + \eta_n > x\right), \tag{29}$$

where ξ_n and η_n are given by

$$\xi_n = \max_{1\le k\le n}\left(\sum_{i=1}^{k} Z_i - (1-\delta)vk\right) \tag{30}$$

and

$$\eta_n = \max_{1\le k\le n}\left((1-\delta)vk - vnU_{(k,n)}\right). \tag{31}$$

For the random variables ξ_n, $n = 1, 2, \ldots$, by Korshunov (19), it holds uniformly over all $n = 1, 2, \ldots$ that

$$\Pr\left(\xi_n > x\right) \lesssim \frac{1}{(1-\delta)v - \mathrm{E}Z_1}\int_x^{x+(v-\mathrm{E}Z_1)n} \overline{B}(u)\mathrm{d}u.$$

Hence for each $\varepsilon > 0$, we may choose some $M > 0$ such that for all $n = 1, 2, \ldots$ and $x \ge M$,

$$\Pr\left(\xi_n > x\right) \le \frac{1+\varepsilon}{(1-\delta)v - \mathrm{E}Z_1}\int_x^{x+(v-\mathrm{E}Z_1)n} \overline{B}(u)\mathrm{d}u. \tag{32}$$

For the random variables η_n, $n = 1, 2, \ldots$, we aim to prove that there exists some nonnegative random variable η independent of $\{\xi_n, n = 1, 2, \ldots\}$ and satisfying

$$\overline{G}(x) = \Pr\left(\eta > x\right) \le C_1 \mathrm{e}^{-C_2 x} \tag{33}$$

with some $C_i = C_i(\delta) > 0$, $i = 1, 2$, such that for all $n = 1, 2, \ldots$ and $x \ge 0$,

$$\Pr\left(\eta_n > x\right) \le \Pr\left(\eta > x\right). \tag{34}$$

In view that the identity $\left(U_{(k,n)} \le u\right) = \left(\sum_{i=1}^{n} 1_{(U_i \le u)} \ge k\right)$ holds for all $k = 1, 2, \ldots, n$ and $u \in [0, 1]$, we have

$$\begin{aligned}\Pr\left(\eta_n > x\right) &= \Pr\left(\bigcup_{k=1}^{n}\left(U_{(k,n)} < \frac{(1-\delta)vk - x}{vn}\right)\right)\\ &\le \sum_{\frac{x}{(1-\delta)v}\le k\le n} \Pr\left(\sum_{i=1}^{n} 1_{\left(U_i \le \frac{(1-\delta)vk-x}{vn}\right)} \ge k\right).\end{aligned}$$

For arbitrarily fixed $h > 0$, an application of Chebyshev's inequality gives that

$$\Pr\left(\eta_n > x\right) \le \sum_{\frac{x}{(1-\delta)v}\le k\le n} \mathrm{e}^{-hk}\mathrm{E}\left(\exp\left\{h\sum_{i=1}^{n} 1_{\left(U_i \le \frac{(1-\delta)vk-x}{vn}\right)}\right\}\right)$$

$$\leq \sum_{\frac{x}{(1-\delta)v}\leq k\leq n} \mathrm{e}^{-hk}\exp\left\{\left(\mathrm{e}^{h}-1\right)\frac{(1-\delta)vk-x}{v}\right\}$$
$$= \exp\left\{-\frac{\left(\mathrm{e}^{h}-1\right)x}{v}\right\}\sum_{\frac{x}{(1-\delta)v}\leq k\leq n}\exp\left\{\left[\left(\mathrm{e}^{h}-1\right)(1-\delta)-h\right]k\right\}.$$

Choose $h_0>0$ so that $\left(\mathrm{e}^{h_0}-1\right)(1-\delta)-h_0<0$. It follows that for all $n=1,2,\ldots$ and $x\geq 0$,

$$\Pr\left(\eta_n>x\right)$$
$$\leq exp\left\{-\frac{\left(\mathrm{e}^{h_0}-1\right)x}{v}\right\}\frac{\exp\left\{\left[\left(\mathrm{e}^{h_0}-1\right)(1-\delta)-h_0\right]\frac{x}{(1-\delta)v}\right\}}{1-\exp\left\{\left(\mathrm{e}^{h_0}-1\right)(1-\delta)-h_0\right\}}$$
$$= \frac{\exp\left\{-\frac{xh_0}{(1-\delta)v}\right\}}{1-\exp\left\{\left(\mathrm{e}^{h_0}-1\right)(1-\delta)-h_0\right\}}.$$

The last inequality illustrates that for some positive numbers C_1 and C_2, the tail probability $\Pr(\eta_n>x)$ is bounded by $\min\left\{1,C_1e^{-C_2x}\right\}$ for all $n=1,2,...$ and $x\geq 0$. We introduce a random variable η independent of the sequence $\{\xi_n,n=1,2,\ldots\}$ and with an exact tail probability $\min\left\{1,C_1e^{-C_2x}\right\}$. In this way, as announced, relations (33) and (34) are fulfilled immediately. Starting from (29) and using (34), (32), and Fubini's theorem in turn, we obtain that for all $n=1,2,\ldots$ and $x\geq M$,

$$\Pr\left(\max_{1\leq k\leq n}\left(\sum_{i=1}^{k}Z_i-vnU_{(k,n)}\right)>x\right)$$
$$\leq \int_{0^-}^{x-M}\Pr\left(\xi_n+y>x\right)G(\mathrm{d}y)+\overline{G}(x-M)$$
$$\leq \frac{1+\varepsilon}{(1-\delta)v-\mathrm{E}Z_1}\int_{0^-}^{x}\int_{x}^{x+(v-\mathrm{E}Z_1)n}\overline{B}(u-y)\mathrm{d}uG(\mathrm{d}y)+\overline{G}(x-M)$$
$$\leq \frac{1+\varepsilon}{(1-\delta)v-\mathrm{E}Z_1}\int_{x}^{x+(v-\mathrm{E}Z_1)n}\overline{B*G}(u)\mathrm{d}u+\overline{G}(x-M). \tag{35}$$

Since $\overline{G}(x)$ decreases exponentially, as described in (33), and $B\in\mathcal{S}_*\subset\mathcal{S}\subset\mathcal{L}$, we know that $\overline{G}(x)=o\left(\overline{B}(x)\right)$ by part (2) of Lemma 2.1, hence that $\overline{B*G}(x)\sim\overline{B}(x)$ by part (3) of Lemma 2.1. Moreover, for all $n=1,2,\ldots$ and $x\geq M$,

$$\frac{\overline{G}(x-M)}{\int_{x}^{x+(v-\mathrm{E}Z_1)n}\overline{B}(u)\mathrm{d}u}$$
$$\leq \frac{\overline{G}(x-M)}{\overline{B}(x-M)}\frac{\overline{B}(x-M)}{(v-\mathrm{E}Z_1)\overline{B}(x+(v-\mathrm{E}Z_1))}\to 0.$$

It follows from (35) that uniformly over all $n = 1, 2, \ldots$,

$$\Pr\left(\max_{1\le k\le n}\left(\sum_{i=1}^{k} Z_i - vnU_{(k,n)}\right) > x\right) \lesssim \frac{1+\varepsilon}{(1-\delta)v - \mathrm{E}Z_1}\int_x^{x+(v-\mathrm{E}Z_1)n} \overline{B}(u)\mathrm{d}u.$$

By the arbitrariness of ε and δ, we obtain the upper bound of Theorem 2. □

3.3 Discussions

*Example 1 (*Special case*).* In the case of Poisson process. We take $t = \infty$. Then it holds that

$$\psi(x;\infty) \sim \frac{1}{\mu}\int_x^\infty \overline{B}(u)\mathrm{d}u, \tag{36}$$

which is consistence with the result of Embrechts and Veraverbeke ([4]). If we take the distribution as $\mathcal{R}_{-\alpha}$ and $\alpha > 1$ specially. Then

$$\psi(x;\infty) \sim \frac{1}{\mu}x^{1-\alpha}. \tag{37}$$

In other words, we can calculate the ruin probability when the claim follows Pareto distribution. Similarly, when the claim follows Weiull distribution, it holds that

$$\psi(x;\infty) \sim \frac{1}{\mu}\int_x^\infty e^{-cu^\tau}\mathrm{d}u. \tag{38}$$

Thus, by calculating the integral in the right hand of (38), we could easily estimate the approximation probability.

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