Sparse Reconstruction in Frequency Domain and DOA Estimation for One-Dimensional Wideband Signals

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Abstract. Previous recovery methods in the literature are usually based on grid partition, which will bring about some perturbation to the eventual result. In the paper, a novel idea for one-dimensional wideband signals by sparse reconstruction in frequency domain is put forward. Firstly, Discrete Fourier Transformation (DFT) is performed on the received data. Then the data of the frequency with the most power is expressed by Fourier serious coefficients. On this basis, the optimization functions and corresponding dual problems are solved. After that the support set is calculated, and the primary sources of this frequency and direction of arrival (DOA) can also be acquired. Comparing with the traditional methods, the proposed approach has further improved the estimation accuracy.

Keywords: Direction of arrival · Sparse reconstruction · Frequency domain
Wideband signals

1 Introduction

Direction of arrival (DOA) estimation methods based on sparse recovery is a hot topics in recent years [1–7], and some good ideas have been put forward successively. Li [8] made full use of the frequency distribution of a received signal to generate the over-complete dictionary and it required no spectral decomposition or focusing. Xu [9] used the Capon spectrum to design a weighted ℓ_1-norm penalty for choosing a proper regularization parameter. He [10] provided a low complexity method for DOA estimation via array covariance matrix sparse representation, the method showed an extended-aperture and leaded to a significant improvement in the resolution limit.

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Based on different optimization problems which were solvable using second-order cone (SOC) programming, Hu [11] introduced a perspective for DOA estimation without knowing the signal number. Jagannath [12] derived a Bayesian Cramer-Rao bound for the grid mismatch problem with the errors in variables model and proposed a block sparse estimator for grid matching and sparse recovery, decreasing the computation complexity properly. Amin [13] established the role of sparse arrays and sparse sampling in antijam global navigation satellite systems and showed that both jammer DOA estimation methods and mitigation techniques benefited from the design flexibility of sparse arrays and their extended virtual apertures or coarrays.

DOA estimation by sparse reconstruction has lowered the demand for SNR and number of snapshots to a large extent. But previous recovery methods in the literature are usually based on grid division, which will bring about some perturbation to the eventual result. Candes [14, 15] discussed the sparse recovery in continuous domain, averted errors when signals were recovered in discrete domain, the estimation precision had been improved greatly, it is a development of the application of compressed sensing, but they did not tell us how to implement super-resolution direction finding for wideband signals.

In the paper, a novel idea for one-dimensional wideband sources by sparse reconstruction in frequency domain is put forward, Firstly, Discrete Fourier Transformation (DFT) is performed on the received data. Then the data of the frequency with the most power is expressed by Fourier serious coefficients. On this basis, the optimization functions and corresponding dual problems are solved. After that the support set is calculated, and the primary sources of this frequency and DOA can also be acquired. Comparing with the traditional methods, the proposed approach has further improved the estimation accuracy.

2 Signal Model

Consider N far field wideband signals $s_n(t)\,(n = 1, 2, \cdots, N)$ with the same energy arriving at the uniform linear array formed by M sensors from $\theta_1, \cdots, \theta_N$, it is illustrated as Fig. 1, the interval of adjacent sensors is d, which equals half of the wavelength of center frequency, here N is known in advance. The signals and noise are

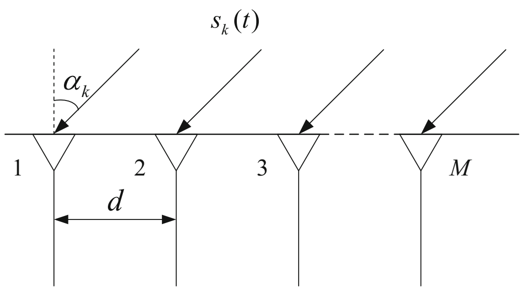

Fig. 1. Array signal model

assumed to be Gaussian distribution, and independent of each other, the first sensor is regarded as the reference, then output of the array can be modeled as

$$\boldsymbol{y}(t) = \left[\sum_{n=1}^{N} s_n(t), \cdots, \sum_{n=1}^{N} s_n\left(t - (m-1)\frac{d}{c}\sin(\theta_n)\right), \cdots, \sum_{n=1}^{N} s_n\left(t - (M-1)\frac{d}{c}\sin(\theta_n)\right)\right]^{\mathrm{T}} + [b_1(t), \cdots, b_m(t), \cdots, b_M(t)]^{\mathrm{T}} \tag{1}$$

Assume that number of the subbands is K, perform DFT on $\boldsymbol{y}(t)$, the wideband sources can be partitioned into K parts:

$$\boldsymbol{Y}(f_k) = \boldsymbol{A}(f_k)\boldsymbol{S}(f_k) + \boldsymbol{B}(f_k)\ k = 1, \cdots, K \tag{2}$$

Here, $\boldsymbol{B}(f_k)$ is the noise vector at f_k with mean 0 and variance $\sigma^2(f_k)$, and the steering vector matrix at f_k is determined as

$$\boldsymbol{A}(f_k) = [\boldsymbol{a}(f_k, \theta_1), \cdots, \boldsymbol{a}(f_k, \theta_N)] = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ \mathrm{e}^{\left(-\mathrm{j}2\pi f_k m \frac{d}{c}\sin(\theta_1)\right)} & \cdots & \mathrm{e}^{-\mathrm{j}2\pi f_k m \frac{d}{c}\sin(\theta_N)} \\ \vdots & & \vdots \\ \mathrm{e}^{\left(-\mathrm{j}2\pi f_k (M-1)\frac{d}{c}\sin(\theta_1)\right)} & \cdots & \mathrm{e}^{\left(-\mathrm{j}2\pi f_k (M-1)\frac{d}{c}\sin(\theta_N)\right)} \end{bmatrix} \tag{3}$$

Here, $\boldsymbol{a}(f_k, \theta_n)$ is the direction vector of wideband source coming from θ_n $(n = 1, \cdots, N)$ at frequency f_k. Suppose $\boldsymbol{S}(f_k)$ is sparse, for example, the signal vector is composed by some spikes [14], it can be expressed as the sparse model:

$$\boldsymbol{S}(f_k) = \begin{bmatrix} S_1(f_k) \\ \vdots \\ S_n(f_k) \\ \vdots \\ S_N(f_k) \end{bmatrix} = \begin{bmatrix} \upsilon_1^2(f_k)\delta_{\varphi_1(f_k)} \\ \vdots \\ \upsilon_n^2(f_k)\delta_{\varphi_n(f_k)} \\ \vdots \\ \upsilon_N^2(f_k)\delta_{\varphi_N(f_k)} \end{bmatrix} \tag{4}$$

Where

$$\varphi_n(f_k) = \frac{df_k}{c}(1 - \sin(\theta_n)) \tag{5}$$

And $\delta_{\varphi_n(f_k)}$ is a dirac measure at $\varphi_n(f_k)$, define $\{\varphi_1(f_k), \cdots, \varphi_N(f_k)\}$ as the sparse support set of $\boldsymbol{S}(f_k)$, where $\varphi_n(f_k)$ contains direction of the nth signal, $\upsilon_n(f_k)$ is corresponding amplitude.

3 Estimation Theory

3.1 Infinite Samples

Assume that information of f_0 has the more power than that of the other frequencies, the covariance matrix at f_0 is

$$\begin{aligned} \boldsymbol{R}_Y(f_0) &= E\left[\boldsymbol{Y}(f_0)\boldsymbol{Y}^{\mathrm{H}}(f_0)\right] \\ &= \boldsymbol{A}(f_0)\boldsymbol{R}_S(f_0)\boldsymbol{A}^{\mathrm{H}}(f_0) + \boldsymbol{\Omega}(f_0) \\ &= \sum_{n=1}^{N} \upsilon_n^2(f_0)\,\boldsymbol{a}(f_0,\theta_n)\boldsymbol{a}^{\mathrm{H}}(f_0,\theta_n) + \boldsymbol{\Omega}(f_0) \end{aligned} \tag{6}$$

Where

$$\boldsymbol{R}_S(f_0) = E\left[\boldsymbol{S}(f_0)\boldsymbol{S}^{\mathrm{H}}(f_0)\right] = \mathrm{diag}(\boldsymbol{\Sigma}_S(f_0)) = \mathrm{diag}\left(\left[\upsilon_1^2(f_0),\cdots,\upsilon_N^2(f_0)\right]^{\mathrm{T}}\right) \tag{7}$$

Here, $\boldsymbol{\Sigma}_S(f_0) = \left[\upsilon_1^2(f_0),\cdots,\upsilon_N^2(f_0)\right]^{\mathrm{T}}$, and

$$\boldsymbol{\Omega}(f_0) = E\left[\boldsymbol{B}(f_0)\boldsymbol{B}^{\mathrm{H}}(f_0)\right] = \mathrm{diag}\left(\left[\sigma^2(f_0),\cdots,\sigma^2(f_0)\right]_{1\times M}^{\mathrm{T}}\right) \tag{8}$$

Vectoring (6), we have

$$\boldsymbol{p}(f_0) = \mathrm{vec}(\boldsymbol{R}_Y(f_0)) = \boldsymbol{\Theta}(f_0)\boldsymbol{\Sigma}_S(f_0) + \boldsymbol{\Gamma}(f_0) \tag{9}$$

Where $\boldsymbol{\Gamma}(f_0) = \left[\sigma^2(f_0)\boldsymbol{e}_1^{\mathrm{T}},\cdots,\sigma^2(f_0)\boldsymbol{e}_M^{\mathrm{T}}\right]^{\mathrm{T}}$, $\boldsymbol{e}_m$ is the vector with all zero elements, except for the mth element, which equals one, and

$$\boldsymbol{\Theta}(f_0) = \boldsymbol{A}^*(f_0)\odot\boldsymbol{A}(f_0) = [\boldsymbol{a}^*(f_0,\theta_1)\otimes\boldsymbol{a}(f_0,\theta_1),\cdots,\boldsymbol{a}^*(f_0,\theta_N)\otimes\boldsymbol{a}(f_0,\theta_N)] \tag{10}$$

Get rid of duplicate items in (9), then arrange them in order, we have

$$\bar{\boldsymbol{p}}(f_0) = \overline{\boldsymbol{\Theta}}(f_0)\boldsymbol{\Sigma}_S(f_0) + \overline{\boldsymbol{\Gamma}}(f_0) \tag{11}$$

Where

$$\overline{\boldsymbol{\Theta}}(f_0) = \begin{bmatrix} \mathrm{e}^{\left(-\mathrm{j}2\pi f_0(2M-1)\frac{d}{c}\sin(\theta_1)\right)} & \cdots & \mathrm{e}^{\left(-\mathrm{j}2\pi f_0(2M-1)\frac{d}{c}\sin(\theta_N)\right)} \\ \mathrm{e}^{\left(-\mathrm{j}2\pi f_0(2M-2)\frac{d}{c}\sin(\theta_1)\right)} & \cdots & \mathrm{e}^{\left(-\mathrm{j}2\pi f_0(2M-2)\frac{d}{c}\sin(\theta_N)\right)} \\ \vdots & \ddots & \vdots \\ \mathrm{e}^{\left(\mathrm{j}2\pi f_0(2M-1)\frac{d}{c}\sin(\theta_1)\right)} & \cdots & \mathrm{e}^{\left(\mathrm{j}2\pi f_0(2M-1)\frac{d}{c}\sin(\theta_N)\right)} \end{bmatrix} \tag{12}$$

$\overline{\boldsymbol{\Gamma}}(f_0)$ is acquired after rearranging $\boldsymbol{\Gamma}(f_0)$. Given a measure $S(\varphi)$ with $\varphi\in[0,1]$, the Fourier serious coefficients of $S^2(\varphi)$ can be expressed as

$$q(m) = \int_0^1 \exp(-\mathrm{j}2\pi m\varphi)S^2(\varphi)\mathrm{d}\varphi \; m = -(2M-1), -(2M-2), \cdots, (2M-1) \tag{13}$$

Combine (4) and (13)

$$q(m,f_0) = \sum_{n=1}^{N} \exp(-\mathrm{j}2\pi m\varphi_n(f_0))\, \upsilon_n^2(f_0),\; m = -(2M-1), -(2M-2), \cdots, (2M-1) \tag{14}$$

So we have

$$\boldsymbol{Q}(f_0) = \boldsymbol{F}(f_0)\boldsymbol{\Sigma}_S(f_0) \tag{15}$$

Where

$$\boldsymbol{Q}(f_0) = [q(-(2M-1),f_0), q(-(2M-2),f_0), \cdots, q((2M-1),f_0)]^{\mathrm{T}} \tag{16}$$

and

$$\boldsymbol{F}(f_0) = \begin{bmatrix} \mathrm{e}^{(\mathrm{j}2\pi(2M-1)\varphi_1(f_0))} & \cdots & \mathrm{e}^{(\mathrm{j}2\pi(2M-1)\varphi_N(f_0))} \\ \mathrm{e}^{(\mathrm{j}2\pi(2M-2)\varphi_1(f_0))} & \cdots & \mathrm{e}^{(\mathrm{j}2\pi(2M-2)\varphi_N(f_0))} \\ \vdots & \ddots & \vdots \\ \mathrm{e}^{(-\mathrm{j}2\pi(2M-1)\varphi_1(f_0))} & \cdots & \mathrm{e}^{(-\mathrm{j}2\pi(2M-1)\varphi_N(f_0))} \end{bmatrix} \tag{17}$$

In order to reconstruct primary sources, it is possible to solve the following question

$$\min_{\boldsymbol{\Sigma}_S(f_0)} \|\boldsymbol{\Sigma}_S(f_0)\|_{\mathrm{TV}}, \text{s.t.} \boldsymbol{Q}(f_0) = \boldsymbol{F}(f_0)\boldsymbol{\Sigma}_S(f_0) \tag{18}$$

where $\|\boldsymbol{\Sigma}_S(f_0)\|_{\mathrm{TV}} = \sum_{n=1}^{N} \upsilon_n^2(f_0)$, then we can recover the source $\mathbf{S}(f_0)$ if the interval between $\varphi_\alpha(f_0)$ and $\varphi_\beta(f_0)$ is wider than $2/f_0$ for $1 \le \alpha, \beta \le N$ [14].

3.2 Finite Samples

In real systems, suppose the sampling times at every frequency is KP, the covariance matrix can be acquired by

$$\hat{\boldsymbol{R}}_{\boldsymbol{Y}}(f_0) = \frac{1}{KP} \sum_{kp=1}^{KP} \boldsymbol{Y}(f_0)\boldsymbol{Y}^{\mathrm{H}}(f_0) \tag{19}$$

Get rid of the noise item, we have

$$\hat{\boldsymbol{R}}_Y(f_0) - \sigma^2(f_0)\boldsymbol{I} = \boldsymbol{A}(f_0)\boldsymbol{R}_S(f_0)\boldsymbol{A}^{\mathrm{H}}(f_0) + \boldsymbol{D}(f_0) \tag{20}$$

Where $\boldsymbol{D}(f_0)$ is the corresponding error, $\sigma^2(f_0)$ can be estimated according to the eigenvalue, then

$$\hat{\boldsymbol{p}}(f_0) = \mathrm{vec}\big(\hat{\boldsymbol{R}}_{\boldsymbol{Y}}(f_0)\big) = \boldsymbol{\Theta}(f_0)\boldsymbol{\Sigma}_S(f_0) + \boldsymbol{\Gamma}(f_0) + \boldsymbol{\Psi}(f_0) \tag{21}$$

Here $\boldsymbol{\Psi}(f_0) = \mathrm{vec}(\boldsymbol{D}(f_0))$, take out repeated items in (21) and arrange them in sequence, we have

$$\bar{\boldsymbol{p}}(f_0) = \overline{\boldsymbol{\Theta}}(f_0)\boldsymbol{\Sigma}_S(f_0) + \overline{\boldsymbol{\Gamma}}(f_0) + \overline{\boldsymbol{\Psi}}(f_0) \tag{22}$$

Where $\overline{\boldsymbol{\Psi}}(f_0)$ can be obtained by rearranging $\boldsymbol{\Psi}(f_0)$. Referring to (4), the linear transform of (22) is

$$\begin{aligned} q(m,f_0) &= \exp\left(-\mathrm{j}2\pi m\frac{df_0}{c}\right)\left(\bar{\boldsymbol{p}}_m(f_0) - \overline{\boldsymbol{\Gamma}}_m(f_0)\right) \\ &= \exp\left(-\mathrm{j}2\pi m\frac{df_0}{c}\right)\left(\sum_{n=1}^{N}\exp\left(\mathrm{j}2\pi m\frac{df_0}{c}\sin(\theta_n)\right)\upsilon_n^2(f_0) + \overline{\boldsymbol{\Psi}}(m,f_0)\right) \\ &= \sum_{n=1}^{N}\exp\left(-\mathrm{j}2\pi m\frac{df_0}{c}(1-\sin\theta_n)\right)\upsilon_n^2(f_0) + \exp\left(-\mathrm{j}2\pi m\frac{df_0}{c}\right)\overline{\boldsymbol{\Psi}}(m,f_0) \\ &= \sum_{n=1}^{N}\exp(-\mathrm{j}2\pi m\varphi_n(f_0))\,\upsilon_n^2(f_0) + \boldsymbol{\omega}(m,f_0) \end{aligned} \tag{23}$$

Where $\boldsymbol{\omega}(m,f_0) = \exp\left(-\mathrm{j}2\pi m\frac{df_0}{c}\right)\overline{\boldsymbol{\Psi}}(m,f_0)$, so we have

$$\boldsymbol{Q}(f_0) = \boldsymbol{F}(f_0)\boldsymbol{\Sigma}_s(f_0) + \boldsymbol{\omega}(f_0) \tag{24}$$

Where $\boldsymbol{\omega}(f_0) = [\boldsymbol{\omega}(-(2M-1),f_0),\cdots,\boldsymbol{\omega}(m,f_0),\cdots,\boldsymbol{\omega}((2M-1),f_0)]$, similarly, primary sources can be recovered by the question (25)

$$\min_{\boldsymbol{\Sigma}_S(f_0)}\|\boldsymbol{\Sigma}_S(f_0)\|_{\mathrm{TV}} \text{ s.t. } \|\boldsymbol{Q}(f_0) - \boldsymbol{F}(f_0)\boldsymbol{\Sigma}_s(f_0)\|_2 \le |\varsigma(f_0)| \tag{25}$$

It can be recast as the formula below [14]

$$\begin{aligned} &\max_{\boldsymbol{\Phi}(f_0),\boldsymbol{Z}} \mathrm{Re}[\boldsymbol{Q}^*(f_0)\boldsymbol{\Phi}(f_0)] - \varsigma(f_0)\|\boldsymbol{\Phi}(f_0)\|_2 \text{ s.t.} \\ &\begin{bmatrix} \boldsymbol{Z} & \boldsymbol{\Phi}(f_0) \\ \boldsymbol{\Phi}^*(f_0) & 1 \end{bmatrix} \succ 0, \|\boldsymbol{F}^*(f_0)\boldsymbol{\Phi}(f_0)\|_{L\infty} \le 1 \end{aligned} \tag{26}$$

Where $\sum_{\alpha=1}^{4M+1-\beta} \boldsymbol{Z}_{\alpha,\alpha+\beta} = \begin{cases} 1 & \beta = 0 \\ 0 & \beta = 1, 2, \cdots, 4M \end{cases}$, $\boldsymbol{Z} \in C^{(4M-1)\times(4M-1)}$ is a Hermitian matrix, and $\boldsymbol{\Phi}(f_0) \in C^{(4M-1)\times 1}$ is the corresponding Lagrangian matrix for $\boldsymbol{Q}(f_0) = \boldsymbol{F}(f_0)\boldsymbol{\Sigma}_s(f_0) + \boldsymbol{\omega}(f_0)$, it can be acquired according to some softwares [16, 17].

We can use the equation below [15] to state the relationship between (25) and (26):

$$(\boldsymbol{F}^{*}\boldsymbol{\Phi})(f_0) = \operatorname{sign}\left(\|\boldsymbol{\Sigma}_s(f_0)\|_{\mathrm{TV}}\right) \tag{27}$$

So

$$\left|\boldsymbol{F}^{\mathrm{H}}(n, f_0)\boldsymbol{\Phi}(f_0)\right| = 1, \ (n = 1, \cdots, N) \tag{28}$$

Then DOA can be determined by integrating (5), (17) and (28), then the primary sources can also be reconstructed according to (4). As the proposed sparse reconstruction method is implemented in frequency domain, and suitable for one-dimensional sources, we can call it SFO method for short.

4 Simulations

Here, some simulations are presented, consider some wideband sources impinge on a uniform linear array with 12 omnidirectional sensors, the frequencies of these sources are 4 GHz–5 GHz, frequency bins K = 10, sparse recovery methods in discrete domain (SRD) [18] and SFO are respectively employed, $\varepsilon(f_0)$ in the sparse reconstruction method is taken as 1.2. The grids in discrete domain are partitioned according to $\sin(\theta)$, the step size is 0.004, 300 Monte-Carlo simulations have run for each condition.

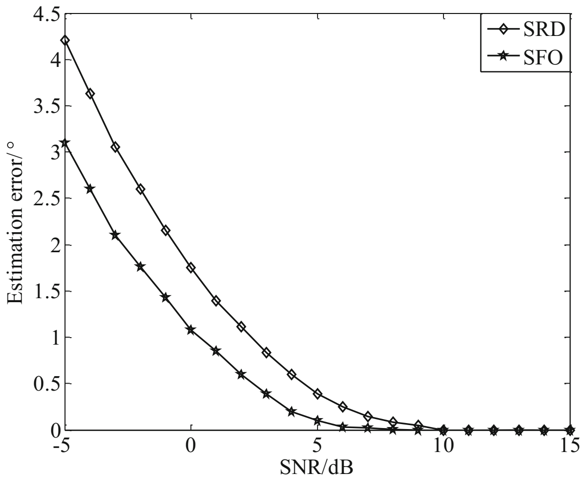

Fig. 2. Estimation error versus SNR

4.1 Estimation Error Versus SNR

Suppose that four far-field wideband sources simultaneously arrive at the array with the same energy from $\sin(\theta) = [0.213, 0.459, 0.576, 0.624]$, Fig. 2 shows the estimation errors versus signal to noise ratio (SNR) when sampling times at every frequency is 20; Fig. 3 provides that versus sampling times at every frequency when SNR is 4 dB, and we can see from the simulations, the estimation precision of SFO method is higher than SRD.

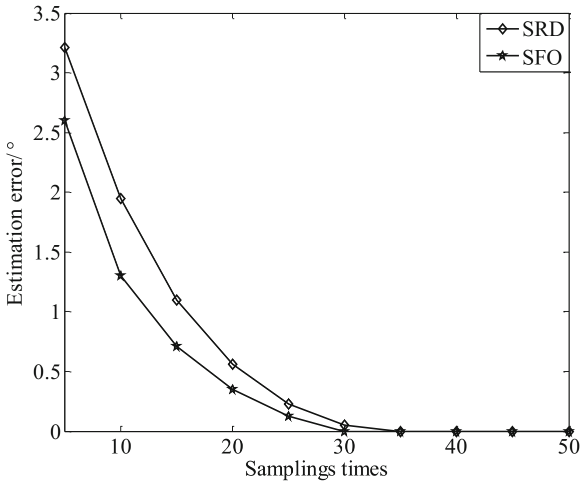

Fig. 3. Estimation error versus sampling times

4.2 Resolution

Suppose that two wideband sources simultaneously impinge at the array with the same energy from $\sin(\theta) = [0.76, 0.82]$, SNR is 6 dB, sampling times at every frequency is 20,

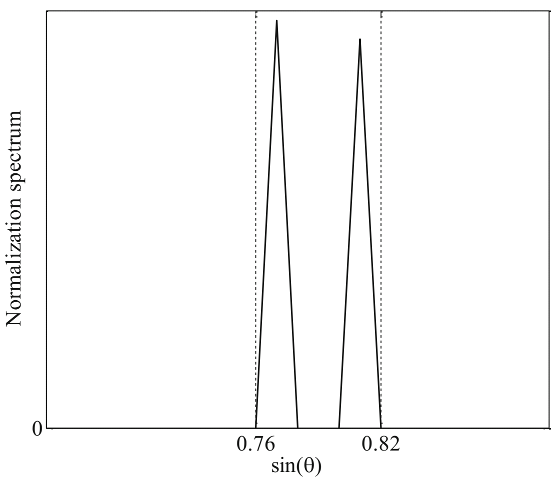

Fig. 4. Normalization spectrum of SRD for near sources

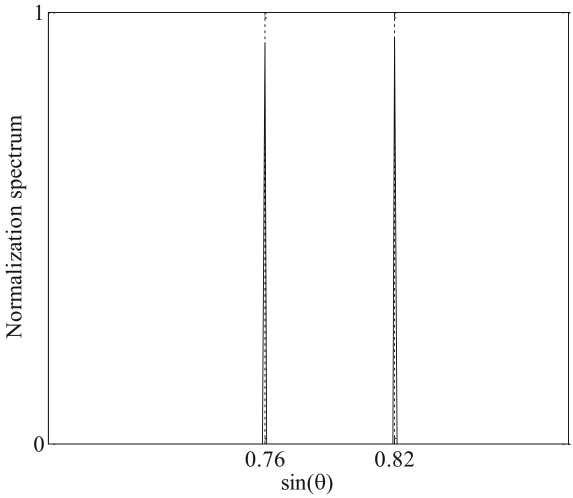

Fig. 5. Normalization spectrum of SFO for near sources

the normalization spectrum are illustrated in Figs. 4 and 5. It is seen that when the two sources are near to each other, SFO can still resolve them more accurately than SRD.

5 Conclusion

The DOA estimation for one-dimensional wideband sources by sparse reconstruction in frequency domain is put forward in this paper, only the information of the frequency with the most energy is used. This method averts the uncertainty brought by sparse reconstruction based on grid partition, and it still has a good property when the sampling times are not enough or SNR is low.

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