Impact of Uncertainty About a User to be Active on OFDM Transmission Strategies

Andrey Garnaev[✉], Wade Trappe, Ratnesh Kumbhkar, and Narayan B. Mandayam

WINLAB, Rutgers University, North Brunswick, USA
garnaev@yahoo.com, {trappe,ratnesh,narayan}@winlab.rutgers.edu

Abstract. In this paper we investigate the impact that incomplete knowledge regarding user activity can have on the equilibrium transmission strategy for an OFDM-based communication system. The problem is formulated as a two user non-zero sum game for independent fading channel gains, where the equilibrium strategies are derived in closed form. This allows one to show that a decrease in uncertainty about the user activity could reduce the number of subcarriers jointly used by the users. For the boundary case (with complete information, which reflects a classical water-filling game) the equilibrium strategies are given explicitly. The necessary and sufficient conditions, when channels sharing strategies are optimal, is established as well as the set of shared subcarriers is identified. The stability of the upper bound of the size of this set with respect to power budgets is derived.

Keywords: OFDM · Nash equilibrium · NC-OFDM
Multicarrier modulation

1 Introduction

Multiuser power control problems in wireless networks employing orthogonal frequency division multiplexing (OFDM) technology [15] and its variants like noncontiguous orthogonal frequency division multiplexing (NC-OFDM), where only a subset of all subcarriers are used due to either to avoid incumbent transmissions or tactical considerations [19] have received significant research interest in current and future wireless communication systems due to their reliability, adaptability and spectral efficiency. Selfish behaviour of users in OFDM style systems has been extensively studied in the literature (see, for example [11], and references therein). An important tool for designing optimal power allocation as well as estimating their effectiveness is game theory. This is due to the fact that, in general, such systems are multi-agent systems where each agent has its own (selfish) goal to achieve. Game theory supplies solutions for such multi-agent problems as well as methods to find them (see, for example [11] as a survey for such concepts and applications to wireless problems). In [21], a multiuser power control problem in a frequency-selective interference channel was modeled by a

P. Marques et al. (Eds.): CROWNCOM 2017, LNICST 228, pp. 211–222, 2018.
https://doi.org/10.1007/978-3-319-76207-4_18

two-player game. A condition on fading channel gains was derived to guarantee existence of a Nash equilibrium as well as that this equilibrium is unique and stable, i.e., an iterative water-filling algorithm can efficiently reach the Nash equilibrium. In [12], a power allocation problem in the downlink of wireless networks, where multiple access points send independent coded network information to multiple mobile terminals through orthogonal channels was formulated as non-zero sum game. It was proven that this is a potential game having a unique equilibrium with probability one. In [16], a problem to maximize information rates for the Gaussian frequency-selective interference channel was formulated as a noncooperative game of complete information and an asynchronous iterative water-filling algorithm was proposed to achieve the Nash equilibria. In [1], closed form solutions for the symmetric water-filling game with equal crosstalk coefficients was obtained. This allows one to derive the conditions for which there is a unique solution or multiple solutions. In [17], the saddle point was found explicitly in a jamming game where a user and a jammer has enough energy to employ all the channels in their optimal behaviour. In [3,4], a game theoretic analysis of secret and reliable communication under combined jamming and eavesdropping attack was given. In [18], a problem of multiband transmission under hostile jamming modeled by zero-sum game was solved. In [9], a bargaining solution over the fair trade-off between secrecy and throughput was derived.

In this paper, we investigate the impact that incomplete knowledge about user activity can have on the equilibrium OFDM transmission strategy. The problem is formulated as a two user non-zero sum game for independent fading channel gains. This impact is investigated by means of two algorithms developed to find equilibrium strategies. The first is the best response strategies algorithm illustrating learning mechanism to reach an equilibria. The second one is a superposition of two bisection methods based describing the equilibrium in closed form. For the case of complete knowledge about whether a user is active (which corresponds to classical water-filling problem) the equilibrium strategies are given explicitly. To the best of our knowledge, this classical water-filling game has not been yet solved explicitly in the literature. The necessary and sufficient conditions for when subcarrier sharing strategies are optimal is established, as well as the set of shared subcarriers is identified.

The organization of this paper is as follows: in Sect. 2, a model of transmission with incomplete information is formulated, and the convergence of the best response algorithm is proven. In Sect. 3, the equilibrium strategies are derived in closed form as well as an algorithm to find them based on superposition of bisection methods is given. In Sect. 4, for the boundary case of complete information the strategies are found explicitly. Finally, in Sect. 5, conclusions are offered, and in Appendix sketch of the proof of the obtain results are given.

2 Formulation of the Problem

We assume that the total spectrum band that can be used jointly by *two users* for communication with *one receiver* is split into n subcarriers. *One of the users (called, user 1) is active, i.e., he communicates with certainty.* User 1 has only a priori knowledge about the other user (called, user 2) being active. Namely, *user 1 knows that with a priori probability q^1 that user 2 will be active, while with probability q^0 user 2 will not be active.*

The strategy of user j ($j = 1, 2$) is a power allocation vector $\boldsymbol{P}^j = (P_1^j, \ldots, P_n^j)$ with $P_i^j \geq 0$ is the power assigned to transmit in subcarrier i, $\sum_{i=1}^n P_i^j = \overline{P}^j$ where $\overline{P}^j$ is the total power to transmit. Let Π^j be the set of all feasible strategies for user j. The payoff v^1 to user 1 is the expected throughput, while the payoff v^2 to user 2 is the throughput given as follows:

$$\begin{aligned} v^1(\boldsymbol{P}^1, \boldsymbol{P}^2) &= q^1 \sum_{i=1}^n \ln\left(1 + \frac{h_i^1 P_i^1}{\sigma^2 + h_i^2 P_i^2}\right) + q^0 \sum_{i=1}^n \ln\left(1 + \frac{h_i^1 P_i^1}{\sigma^2}\right), \\ v^2(\boldsymbol{P}^1, \boldsymbol{P}^2) &= \sum_{i=1}^n \ln\left(1 + \frac{h_i^2 P_i^2}{\sigma^2 + h_i^1 P_i^1}\right), \end{aligned} \tag{1}$$

where h_i^j is the fading channel gains and σ^2 is the background noise power. Thus, in (1), we deal with the scenario involving independent fading channels.

Since user 1 has only a priori knowledge about whether user 2 is active, while user 2 knows about his activity, this is a *Bayesian game* [11]. Bayesian approaches have been widely used for modeling network problems, such as to incorporate an incentive mechanism in a cooperative medium access scheme in a wireless relaying network [13], to design anti-eavesdropping strategies when eavesdropper might be an active adversary [6], and for intrusion detection in wireless ad hoc networks [20].

We look for (Nash) equilibrium strategies. Recall that $(\boldsymbol{P}_*^1, \boldsymbol{P}_*^2)$ is an equilibrium if and only if for any $(\boldsymbol{P}^1, \boldsymbol{P}^2)$ the following inequalities holds: $v^1(\boldsymbol{P}^1, \boldsymbol{P}_*^2) \leq v^1(\boldsymbol{P}_*^1, \boldsymbol{P}_*^2)$ and $v^2(\boldsymbol{P}_*^1, \boldsymbol{P}^2) \leq v^2(\boldsymbol{P}_*^1, \boldsymbol{P}_*^2)$. Thus, $(\boldsymbol{P}_*^1, \boldsymbol{P}_*^2)$ is an equilibrium if and only if they are the best response strategy to each other. i.e., $\boldsymbol{P}_*^1 = \mathrm{BR}^1(\boldsymbol{P}_*^2) := \arg_{\boldsymbol{P}^1 \in \Pi^1} \max v^1(\boldsymbol{P}^1, \boldsymbol{P}_*^2)$ and $\boldsymbol{P}_*^2 = \mathrm{BR}^2(\boldsymbol{P}_*^1) := \arg_{\boldsymbol{P}^2 \in \Pi^2} \max v^2(\boldsymbol{P}_*^1, \boldsymbol{P}^2)$.

Theorem 1

(a) The considered game has an equilibrium.

(b) The best response strategies $(\boldsymbol{P}^1, \boldsymbol{P}^2) = (BR^1(\boldsymbol{P}^2), BR^2(\boldsymbol{P}^1))$ can be found in water-filling form as follows:

$$P_i^1 = P_i^1(\omega) := \left\lfloor \frac{1}{2\omega} - \frac{h_i^2 P_i^2 + 2\sigma^2}{2h_i^1} + \frac{1}{2}\sqrt{\left(\frac{1}{\omega} - \frac{h_i^2 P_i^2}{h_i^1}\right)^2 + 4q^0 \frac{h_i^2 P_i^2}{(h_i^1)^2}} \right\rfloor_+, i = 1, \ldots, n$$

with ω being the unique root of the equation $\sum_{i=1}^n P_i^1(\omega) = \overline{P}^1$ and

$$P_i^2 = P_i^2(\omega) := \left\lfloor 1/\omega - (\sigma^2 + h_i^1 P_i^1)/h_i^2 \right\rfloor_+, i = 1, \ldots, n$$

with ω being the unique root of the equation $\sum_{i=1}^n P_i^2(\omega) = \overline{P}^2$.

(c) The best-response algorithm converges to an equilibrium. Namely, let $\boldsymbol{P}_0^2$ be any strategy of user 2, $\boldsymbol{P}_1^1 = BR^1(\boldsymbol{P}_0^2)$, $\boldsymbol{P}_1^2 = BR^2(\boldsymbol{P}_1^2)$ and so on. Then, $(\boldsymbol{P}_k^1, \boldsymbol{P}_k^2)$ converges to an equilibrium.

3 Equilibrium Strategies in Closed Form

In this section, we obtain the solution in closed form as a function of two auxiliary parameters. This allows us to examine the structure of the strategies as well as to design an alternative algorithm based on the bisection method, which can find these parameters and thereby determine the equilibrium strategies.

Theorem 2. *The equilibrium strategies* $(\boldsymbol{P}^1, \boldsymbol{P}^2)$ *of the considered game with* $q^0 > 0$ *must have the following form with* ω^1 *and* ω^2 *as positive parameters:*

$$P_i^1 = P_i^1(\omega^1, \omega^2) := \begin{cases} \dfrac{q^0}{\omega^1 - q^1 h_i^1 \omega^2 / h_i^2} - \dfrac{\sigma^2}{h_i^1}, & i \in I_{11}(\omega^1, \omega^2), \\ \dfrac{1}{\omega^1} - \dfrac{\sigma^2}{h_i^1}, & i \in I_{10}(\omega^1, \omega^2), \\ 0, & i \in I_{00}(\omega^1, \omega^2) \cup I_{01}(\omega^1, \omega^2), \end{cases} \tag{2}$$

$$P_i^2 = P_i^2(\omega^1, \omega^2) := \begin{cases} \dfrac{1}{\omega^2} - \dfrac{h_i^1}{h_i^2} \dfrac{q^0}{\omega^1 - q^1 h_i^1 \omega^2 / h_i^2}, & i \in I_{11}(\omega^1, \omega^2), \\ \dfrac{1}{\omega^2} - \dfrac{\sigma^2}{h_i^2}, & i \in I_{01}(\omega^1, \omega^2), \\ 0, & i \in I_{00}(\omega^1, \omega^2) \cup I_{10}(\omega^1, \omega^2), \end{cases} \tag{3}$$

with

$$\begin{aligned} I_{00}(\omega^1, \omega^2) &= \left\{ i : 1/\sigma^2 \le \omega^1 / h_i^1, 1/\sigma^2 \le \omega^2 / h_i^2 \right\}, \\ I_{10}(\omega^1, \omega^2) &= \left\{ i : 1/\sigma^2 > \omega^1 / h_i^1, \omega^1 / h_i^1 \le \omega^2 / h_i^2 \right\}, \\ I_{01}(\omega^1, \omega^2) &= \left\{ i : 1/\sigma^2 > \omega^2 / h_i^2, q^1 \omega^2 / h_i^2 + q^0/\sigma^2 \le \omega^1 / h_i^1 \right\}, \\ I_{11}(\omega^1, \omega^2) &= \left\{ i : \omega^2 / h_i^2 < \omega^1 / h_i^1 < q^1 \omega^2 / h_i^2 + q^0 / \sigma^2 \right\}. \end{aligned}$$

In particular, Theorem 2 (and subsequently Theorem 4) specify the subcarriers that are either not used (I_{00}), or used by just one of the users (I_{10}) and (I_{10}), or by both users (I_{11}). The strategies can be considered *subcarrier-sharing* if the set of the shared subcarriers I_{11} is empty.

Theorem 3. *The set of subcarriers* $I_{11}(\omega^1, \omega^2)$ *employed by the users for joint use is non-decreasing in probability* q^0*.*

The value of the parameters ω^1 and ω^2 are defined based on the condition that the power resources $H^1(\omega^1, \omega^2)$ and $H^2(\omega^1, \omega^2)$ employed by $\boldsymbol{P}^1(\omega^1, \omega^2)$ and $\boldsymbol{P}^2(\omega^1, \omega^2)$ have to be equal to $\overline{P}^1$ and $\overline{P}^2$, i.e.,

$$H^k(\omega^1, \omega^2) := \sum_{i=1}^{n} P_i^k(\omega^1, \omega^2) = \overline{P}^k, k = 1, 2. \tag{4}$$

In the following Proposition, which follows directly from Theorem 2, auxiliary properties of the functions H^1 and H^2 are given.

Proposition 1

(a) For a fixed ω^2, $H^1(\omega^1,\omega^2)$ is continuous on ω^1 and decreasing from infinity for $\omega^1 \downarrow 0$ to zero for $\omega^1 \ge \max_i(q^0/\sigma^2 + q^1\omega^2/h_i^2)$.

(b) For a fixed ω^1, $H^1(\omega^1,\omega^2)$ is continuous and increasing on ω^2 such that

$$H^1(\omega^1,0) = \sum_{i=1}^{n} \lfloor q^0/\omega^1 - \sigma^2/h_i^1 \rfloor_+,$$

$$H^1(\omega^1,\omega^2) = \sum_{i=1}^{n} \lfloor 1/\omega^1 - \sigma^2/h_i^1 \rfloor_+ \text{ for } \omega^2 \ge \omega^1 \max_i(h_i^2/h_i^1).$$

(c) For a fixed ω^2 there is an $\Omega^1(\omega^2)$ such that

$$H^1(\Omega^1(\omega^2),\omega^2) = \overline{P}^1. \tag{5}$$

(d) $\Omega^1(\omega^2)$ is continuous and increasing on ω^2 such that $\Omega^1(0) = \underline{\omega}^1$ and $\Omega^1(\infty) = \overline{\omega}^1$ with $\underline{\omega}^1$ and $\overline{\omega}^1$ uniquely given as roots of the equations:

$$\sum_{i=1}^{n} \left\lfloor q^0/\underline{\omega}^1 - \sigma^2/h_i^1 \right\rfloor_+ = \overline{P}^1 \text{ and } \sum_{i=1}^{n} \left\lfloor 1/\overline{\omega}^1 - \sigma^2/h_i^1 \right\rfloor_+ = \overline{P}^1. \tag{6}$$

(e) For a fixed ω^1, $H^2(\omega^1,\omega^2)$ is continuous on ω^2 and decreasing from infinity for $\omega^2 \downarrow 0$ to zero for $\omega^2 \ge \underline{\omega}^2 := \max_i h_i^2 \max\{1/\sigma^2, \omega^1/h_i^1\}$.

(f) $H^2(\Omega^1(\omega^2),\omega^2)$ is continuous on ω^2 such that $H(\Omega^1(\omega^2),\omega^2)$ tends to infinity for ω^2 tending to zero, and $H^2(\Omega^1(\omega^2),\omega^2) = 0$ for $\omega^2 \ge \underline{\omega}^2$. Thus, the root of the following equation exists and it can be found by bisection method:

$$H^2(\Omega^1(\omega^2),\omega^2) = \overline{P}^2. \tag{7}$$

Proposition 2 and Theorem 2 directly imply the following main result:

Theorem 4. *For $q^0 > 0$ the equilibrium strategies are given by (2) and (3), where $\omega^1 = \Omega^1(\omega^2)$ with Ω^1 given by (5), while ω^2 is given by (7). Due to the monotonic properties of H^1 and H^2 the $\Omega^1(\omega^2)$ can be found by the bisection method for each fixed ω^2, while the optimal ω^2 can be found by the superposition of two bisection methods.*

As an illustrative example throughout the paper we consider the total spectrum band consisting of five subcarriers, i.e., $n = 5$, the background noise power is $\sigma^2 = 1$ and the fading channel gains are $h^1 = (0.2, 0.5, 0.4, 0.1, 0.6)$, $h^2 = (0.23, 0.1, 0.5, 0.15, 1)$. Figure 1 illustrates an increase in the payoff to user 1 and a decrease in the payoff to user 2 with an increase in a priori probability q^0 for user 2 to be non-active. Of course, an increase in the power resource of user 2 leads to an increase in his payoff and in a reduction of the payoff to user 1. Figure 2 illustrates that for small power budget of user 2 ($\overline{P}^2 = 0.5$) the users employ subcarrier sharing strategies (i.e. I_{11} is empty). An increase in his power budget makes the user to employ the subcarriers user 1 also uses. Namely, for

$\overline{P}^2 = 1$ the set I_{11} is empty for $q^0 < 0.57$ and $I_{11} = \{5\}$ for $q^0 > 0.57$. While for $\overline{P}^2 = 1.5$ the set $I_{11} = \{3\}$ for $q^0 < 0.75$ and $I_{11} = \{3, 5\}$ for $q^0 > 0.75$. Thus, an increase in the probability q^0 leads to an increase in interference reflected by an increase in number of the subcarriers involved in being jointly employed by both users.

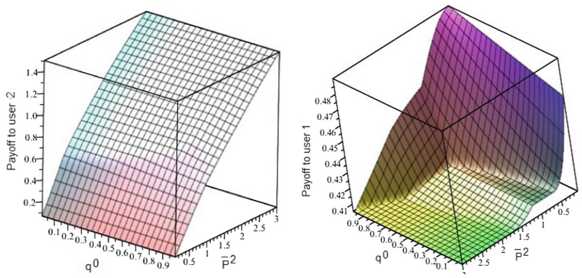

Fig. 1. The payoff to user 2 (left) and the payoff to user 1 (right) as functions on a priori probability q^0 and power budget $\overline{P}^2$ with $\overline{P}^1 = 1$.

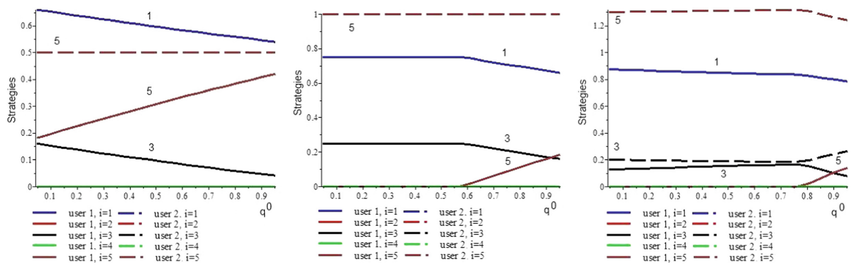

Fig. 2. Strategies of users for $\overline{P}^2 = 0.5$ (left), $\overline{P}^2 = 1$ (center) and $\overline{P}^2 = 1.5$ (right) with $\overline{P}^1 = 1$.

4 Both Users Always are Active: Explicit Solution

In this section we obtain the equilibrium strategies explicitly in an important boundary case for the a priori probability $q^0 = 0$, i.e., when both users always are active. This case coincides with two-person water-filling game in classical framework with independent fading channel gains. To get the equilibrium explicitly let us introduce an auxiliary notations. First, to avoid bulkiness in formulas we assume that all the subcarriers are different in ratio of fading channel gains for the users, i.e., $h_i \neq h_j$ with $i \neq j$, where $h_i := h_i^2/h_i^1$. Then, without loss of generality we can assume that the subcarriers are arranged in increasing order on ratio

$$h_1 < h_2 < \ldots < h_n < h_{n+1} := \infty. \tag{8}$$

$$\sum_{i=1}^{k}\left\lfloor\frac{1}{\omega_k^1}-\frac{\sigma^2}{h_i^1}\right\rfloor_+=\overline{P}^1 \text{ and } \sum_{i=k+1}^{n}\left\lfloor\frac{1}{\omega_k^2}-\frac{\sigma^2}{h_i^2}\right\rfloor_+=\overline{P}^2. \tag{9}$$

Due to the left side of the first equation (9) is decreasing on ω and increasing on k we have that $\omega_{k+1}^1 > \omega_k^1$. While due to the left side of the second equation (9) is decreasing on ω and k we have that $\omega_{k+1}^2 < \omega_k^2$. Thus, ξ_k is decreasing on k where

$$\xi_k := \omega_k^2/\omega_k^1 \text{ for } k=1,\dots,n-1 \text{ and } \xi_0=\infty \text{ and } \xi_n=0. \tag{10}$$

Theorem 5. *The considered game with $q^0 = 0$ has the unique equilibrium $(\boldsymbol{P}^1, \boldsymbol{P}^2)$.*

(a) If

$$h_k \le \xi_k < h_{k+1} \tag{11}$$

then

$$P_i^1 = \begin{cases}\left\lfloor\frac{1}{\omega_k^1}-\frac{\sigma^2}{h_i^1}\right\rfloor_+, & i\le k,\\ 0, & i\ge k+1,\end{cases} \text{ and } P_i^2 = \begin{cases}0, & i\le k,\\ \left\lfloor\frac{1}{\omega_k^2}-\frac{\sigma^2}{h_i^2}\right\rfloor_+, & i\ge k+1.\end{cases} \tag{12}$$

(b) If

$$\xi_k < h_k < \xi_{k-1} \tag{13}$$

then

$$P_i^1 = \begin{cases}\left\lfloor\frac{1}{\omega^1}-\frac{\sigma^2}{h_i^1}\right\rfloor_+, & i\le k-1,\\ \overline{P}^1-\sum_{j=1}^{k-1}\left\lfloor\frac{1}{\omega^1}-\frac{\sigma^2}{h_i^1}\right\rfloor_+ & i=k,\\ 0, & i\ge k+1,\end{cases} \quad P_i^2 = \begin{cases}0, & i\le k-1,\\ \overline{P}^2-\sum_{j=1}^{k-1}\left\lfloor\frac{1}{\omega^2}-\frac{\sigma^2}{h_i^2}\right\rfloor_+ & i=k,\\ \left\lfloor\frac{1}{\omega^2}-\frac{\sigma^2}{h_i^2}\right\rfloor_+, & i\ge k+1,\end{cases} \tag{14}$$

with

$$\omega^2 = h_k\omega^1 \tag{15}$$

and ω^1 is the unique positive root of the equation

$$F(\omega^1) := \sum_{j=1}^{k}\left\lfloor\frac{1}{\omega^1}-\frac{\sigma^2}{h_i^1}\right\rfloor_+ + \sum_{j=k+1}^{n}\left\lfloor\frac{1}{\omega^1}-\frac{\sigma^2 h_k}{h_i^2}\right\rfloor_+ = \overline{P}^1 + h_k\overline{P}^2. \tag{16}$$

Since ξ_i is decreasing while h_i is increasing, the condition (11) and (13) uniquely define the switching subcarrier k. The equilibrium strategies cannot jointly employ more than one subcarrier. It is interesting to note that a similar band sharing phenomena we can observe in bandwidth scanning strategy under incomplete information about adversary's activity [5,7,8]. If (11) holds than the strategies are subcarrier sharing while if (13) holds the equilibrium strategies subcarrier sharing except the only subcarrier k which they use jointly. Thus, (11) is the necessary and sufficient condition for the equilibrium strategy to be subcarrier sharing. Figure 3(left) illustrates that an increase in power budget of user 2 leads to a decrease in switching subcarrier while an increase in power budget of user 1 impacts on switching subcarrier in opposite way. Also, it illustrates sequential switching between the criteria with an increase of users' power budgets. Figure 3(center) illustrates that an increase in power budget to user 2 leads to an increase in switching subcarrier k, while an decrease in power budget to user 1 yields into an increase in k. An increase of power budget to a user leads to an increase in his payoff and in a decrease in the payoff to the other (Fig. 3(right)). A surprising property of the equilibrium strategies is that an increase in power budgets cannot lead to employing more than one subcarrier for joint using. This is quite different from the scenario with one of the users being malicious where an increase in power budgets make the users employ more and more subcarriers [10,17].

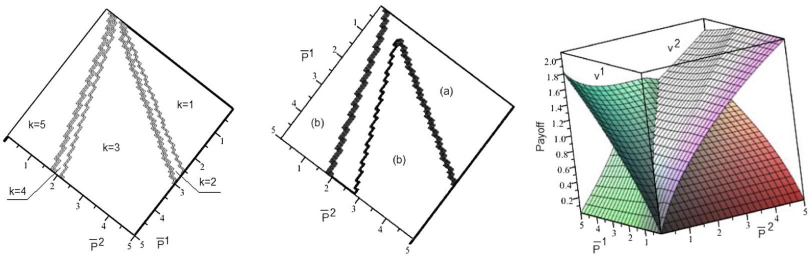

Fig. 3. The switching subcarrier k (left) and the cases of Theorem 5 (center) and payoffs to the users (right) as functions on $\overline{P}^1$ and $\overline{P}^2$.

5 Conclusions

In this paper, by means of a two users non-zero sum OFDM transmission game with independent fading channel gains, we investigate an impact of incomplete knowledge about whether a user is active on the equilibrium transmission strategy. Two algorithms to find equilibrium strategies are given. The first is the best response strategies algorithm illustrating learning mechanism to reach an equilibria. The second one is a superposition of two bisection methods based describing the equilibrium in on closed form. It allows to show that an decrease in uncertainty about the user to be active reduces size of the set of shared subcarriers. For the boundary case (i.e. complete knowledge about a user to be present, which reflects a classical water-filling game) the equilibrium strategies are given explicitly. The necessary and sufficient conditions, when subcarrier sharing strategies are optimal, is established, as well as the set of shared subcarriers is identified. Stability of the upper bound of size of this subcarriers' set to an increase of users' power budgets is proven, what can be applicable for NC-OFDM networks.

Acknowledgments. This work is supported in part by a grant from the U.S. Office of Naval Research (ONR) under grant number N00014-15-1-2168.

A Appendix

Proof of Theorem 1: Since $v^j(\boldsymbol{P}^1, \boldsymbol{P}^2)$ is concave on $\boldsymbol{P}^j$, (a) follows [2]. The KKT Theorem straightforward implies (b).

(c) It is clear that the sets of equilibrium coincide for the games with payoffs scaled by positive multiplies. That is why, to find equilibrium instead of the original game with payoffs (v^1, v^2) we can consider equivalent game with payoffs $(V^1, V^2) = (v^1, q^1 v^2)$. The last game is an exact potential game [14], and so, the best response algorithm converges. Recall that the game with payoffs (V^1, V^2) is an exact potential game if and only if there is a function $V(\boldsymbol{P}^1, \boldsymbol{P}^2)$ such that for any strategies $(\boldsymbol{P}^1, \boldsymbol{P}^2)$ and $(\boldsymbol{P}^1_*, \boldsymbol{P}^2_*)$ the following conditions hold:

$$\begin{aligned} V(\boldsymbol{P}^1_*, \boldsymbol{P}^2) - V(\boldsymbol{P}^1, \boldsymbol{P}^2) &= V^1(\boldsymbol{P}^1_*, \boldsymbol{P}^2) - V^1(\boldsymbol{P}^1, \boldsymbol{P}^2), \\ V(\boldsymbol{P}^1, \boldsymbol{P}^2_*) - V(\boldsymbol{P}^1, \boldsymbol{P}^2) &= V^2(\boldsymbol{P}^1, \boldsymbol{P}^2_*) - V^2(\boldsymbol{P}^1, \boldsymbol{P}^2). \end{aligned} \tag{17}$$

It is clear for the function

$$V(\boldsymbol{P}^1, \boldsymbol{P}^2) = q^1 \sum_{i=1}^{n} \ln(\sigma^2 + h^1_i P^1_i + h^2_i P^2_i) + q^0 \sum_{i=1}^{n} \ln(\sigma^2 + h^1_i P^1_i)$$

the condition (17) holds, and the result follows. ■

Proof of Theorem 2: Since $v^j(\boldsymbol{P}^1, \boldsymbol{P}^2)$ is concave on $\boldsymbol{P}^j$, by KKT Theorem, $(\boldsymbol{P}^1, \boldsymbol{P}^2)$ is an equilibrium if and only if there are ω^1 and ω^2 (Lagrangian multipliers) such that the following conditions hold:

$$\frac{q^1 h^1_i}{\sigma^2 + h^1_i P^1_i + h^2_i P^2_i} + \frac{q^0 h^1_i}{\sigma^2 + h^1_i P^1_i} \begin{cases} = \omega^1, & P^1_i > 0, \\ \le \omega^1, & P^1_i = 0, \end{cases} \tag{18}$$

$$\frac{h^2_i}{\sigma^2 + h^1_i P^1_i + h^2_i P^2_i} \begin{cases} = \omega^2, & P^2_i > 0, \\ \le \omega^2, & P^2_i > 0. \end{cases} \tag{19}$$

Thus, by (18) and (19), we have that

(a) if $P^1 = 0$ and $P^2 = 0$ then $h^1_i/\sigma^2 \le \omega^1$ and $h^2_i/\sigma^2 \le \omega^2$,

(b) if $P^1 > 0$ and $P^2 = 0$ then $P^1_i = 1/\omega^1 - \sigma^2/h^1_i$ with $h^1_i/\sigma^2 > \omega^1$ and $h^2_i/h^1_i \le \omega^2/\omega^1$.

(c) if $P^1 = 0$ and $P^2 > 0$ then $P^2_i = 1/\omega^2 - \sigma^2/h^2_i$ with $h^2_i/\sigma^2 > \omega^2$ and $q^1 h^1_i/h^2_i \omega^2 + q^0 h^1_i/\sigma^2 \le \omega^1$.

(d) if $P^1 > 0$ and $P^2 > 0$ then

$$P^1_i = \frac{q^0}{\omega^1 - q^1 h^1_i \omega^2/h^2_i} - \frac{\sigma^2}{h^1_i} \text{ and } P^2_i = \frac{1}{\omega^2} - \frac{h^1_i}{h^2_i} \frac{q^0}{\omega^1 - q^1 h^1_i \omega^2/h^2_i},$$

and the result follows. ■

Proof of Theorem 3: The set of the channels jointly used by both users is $I_{11}(\omega^1, \omega^2) = \{i : \omega^2/h^2_i < \omega^1/h^1_i < q^1\omega^2/h^2_i + q^0/\sigma^2\}$. First, note that due to $q^1\omega^2/h^2_i + q^0/\sigma^2 > \omega^2/h^2_i$ and $q^0 + q^1 = 1$ yield that $\sigma^2 > \omega^2/h^2_i$. Then, $q^1\omega^2/h^2_i + q^0/\sigma^2 = q^0(1/\sigma^2 - \omega^2/h^2_i) + \omega^2/h^2_i$ is increasing on q^0, and the result follows. ■

Proof of Theorem 5: Since $v^j(\boldsymbol{P}^1, \boldsymbol{P}^2)$ is concave on $\boldsymbol{P}^j$, by KKT Theorem, $(\boldsymbol{P}^1, \boldsymbol{P}^2)$ is an equilibrium if and only if there are ω^1 and ω^2 (Lagrangian multipliers) such that the following conditions hold for $m = 1, 2$:

$$\frac{h_i^m}{\sigma^2 + h_i^1 P_i^1 + h_i^2 P_i^2} \begin{cases} = \omega^m, & P_i^m > 0, \\ \le \omega^m, & P_i^m = 0. \end{cases} \tag{20}$$

Then, by (20),

$$(P_i^1, P_i^2) = \begin{cases} (0,0), & h_i^1/\omega^1 \le \sigma^2 \text{ and } h_i^2/\omega^2 \le \sigma^2, \\ \left(\dfrac{1}{\omega^1} - \dfrac{\sigma^2}{h_i^1}, 0\right), & h_i^1/\omega^1 > \sigma^2 \text{ and } h_i^2/\omega^2 \le h_i^1/\omega^1, \\ \left(0, \dfrac{1}{\omega^2} - \dfrac{\sigma^2}{h_i^2}\right), & h_i^2/\omega^2 > \sigma^2 \text{ and } h_i^1/\omega^1 \le h_i^2/\omega^2, \\ \sigma^2 + h_i^1 P_i^1 + h_i^2 P_i^2 = \dfrac{h_i^1}{\omega^1} = \dfrac{h_i^2}{\omega^2}, & h_i^1/\omega^1 = h_i^2/\omega^2 > \sigma^2. \end{cases} \tag{21}$$

By (21), if $P_i^1 > 0$ and $P_i^2 > 0$ then $h_i^2/h_i^1 = \omega^2/\omega^1$. Thus, by (8), both strategies can employ only at most one channel for joint use. Moreover, there is a k such that

$$P_i^1 \begin{cases} > 0, & i < k-1, \\ \ge 0, & i = k, \\ = 0, & i > k, \end{cases} \quad \text{and } P_i^2 \begin{cases} = 0, & i < k-1, \\ \ge 0, & i = k, \\ > 0, & i > k, \end{cases} \tag{22}$$

where (a) $P_k^1 > 0$ and $P_k^2 > 0$ if $h_k^2/h_k^1 = \omega^2/\omega^1$, (b) $P_k^1 > 0$ and $P_k^2 = 0$ if $h_k^2/h_k^1 < \omega^2/\omega^1$, and (c) $P_k^1 = 0$ and $P_k^2 > 0$ if $h_k^2/h_k^1 > \omega^2/\omega^1$.

Thus, by assumption (8), we have to consider separately two cases: (A) there is a k such that $h_k < \omega^2/\omega^1 < h_{k+1}$, (B) there is a k such that $h_k = \omega^2/\omega^1$.

(A) Let there exist a k such that $h_k < \omega^2/\omega^1 < h_{k+1}$. Then, by (21), (22) and the fact that $\boldsymbol{P}^1 \in \Pi^1$ and $\boldsymbol{P}^2 \in \Pi^2$, we have that $\boldsymbol{P}^1$ and $\boldsymbol{P}^2$ have to be given by (12), and $\omega^1 = \omega_k^1$ and $\omega^2 = \omega_k^2$. Thus, (11) also has to hold.

(B) Let there exist a k such that $h_k = \omega^2/\omega^1$. Thus, (15) holds. Also, by (21), (22) and the fact that $\boldsymbol{P}^1 \in \Pi^1$ and $\boldsymbol{P}^2 \in \Pi^2$, we have that $\boldsymbol{P}^1$ and $\boldsymbol{P}^2$ have to be given by (14) and also the following condition has to hold:

$$\sigma^2 + h_k^1 P_k^1 + h_k^2 P_k^2 = \frac{h_k^1}{\omega^1} = \frac{h_k^2}{\omega^2} \tag{23}$$

By (23) with right side h_k^1/ω^1, $P_k^1 = 1/\omega^1 - (\sigma^2 + P_k^2)/h_k^1 < 1/\omega^1 - \sigma^2/h_k^1$. Substituting this P_k^1 into (14) and taking into account that $\boldsymbol{P}^1 \in \Pi^1$ yield that

$$\omega^1 \le \omega_k^1. \tag{24}$$

Similarly, dealing with strategy $\boldsymbol{P}^2$ in condition (23) with right side h_k^2/ω^2 implies that

$$\omega^2 \le \omega_{k-1}^2. \tag{25}$$

By (14), the condition (23) with right side h_k^1/ω^1 is equivalent to

$$\sigma^2 + h_k^1\left(\overline{P}^1 - \sum_{j=1}^{k-1}\left\lfloor\frac{1}{\omega^1} - \frac{\sigma^2}{h_i^1}\right\rfloor_+\right) + h_k^2\left(\overline{P}^2 - \sum_{j=k+1}^{n}\left\lfloor\frac{1}{\omega^2} - \frac{\sigma^2}{h_i^2}\right\rfloor_+\right) = \frac{h_k^1}{\omega^1}. \tag{26}$$

Substituting (15) into (26) implies (16).

Since the left side of Eq. (16) is decreasing on ω^1, by (24), it has a root if and only of $F(\omega_k^1) < \overline{P}^1 + h_k\overline{P}^2$. This condition is equivalent to

$$\overline{P}^2 > \sum_{j=k+1}^{n}\left\lfloor\frac{1}{\omega^1 h_k} - \frac{\sigma^2}{h_i^2}\right\rfloor_+ = (\text{by } (15)) = \sum_{j=k+1}^{n}\left\lfloor\frac{1}{\omega^2} - \frac{\sigma^2}{h_i^2}\right\rfloor_+ .$$

Thus, $\omega^2 > \omega_k^2$. Substituting (15) in the last inequality and taking into account (24) implies that

$$\xi_k < h_k. \tag{27}$$

By (14) and (15), the condition (23) with right side h_k^2/ω^2 is equivalent to

$$G(\omega^2) := \sum_{j=1}^{k-1}\left\lfloor\frac{1}{\omega^2} - \frac{\sigma^2}{h_i^1 h_k}\right\rfloor_+ + \sum_{j=k}^{n}\left\lfloor\frac{1}{\omega^2} - \frac{\sigma^2}{h_i^2}\right\rfloor_+ = \overline{P}^1/h_k + \overline{P}^2. \tag{28}$$

Thus, by (25), this equation has a positive root if and only if $G(\omega_{k-1}^2) < \overline{P}^1/h_k + \overline{P}^2$. By (15) and (28), this is equivalent to $\omega_{k-1}^1 < \omega^1$. This, jointly with (15) and (25), implies that $\xi_{k-1} > h_k$. Then, taking into account (27) yields (13), and the result follows. ■

References

1. Altman, E., Avrachenkov, K., Garnaev, A.: Closed form solutions for water-filling problems in optimization and game frameworks. Telecommun. Syst. **47**, 153–164 (2011)
2. Fudenberg, D., Tirole, J.: Game Theory. MIT Press, Boston (1991)
3. Garnaev, A., Baykal-Gursoy, M., Poor, H.V.: A game theoretic analysis of secret and reliable communication with active and passive adversarial modes. IEEE Trans. Wirel. Commun. **15**, 2155–2163 (2016)
4. Garnaev, A., Trappe, W.: The eavesdropping and jamming dilemma in multi-channel communications. In: IEEE International Conference on Communications (ICC), pp. 2160–2164. Budapest, Hungary (2013)
5. Garnaev, A., Trappe, W.: Stationary equilibrium strategies for bandwidth scanning. In: Jonsson, M., Vinel, A., Bellalta, B., Marina, N., Dimitrova, D., Fiems, D. (eds.) MACOM 2013. LNCS, vol. 8310, pp. 168–183. Springer, Cham (2013). https://doi.org/10.1007/978-3-319-03871-1_15
6. Garnaev, A., Trappe, W.: Secret communication when the eavesdropper might be an active adversary. In: Jonsson, M., Vinel, A., Bellalta, B., Belyaev, E. (eds.) MACOM 2014. LNCS, vol. 8715, pp. 121–136. Springer, Cham (2014). https://doi.org/10.1007/978-3-319-10262-7_12

7. Garnaev, A., Trappe, W.: One-time spectrum coexistence in dynamic spectrum access when the secondary user may be malicious. IEEE Trans. Inf. Forensics Secur. **10**, 1064–1075 (2015)
8. Garnaev, A., Trappe, W.: A bandwidth monitoring strategy under uncertainty of the adversary's activity. IEEE Trans. Inf. Forensics Secur. **11**, 837–849 (2016)
9. Garnaev, A., Trappe, W.: Bargaining over the fair trade-off between secrecy and throughput in OFDM communications. IEEE Trans. Inf. Forensics Secur. **12**, 242–251 (2017)
10. Garnaev, A., Trappe, W., Petropulu, A.: Equilibrium strategies for an OFDM network that might be under a jamming attack. In: 51st Annual Conference on Information Systems and Sciences (CISS) (2017)
11. Han, Z., Niyato, D., Saad, W., Basar, T., Hjrungnes, A.: Game Theory in Wireless and Communication Networks: Theory, Models, and Applications. Cambridge University Press, New York (2012)
12. He, G., Cottatellucci, L., Debbah, M.: The waterfilling game-theoretical framework for distributed wireless network information flow. EURASIP J. Wirel. Commun. Netw. **1**, 124–143 (2010)
13. Ju, P., Song, W., Jin, A.L.: A Bayesian game analysis of cooperative MAC with incentive for wireless networks. In: IEEE Global Communications Conference (GLOBECOM) (2014)
14. Monderer, D., Shapley, L.S.: Potential games. Games Econ. Behav. **14**, 124–143 (1996)
15. Prasad, R.: OFDM for Wireless Communications Systems. Artech House, Boston (2004)
16. Scutari, G., Palomar, D.P., Barbarossa, S.: Asynchronous iterative water-filling for Gaussian frequency-selective interference channels. IEEE Trans. Inf. Theory **54**, 2868–2878 (2008)
17. Slimeni, F., Scheers, B., Le Nir, V., Chtourou, Z., Attia, R.: Closed form expression of the saddle point in cognitive radio and Jammer power allocation game. In: Noguet, D., Moessner, K., Palicot, J. (eds.) CrownCom 2016. LNICSSITE, vol. 172, pp. 29–40. Springer, Cham (2016). https://doi.org/10.1007/978-3-319-40352-6_3
18. Song, T., Stark, W.E., Li, T., Tugnait, J.K.: Optimal multiband transmission under hostile jamming. IEEE Trans. Commun. **64**, 4013–4027 (2016)
19. Sridharan, G., Kumbhkar, R., Mandayam, N.B., Seskar, I., Kompella, S.: Physical-layer security of NC-OFDM-based systems. In: IEEE Military Communications Conference (MILCOM), pp. 1101–1106 (2016)
20. Wei, H., Sun, H.: Using Bayesian game model for intrusion detection in wireless Ad Hoc networks. Int. J. Commun. Netw. Syst. Sci. **3**(7), 602–607 (2010)
21. Yu, W., Ginis, G., Cioffi, J.M.: Distributed multiuser power control for digital subscriber lines. IEEE J. Sel. Areas Commun. **20**, 1105–1115 (2002)