Variable Dimension Measurement Matrix Construction for Compressive Sampling via m Sequence

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Abstract. Signal acquisition in ultra-high frequency is a challenging problem due to high cost of analog-digital converter. While compressed sensing (CS) provides an alternative way to sample signal with low sampling rate, the construction of measurement matrix is still challenging due to hardware complexity and random generation. To address this challenge, a variable dimension deterministic measurement matrix construction method is proposed in this paper based on cross-correlation characteristics of m sequences. Specifically, a lower bound of the spark of measurement matrix is derived theoretically. The proposed measurement matrix construction method is applicable to compressive sampling system to improve the quality of signal reconstruction, especially for modulated wideband converter (MWC) architecture. Simulation results demonstrate that the proposed measurement matrix is superior to random Gauss matrix and random Bernoulli matrix.

Keywords: Measurement matrix · Compressed sensing
Modulated wideband converters · M sequence optimum pairs

1 Introduction

Signal processing and communication technology will inevitably involve the sampling process. Nyquist sampling theorem is recognized as basic theory in sampling theory, and it reveals that the required sampling frequency must be greater than or equal to twice the highest frequency sampling signal.

Communication signals always have certain structures and characteristics. For sparse signal processing, compressed sensing (CS) [1] is a revolutionary technology in recent years. It is a kind of effective signal acquisition method that sample signal at a much lower frequency than the Nyquist sampling frequency if signal is sparse in some domain. According to small amount of observations, the original signal can be recovered with high accuracy.

Designing measurement matrix is an important research direction. In order to ensure that the signal is not lost in the process of observation, measurement matrix needs to satisfy certain properties. RIP (Restricted Isometry Property) [2] is a sufficient condition for measurement matrix to be satisfied. However, it is proved that

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measurement matrix satisfies RIP property is a combinatorial problem, and there is no effective method to verify whether measurement matrix satisfies RIP property in polynomial time. A feasible alternative is to evaluate mutual coherence of measurement matrix. It is proved that the smaller cross-correlation value of the measurement matrix is, the more likely the measurement matrix satisfies RIP. The widely used measurement matrix are random Gauss matrix, Bernoulli matrix, partial Fourier matrix, partial Hadamard matrix, Toeplitz matrix and so on. Random matrix such as random Gauss matrix and Bernoulli matrix, their dimension can be arbitrarily generated and their performance are good. It requires large storage space and high complexity of hardware implementation, which limit CS using in practical application. Partial orthogonal matrix such as partial Fourier matrix, partial Hadamard matrix and structured matrix such as Toeplitz matrix, their hardware complexity are greatly reduced compared with random matrix. However, the dimension of these measurement matrix is fixed, which limits actual engineering application in signal acquisition. Moreover, the accurate probability of signal recovery needs to be improved.

In this paper, a variable dimension deterministic measurement matrix construction method is proposed based on cross-correlation characteristics of m sequences. Firstly, the proposed method can sample signal with variable dimension compared with existing methods [3]. Then, it reduces hardware complexity especially in the block diagram of MWC system. Meanwhile, the signal reconstruction performance can be further improved by using proposed method, so as to alleviate the number of required measurements.

2 Compressed Sensing Overview

If $\mathbf{x} \in \mathbb{R}^N$ can be sparsely represented in orthonormal basis $\boldsymbol{\Psi} \in \mathbb{R}^{N \times N}$, $\mathbf{f} \in \mathbb{R}^N$ can be recovered from $\mathbf{y} \in \mathbb{R}^M$ which is a small number of data $M(M \ll N)$.

The sampled signal via compressive sensing can be expressed as:

$$\mathbf{y} = \boldsymbol{\Phi}\mathbf{f} + \mathbf{z} = \boldsymbol{\Phi}\Psi\mathbf{x} + \mathbf{z} \tag{1}$$

If the N dimensional time domain signal $\mathbf{f} \in \mathbb{R}^{N \times 1}$ can be expanded in a linear group $\boldsymbol{\psi} = \{\psi_i\}_{i=1}^N$, we can get formula (2).

$$\mathbf{f} = \sum_{i=1}^{N} \psi_i x_i = \boldsymbol{\psi}\mathbf{x} \tag{2}$$

where $\mathbf{x}$ is coefficient of the N dimensional vector, $K(K \ll N)$ is the number of nonzero elements. The measurement matrix $\boldsymbol{\Phi}_{M \times N}$ is used to observe $\mathbf{x}$ in time domain.

$$\mathbf{y} = \boldsymbol{\Phi}\mathbf{f} = \boldsymbol{\Phi}\Psi\mathbf{x} = \boldsymbol{\Theta}\mathbf{x} \tag{3}$$

where $\Phi\Psi = \boldsymbol{\Theta}$.

Because $\mathbf{x}$ is K-sparse, the process of recovering $\mathbf{x}$ through observations $\mathbf{y}$ can be transformed into solving the following linear programming problem:

$$\min_{\mathbf{x}}\|\mathbf{x}\|_0 \quad \mathbf{s.\ t.}\ \mathbf{y} = \mathbf{\Theta x} \tag{4}$$

where $\min_{\mathbf{x}}\|\mathbf{x}\|_0 \triangleq |\{i\ :\ x_i\ \neq\ 0\}|$ denotes l_0 -norm of $\mathbf{x}$. l_0 -minimization problem is a NP hard problem. In CS, it is usually transformed into l_1 -minimization problem which is tractable.

$$\min_{\mathbf{x}}\|\mathbf{x}\|_1 \quad \mathbf{s.\ t.}\ \mathbf{y} = \mathbf{\Theta x} \tag{5}$$

where $\min_{\mathbf{x}}\|\mathbf{x}\|_1 \triangleq |\{i\ :\ x_i\ \neq\ 0\}|$ denotes l_1 -norm of $\mathbf{x}$. (5) is l_1 -norm optimization to solve $\mathbf{x}$, and we can get reconstruction signal $\widehat{\mathbf{f}}$ at the same time. In order to obtain the accurate reconstruction of $\mathbf{f}$, measurement matrix should satisfy RIP.

Definition 1. If and only if a given constant $\varepsilon \in (0, 1)$ meet:

$$(1-\varepsilon)\|\mathbf{f}\|_2^2 \leq \|\mathbf{\Phi f}\|_2^2 \leq (1+\varepsilon)\|\mathbf{f}\|_2^2 \tag{6}$$

If K-sparse signal $\mathbf{f}$ satisfies (6), we call matrix $\mathbf{\Phi}$ satisfies RIP (N, K, ε). It is quite difficult to judge whether a matrix satisfies RIP. In addition to RIP, mutual correlation can be utilized to measure the ability of measurement matrix $\mathbf{\Phi}$ for reconstructing sparse signal [4], which is defined as follows:

$$\rho = \max_{s\neq t}\frac{|\langle \mathbf{\Phi}(s), \mathbf{\Phi}(t)\rangle|}{\|\mathbf{\Phi}(s)\|_2\ \|\mathbf{\Phi}(t)\|_2} \tag{7}$$

where $\mathbf{\Phi}(s)$, $\mathbf{\Phi}(t)$ is column s and column t of $\mathbf{\Phi}$ respectively, $\langle \bullet\ ,\ \bullet \rangle$ is inner product of two column vectors. The smaller ρ is, the stronger non-correlation of $\mathbf{\Phi}$ is.

The RIP condition is consistent with the uncorrelated constraint condition in the physical sense. (6) requires that sub matrix composed of arbitrary K columns should be approximately orthogonal. That is to say, correlation coefficient of $\mathbf{\Phi}$ is small [5].

Definition 2. The spark of $\mathbf{\Phi}$ is

$$Spark(\mathbf{\Phi}) \stackrel{def}{=} \min\{\|\boldsymbol{\xi}\|_0 \colon \boldsymbol{\xi} \in \mathbf{\Phi}_{Nullsp_{\mathbf{R}^*}}\} \tag{8}$$

where $\mathbf{\Phi}_{Nullsp_{\mathbf{R}^*}} \stackrel{def}{=} \{\boldsymbol{\xi} \in \mathbf{R}^N : \mathbf{\Phi}\boldsymbol{\xi} = 0, \boldsymbol{\xi} \neq \mathbf{0}\}$. It can be proved that if and only if $Spark(\mathbf{\Phi}) > 2k$, k sparse signal $\mathbf{x}$ can be obtained by solving l_0- minimization problem with exact approximation [6].

3 Variable Dimension CS Matrix Construction and Analysis

The m sequence is also called the longest linear feedback shift register sequence, which is pseudo randomness, sharp autocorrelation and small cross-correlation. In the construction of CS measurement matrix, we mainly use the good correlation property. The cross-correlation function of a sequence is defined as follows:

$$R_{a,b}(\tau) = \sum_{i=0}^{M-1} a_i b_{i+\tau} \tag{9}$$

where the cycle of $a(a_0, a_1, \cdots, a_{M-1})$ and $b(b_0, b_1, \cdots, b_{M-1})$ are M. The cross-correlation coefficient is defined as:

$$\rho_{a,b}(\tau) = \frac{1}{n} \sum_{i=0}^{M-1} a_i b_{i+\tau} \tag{10}$$

3.1 Construction Process of CS Matrix

The construction process of measurement matrix in this paper is shown in Fig. 1.

According to the actual length of signal is N, appropriate sub matrix dimension $P \times P$ is selected which satisfies $N/2 \leq P \leq N$.

On the basis of signal cycle P, m sequence optimum pairs $\mathbf{a} = \{a_1\, a_2\, a_3\, \cdots\, a_{P-1}\}$ and $\mathbf{b} = \{b_1\, b_2\, b_3\, \cdots\, b_{P-1}\}$ are generated by correlation verifying. Where $P = 2^r - 1$ and r is the number of shift registers.

The following Toeblitz matrix A and B are achieved by circular shift of m sequence optimum pairs $\mathbf{a} = \{a_1\, a_2\, a_3\, \cdots\, a_{P-1}\}$ an $\mathbf{b} = \{b_1\, b_2\, b_3\, \cdots\, b_{P-1}\}$ respectively.

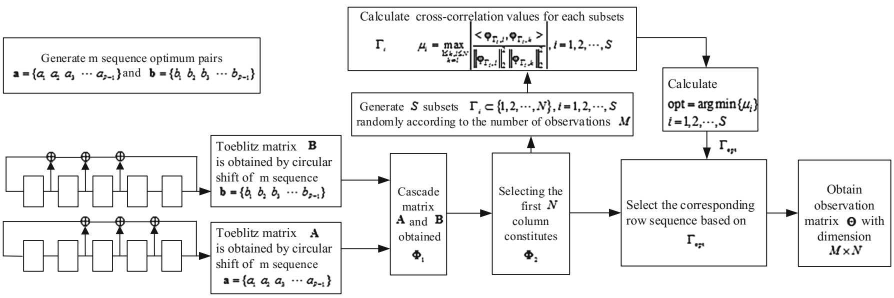

Fig. 1. Construction process of measurement matrix

$$\mathbf{A}=\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{P-1} \\ a_1 & a_2 & a_3 & \cdots & a_0 \\ a_2 & a_3 & a_4 & \cdots & a_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{P-1} & a_0 & a_1 & \cdots & a_{P-2} \end{bmatrix}, \mathbf{B}=\begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{P-1} \\ b_1 & b_2 & b_3 & \cdots & b_0 \\ b_2 & b_3 & b_4 & \cdots & b_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{P-1} & b_0 & b_1 & \cdots & b_{P-2} \end{bmatrix}, \tag{11}$$

Cascade matrix A and B in the following manner to form $\mathbf{\Phi}_1$. The dimension of Matrix $\mathbf{\Phi}_1$ is $P \times 2P$.

$$\mathbf{\Phi}_1 = [\mathbf{A}|\mathbf{B}] = \left[\begin{array}{ccccc|ccccc} a_0 & a_1 & a_2 & \cdots & a_{P-1} & b_0 & b_1 & b_2 & \cdots & b_{P-1} \\ a_1 & a_2 & a_3 & \cdots & a_0 & b_1 & b_2 & b_3 & \cdots & b_0 \\ a_2 & a_3 & a_4 & \cdots & a_1 & b_2 & b_3 & b_4 & \cdots & b_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{P-1} & a_0 & a_1 & \cdots & a_{P-2} & b_{P-1} & b_0 & b_1 & \cdots & b_{P-2} \end{array}\right] = [\boldsymbol{\varphi}_1 \ \boldsymbol{\varphi}_2 \ \boldsymbol{\varphi}_3 \ \cdots \ \boldsymbol{\varphi}_{2P}] \tag{12}$$

Selecting the first N column constitutes $\mathbf{\Phi}_2$.

$$\mathbf{\Phi}_2 = [\boldsymbol{\varphi}_1 \, \boldsymbol{\varphi}_2 \, \boldsymbol{\varphi}_3 \, \cdots \, \boldsymbol{\varphi}_N]. \tag{13}$$

If we have sampled M observations, we can randomly generate S subsets $\Gamma_i \subset \{1, 2, \cdots, N\}, i = 1, 2, \cdots, S$ that meet $|\Gamma_i| = M$. Calculate μ_i:

$$\mu_i = \max_{\substack{1 \le k, l \le N \\ k \ne l}} \left| \frac{<\boldsymbol{\varphi}_{\Gamma_{i,l}}, \boldsymbol{\varphi}_{\Gamma_{i,k}}>}{\|\boldsymbol{\varphi}_{\Gamma_{i,l}}\|_2^2 \|\boldsymbol{\varphi}_{\Gamma_{i,k}}\|_2^2} \right|, i = 1, 2, \cdots, S. \tag{14}$$

Then, choose $\mathrm{opt} = \arg\min\{\mu_i\}, i = 1, 2, \cdots, S$, and Γ_{opt} is the optimal subset. Selecting the corresponding line in $\mathbf{\Phi}_2$ on the basis of Γ_{opt}, measurement matrix is constructed.

$$\mathbf{\Theta} = \mathbf{\Phi}_2(\Gamma_{\mathrm{opt}}, :). \tag{15}$$

3.2 Analysis of the Proposed CS Matrix

It is easily got that m sequence optimum pairs has three cross-correlation values by theory analysis.

$$R_{a,b}(\tau) \in \left\{-1, -1 - 2^{\left\lfloor \frac{(r+2)}{2} \right\rfloor}, -1 + 2^{\left\lfloor \frac{(r+2)}{2} \right\rfloor}\right\} \tag{16}$$

where r is an even number that cannot be divisible by 4. $\lfloor \bullet \rfloor$ is rounding down the objective. The maximum cross-correlation value of the measurement matrix can be obtained. [7]

$$\mu(\mathbf{\Phi}_r)_{\max} = \max_{1 \le l \ne k \le N} \left| \frac{<\boldsymbol{\varphi}_l, \boldsymbol{\varphi}_k>}{\|\boldsymbol{\varphi}_l\|_2^2 \|\boldsymbol{\varphi}_k\|_2^2} \right| = \max\left(\frac{1}{n}, \frac{1 + 2^{\left\lfloor \frac{(r+2)}{2} \right\rfloor}}{n}, \frac{-1 + 2^{\left\lfloor \frac{(r+2)}{2} \right\rfloor}}{n} \right) = \frac{1 + 2^{\left\lfloor \frac{(r+2)}{2} \right\rfloor}}{n} \tag{17}$$

The lower bound of the spark value of the constructed measurement matrix can be calculated.

$$S(\mathbf{\Phi}) \ge 1 + \frac{1}{\mu(\mathbf{\Phi})_{\max}}, \tag{18}$$

where r cannot be divisible by 4.

We can obtain $S(\mathbf{\Phi})/2 \ge k$ from the previous theoretical analysis, then

$$k < \frac{1}{2}\left(1 + \frac{n}{1 + 2^{\left\lfloor \frac{(r+2)}{2} \right\rfloor}}\right). \tag{19}$$

If (18) is satisfied, signal can be reconstructed exactly. For variable dimensional matrices $\mathbf{\Theta}$, rows with small cross correlation values are selected based on M, so that the upper bounds may be reduced.

$$k < \frac{1}{2}\left(1 + \frac{n}{1 + 2^{\left\lfloor \frac{(r+2)}{2} \right\rfloor}}\right) or \frac{1}{2}\left(1 + \frac{n}{-1 + 2^{\left\lfloor \frac{(r+2)}{2} \right\rfloor}}\right). \tag{20}$$

4 Performance Evaluation

In this section, we use MWC to show the performance of the proposed measurement matrix described in the previous section. MWC for sub-Nyquist sampling system [8] is shown in Fig. 2. The sampled signal passes through m parallel channels, and each row of measurement matrix corresponds to each parallel channel. Signals are multiplied in each channel with modulation sequence. Then, they pass through low pass filter, and finally it sample at a low rate. These is the signal acquisition process.

Suppose function expression of the original analog signal is:

$$x(t) = \sum_{n=1}^{N/2} \sqrt{E_n B_n} sinc(B_n(t - \tau_n))cos(2\pi f_n(t - \tau_n)). \tag{21}$$

The energy coefficient E_n and the time delay τ_n are randomly set. Signal bandwidth is $B_n = 50$ MHz, and the carrier frequency f_n is randomly distributed in $[0, 5]$ GHz, which means that Nyquist sampling frequency of the signal is $f_N = 10$ GHz at least. According to the number of parallel channels MWC is 50, the original analog signal

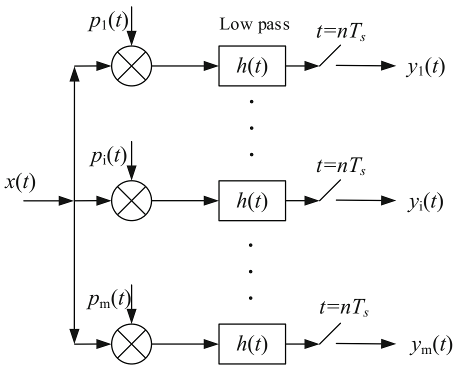

Fig. 2. Block diagram of modulated wideband converter (MWC)

spectrum can be divided into 195 equivalent blocks, and this sets cutoff bandwidth for low-pass filter in each channel and sampling frequency for low speed analog-to-digital converter. Therefore, the total sampling rate of MWC is $50 \times 51.3 \approx 2.565$ GHz, and the dimension of measurement matrix needed for MWC is 50×195. The m sequence cycle is $P = 127 = 2^7 - 1$, which means requiring 7 stage shift registers. Select m sequence optimum pairs $x^7 + x^3 + x^2 + x + 1$ and $x^7 + x^3 + 1$. Then $\boldsymbol{\Theta} \in \mathbb{R}^{50 \times 195}$ is constructed based on the steps of Sect. 3. In this paper, measurement matrix construction requires only two pairs of cyclic shift registers to obtain m sequences. The mixing sequences of other channels can be obtained by cyclic shifts of the generated m sequences. Compared with the traditional random measurement matrix, measurement matrix constructed in this paper greatly reduces the required storage space and is easy to implement by hardware.

In this article, all simulations are based on MWC for sub-Nyquist sampling system. Reconstruction algorithm is Orthogonal Matching Pursuit (OMP) algorithm [9] (Fig. 3).

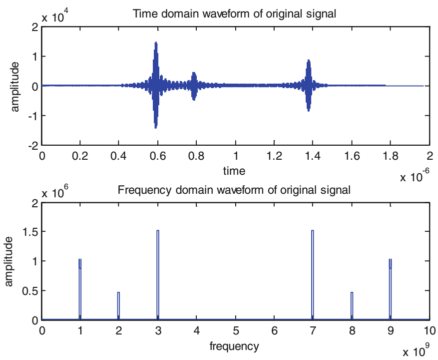

Fig. 3. Original signal waveform

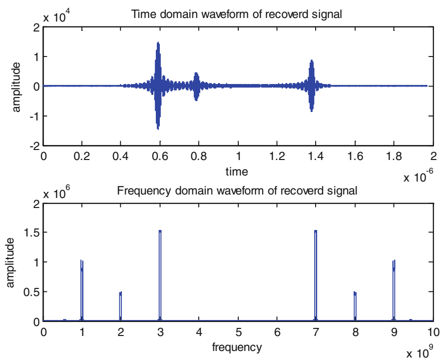

Fig. 4. Reconstructed signal waveform

Figure 4 shows the time domain and frequency domain waveform of reconstructed signal which uses the measurement matrix constructed in this paper. The measurement matrix constructed in this paper is used as mixed function $P_i(t)$. When the sampling frequency is only 1/4 of Nyquist frequency, the original analog signal can be restored almost without distortion.

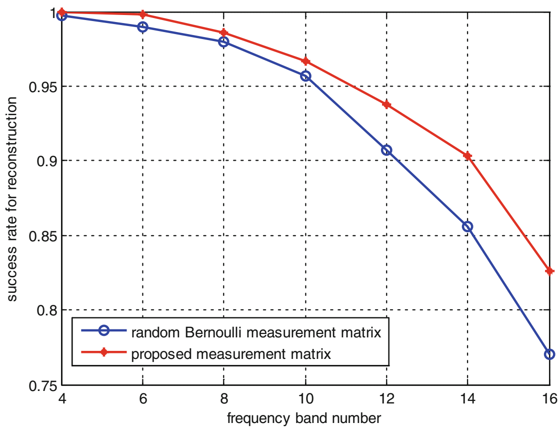

Fig. 5. Performance comparison between proposed measurement matrix and Bernoulli matrix

Figure 5 compares the probability of successful recovery of the signal under different frequency band between the proposed measurement matrix and the Bernoulli measurement matrix for signal acquisition. It can be seen clearly that when frequency band number of analog signals is 6, the signals can still recover almost 100% using the proposed measurement matrix. When frequency band of original analog signal continues to increase, success rate of analog signal restoration using the proposed measurement matrix is still higher than that of Bernoulli measurement matrix. Compared with Bernoulli measurement matrix, the designed measurement matrix occupies less storage and hardware resources. At the same time, it shows better performance, high recovery probability and high reliability in practical applications.

The recovery probability of Gauss signal and 0–1 signal under different sparsity is conducted. Figures 6 and 7 are comparisons of success reconstruction ratio of the proposed measurement matrix and the random Gauss measurement matrix under Gauss signal and 0–1 signal respectively. The dimension of measurement matrix is 50×195. For Gauss signal, random Gauss measurement matrix cannot guarantee accurate reconstruction of the signal when sparsity is 40. However, the measurement matrix designed in this paper still guarantee success rate of reconstruction approaching 100%. For 0–1 signal, performance of signal reconstruction is worse than Gauss signal. However, compared with the random Gauss measurement matrix, the maximum reconstruction probability gain is 45% if using the proposed measurement matrix in this paper.

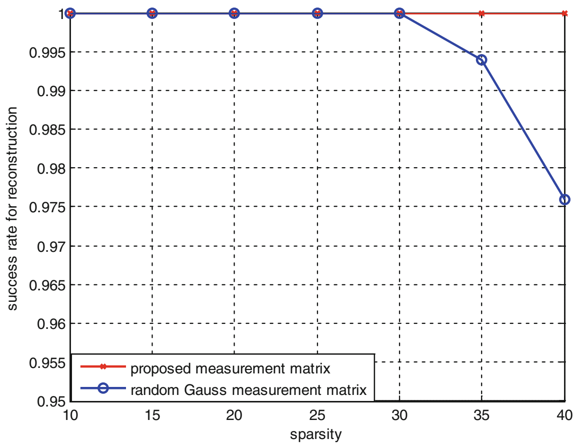

Fig. 6. Recovery probability comparison under Gauss signal

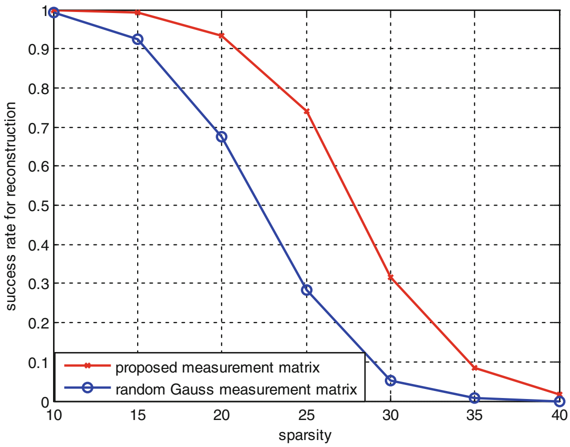

Fig. 7. Recovery probability comparison under 0–1 signal

5 Conclusion

To alleviate the hardware complexity of random measurement matrix. A variable dimension measurement matrix construction method is proposed in this paper based on characteristics of m sequences. A lower bound of the spark for the proposed matrix is obtained by theoretical derivation, which shows the proposed matrix is feasible for signal measurement. Additionally, the method of measurement construction can be extended to partial orthogonal matrices and Toeplitz matrices to realize variable dimension measurement matrix, which is a variable dimension construction framework for measurement matrix. Simulation results demonstrate that the proposed measurement matrix is superior to random Gauss measurement matrix and random Bernoulli matrix for both Gaussian and 0–1 signals in terms of the probability of success reconstruction. For sub-Nyquist sampling architecture MWC, it saves hardware storage resources significantly due to high reconstruction probability.

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References

1. Candes, E.J., Romberg, J., Tao, T.: Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information. J. IEEE Trans. Inf. Theory. **52**, 489–509 (2006)
2. Candes, E.J., Tao, T.: Decoding by linear programming. J. IEEE Trans. Inf. Theory **51**, 4203–4215 (2005)
3. Sun, R., Zhao, H., Xu, H.: The application of improved Hadamard measurement matrix in compressed sensing. In: International Conference on Systems and Informatics, pp. 1994–1997. IEEE, Yantai (2012)
4. Donoho, D.L., Elad, M.: Optimally sparse representation in general (nonorthogonal) dictionaries via $\ell 1$minimization. J. Proc. Natl. Acad. Sci. U.S.A. **100**, 2197–2202 (2003)
5. Xia, S.T., Liu, X.J., Jiang, Y., Zheng, H.T.: Deterministic constructions of binary measurement matrices from finite geometry. J. IEEE Trans. Sig. Process. **63**, 1017–1029 (2013)
6. Tropp, J.A.: Greed is good: algorithmic results for sparse approximation. J. IEEE Trans. Inf. Theory **50**, 2231–2242 (2004)
7. Dang, K., Ma, L., Tian, Y., Zhang, H., Ru, L., Li, X.: Construction of the compressive sensing measurement matrix based on m sequences. J. Xian Dianzi Keji Daxue Xuebao/J. Xidian Univ. **42**, 186–192 (2015)
8. Mishali, M., Eldar, Y.: From theory to practice: sub-nyquist sampling of sparse wideband analog signals. J. IEEE J. Sel. Top. Sig. Process. **4**, 375–391 (2010)
9. Karahanoglu, N.B., Erdogan, H.: A* orthogonal matching pursuit: best-first search for compressed sensing signal recovery. J. Digit. Sig. Process. **22**, 555–568 (2012)