Obtaining Ellipse Common Tangent Line Equations by the Rolling Tangent Line Method

Naizhang Feng, Teng Jiang, Shiqi Duan, and Mingjian Sun(✉)

School of Information and Electrical Engineering,
Harbin Institute of Technology, West Wenhua Road, 2, High-tech District,
Weihai 264209, Shandong, China
sunmingjian@hit.edu.cn

Abstract. In the field of image processing and machine vision, it is sometimes necessary to obtain common tangent line equations and tangent point coordinates from ellipses. A rolling tangent line method was proposed to obtain the 4 common tangent line equations and 8 tangent point coordinates from two ellipses in this paper. The principle of this method is simple and it is easy to program on a computer. Use this method to process two ellipse targets in an image and the experiment results show that the 4 common tangent equations and 8 tangent point coordinates can be obtained in high precision and the maximum execution time is less than 0.1 s.

Keywords: Image processing · Machine vision · Ellipse
Common tangent line

1 Introduction

In some engineering applications, it is necessary to obtain common tangent line equations and tangent point coordinates from two ellipses. For example, Mateos [1] used the tangent points as feature points for camera calibration. Guangjun [2] used the tangent points as feature points to measure the position and orientation of unmanned aerial vehicles.

In order to obtain the 4 tangent line equations and 8 tangent point coordinates, Zhang [2] proposed a method to solve an equation group which consists of two dual conic corresponding to the two ellipse equations. But the equation group is nonlinear and multivariate, it is not easy to solve by computers. Therefore, this method is not applicable to specific engineering practice. Xiaoxiang [3] proposed an iterative method to obtain tangent line equations from two ellipses. This method is sample and highly operational. It also converges quickly. However, in order to ensure convergence and accuracy, it needs to artificially adjust the position of the ellipses in the coordinate system so that the slope of the tangent line is approximately between 0.5 and 1.5. This makes this method no longer applicable to engineering applications with high autonomy requirements.

To solve the above problems, this paper proposes a rolling tangent line method, which can effectively obtain the 4 common tangent line equations and 8 tangent point coordinates from the two ellipses. Compared with the existing methods, this method

X. Gu et al. (Eds.): MLICOM 2017, Part II, LNICST 227, pp. 569–576, 2018.
https://doi.org/10.1007/978-3-319-73447-7_61

has three advantages: (1) The principle of this method is simple, and it is easy to be achieved on a computer; (2) There is no iterative calculation, so this method executes fast; (3) The method has high stability and high precision. This paper will introduce the principle of this method in details, and the effectiveness of the proposed method is verified by experimental results.

2 Basic Principle of the Rolling Tangent Line Method

The general equation of an ellipse can be expressed in the form of formula (1).

$$x^2 + Axy + By^2 + Cx + Dy + E = 0. \tag{1}$$

The ellipse equation shown in formula (1) can be rewritten into a binomial form of homogeneous coordinates, as shown in formula (2). Formula (2) is equivalent to formula (1).

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 1 & A/2 & C/2 \\ A/2 & B & D/2 \\ C/2 & D/2 & E \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0. \tag{2}$$

Further, in order to facilitate analysis, the formula (2) is rewritten into the form of formula (3). In formula (3), d is the vector $[x\ y\ 1]^T$, which represents a point on the ellipse curve.

$$d^T M d = 0. \tag{3}$$

Now assume that there are two ellipses M_1 and M_2 on the image coordinate system. Using the rolling tangent line method to obtain the 4 tangent line equations can be implemented in three steps. Details are as follows:

Step 1: Generate tangent points on ellipse M_2.
Figure 1 shows a Diagrammatic sketch of an image coordinate system. Assuming that the equations for ellipses M_1 and M_2 are known, then tangent points will be generated on the ellipse M_2. Among all the tangent points, the top and the bottom tangent point,

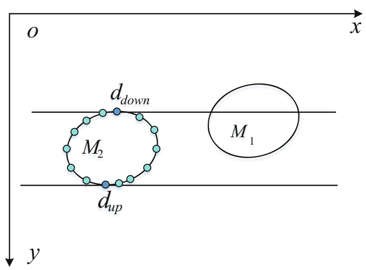


Fig. 1. Tangent points generated on ellipse M_2.

i.e. d_{down} and d_{up} in Fig. 1, will be generated at first. The tangent line through d_{down} or d_{up} is parallel to ox axis.

Assume that the coordinates of d_{down} and d_{up} are (x_{down}, y_{down}) and (x_{up}, y_{up}) respectively. We will use the coordinates of these two points to determine the coordinates of other tangent points.

Divide the interval [y_{down}, y_{up}] into three sub-intervals. The first sub-interval is [y_{down}, $y_{down} + \delta y$], and the y-coordinate of the tangent points in this interval are y_{down}, $y_{down} + \Delta_1$, $y_{down} + 2\Delta_1$, $y_{down} + 3\Delta_1$, $\cdots$, $y_{down} + \delta y$, respectively. The second sub-interval is ($y_{down} + \delta y$, $y_{up} - \delta y$], and the y-coordinate of the tangent points in this interval are $y_{down} + \delta y + \Delta_2$, $y_{down} + \delta y + 2\Delta_2$, $y_{down} + \delta y + 3\Delta_2$, $\cdots$, $y_{up} - \delta y$ respectively. The third sub-interval is ($y_{up} - \delta y$, y_{up}], and the y-coordinate of the tangent points in this interval are $y_{up} - \delta y + \Delta_3$, $y_{up} - \delta y + 2\Delta_3$, $y_{up} - \delta y + 3\Delta_3$, $\cdots$, y_{up} respectively. The δy is a small constant. The Δ_1, Δ_2, Δ_3 are small step lengths. In general, Δ_2 is much larger than Δ_1, and Δ_1 is equal to Δ_3. By dividing different intervals to determine the y-coordinate of the tangent point is to ensure the accuracy of the results and the execution speed of the method.

It is assumed that the y-coordinates of n tangent points are obtained by the above method. For any one of them, take y_i into the equation of ellipse M_2, we can get two x-coordinates, x_{i1} and x_{i2}, which locates on the left and right sides of the connection line between point d_{down} and point d_{up}. That is, each y_i corresponds to the two tangent points (x_{i1}, y_i) and (x_{i2}, y_i) on the ellipse M_2.

Step 2: Solve the tangent line equation for each tangent point generated on ellipse M_2. The coordinates of a tangent point d_i on ellipse M_2 is (x_i, y_i), corresponding to the homogeneous coordinate $[x_i, y_i, 1]^T$. Then a tangent line l_i on the ellipse M_2 that through d_i can be obtained by formula (4) in book [4].

$$l_i = M_2 d_i. \tag{4}$$

Starting at point d_{down}, arrive at point d_{up} in clockwise direction, and continue to go back to point d_{down} in clockwise direction, obtain the equation of the tangent line to each tangent point. N tangent line equations can be obtained in total. This process is like a tangent line rolling on the ellipse M_2, hence we call this method the rolling tangent line method. This process can also be shown in Fig. 2.

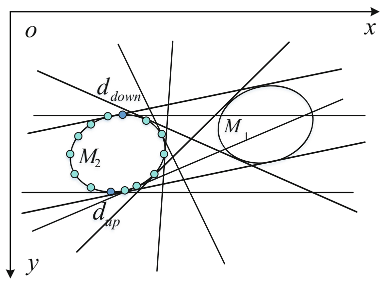


Fig. 2. Each tangent point generated on ellipse M_2 corresponds to a tangent line.

Step 3: Determine all the 4 tangent line equations by the deviation.
The tangent line equations for all tangent points generated on M2 has been obtained in step 2. If the tangent l_i on the ellipse M_2 is also tangent to the ellipse M_1, then the tangent l_i must satisfy the formula (5).

$$l_i^T M_1^{-1} l_i = 0. \tag{5}$$

Here, we define $l_i^T M_1^{-1} l_i$ as the deviation denoted by e. Further analysis shows that if l_i is the tangent line of M_2 but not the tangent line of M_1, then e is not equal to 0, and l_i farther away from M_1, the greater the absolute value of deviation e will be. When a tangent line rolls a circle along the ellipse M_2 from point d_{down}, the diagrammatic sketch that the absolute value of the deviation $|e|$ varies with the tangent line sequence number is shown in Fig. (3).

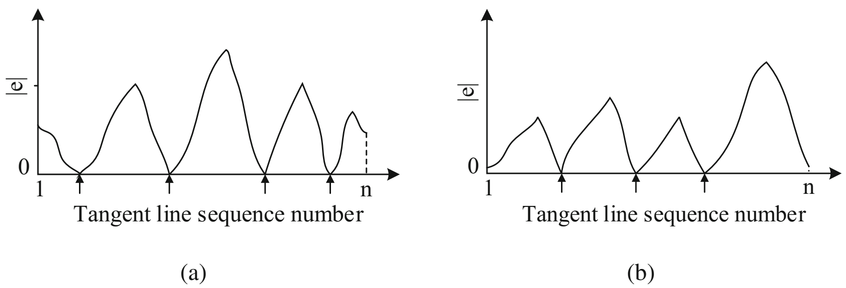


Fig. 3. Diagrammatic sketch that the absolute value of the deviation |e| varies with the tangent line sequence number. (a) represents a general case, and (b) represents a special case.

In Fig. 3(a), the tangent lines corresponding to the 4 valleys between the two peaks indicated by the upper arrow are the common tangent lines. Figure 3(b) shows a special case. In special case, only 3 common tangent lines can be obtained directly. In order to obtain the fourth common tangent line, we can compare the two boundary points, then select the smaller one as the line sequence number corresponding to the fourth common tangent line. Further, we will use formula (6) to obtain the tangent point from the 4 common tangent lines and the ellipse M_1.

$$d = M_1 l. \tag{6}$$

The above three steps illustrates the procedure to obtain the common tangent line equations from two ellipses by the rolling tangent line method. In Sect. 3, we will verify the validity of the method through experiments.

3 Experiment and Results

The experiment was done on a laptop, in which the CPU is Intel Core i3-2350M and the memory size is 6 GB. Figure 4 is a 640 × 480 pixels image captured by a camera, which contains two ellipse targets. After a series of image processing algorithms [5–10], we can get the equations of the outer contour curves of two ellipse targets, which are shown in the Fig. 4. In Fig. 4, the green curves represent the two ellipse and the crossings represent the center of the ellipse. Then, the rolling tangent line method will be used to obtain common tangent line equations and tangent points from the two ellipses.

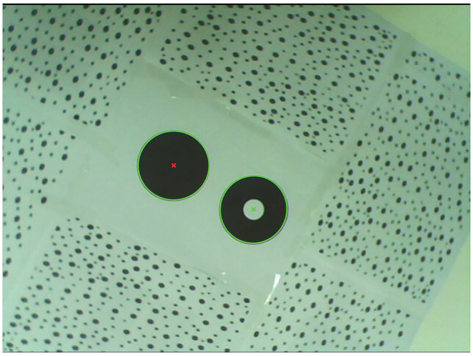


Fig. 4. A 640 × 480 pixels image contains two ellipse targets. In this figure, the green curves represent the two ellipse and the crossings represent the center of the ellipse. (Color figure online)

When writing a program to achieve the rolling tangent line method, we set the parameter δy to 1, Δ_1 and Δ_3 to 0.01, Δ_2 to 0.2. Using the rolling tangent line method generated 1314 tangent points in total, corresponding to 1314 tangent line. The curve

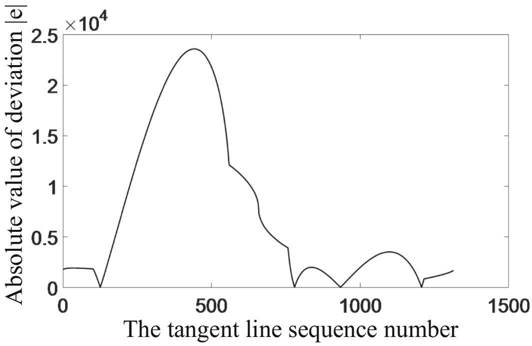


Fig. 5. The absolute value of the deviation |e| varies with the tangent line sequence number. There are 4 valleys in this figure corresponding to the 4 common tangent lines.

Table 1. Tangent line equations obtained from the two ellipses

Tangent line sequence number	Tangent line equations
125	$21.95x - 38.44y + 1302.65 = 0$
779	$-21.115x + 38.89y - 5860.78 = 0$
934	$-44.36x + 9.18y + 10621.07 = 0$
1207	$-14.966x - 42.058y + 15065.51 = 0$

Table 2. Tangent point coordinates obtained from the two ellipses

Tangent point sequence number	Column coordinate	Row coordinate	Tangent point sequence number	Column coordinate	Row coordinate
1	260.60	182.17	5	283.47	214.11
2	367.61	243.80	6	300.49	295.20
3	213.01	265.30	7	252.57	269.04
4	323.25	326.20	8	330.75	240.60

that the absolute value of the deviation $|e|$ varies with the tangent line sequence number is shown in Fig. 5. The tangent line sequence numbers corresponding to the common tangent lines are 125, 779, 934 and 1207. The detailed parameters of the four tangent lines are shown in Table 1. The coordinates of the 8 tangent points are shown in Table 2. Obtaining 4 common tangent line equations and 8 tangent point coordinates took a total of 0.077 s.

In order to intuitively observe the 4 common tangent lines and 8 tangent points obtained from the two ellipses, the tangent lines and the tangent points was marked in

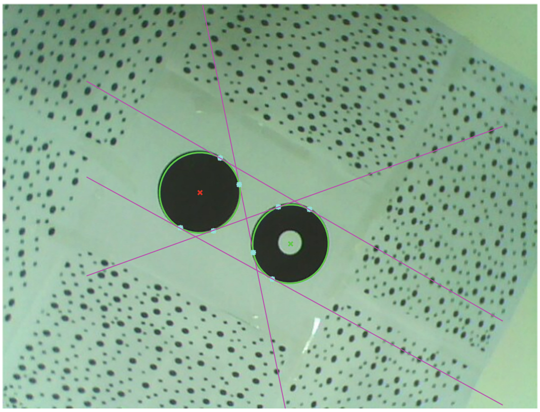
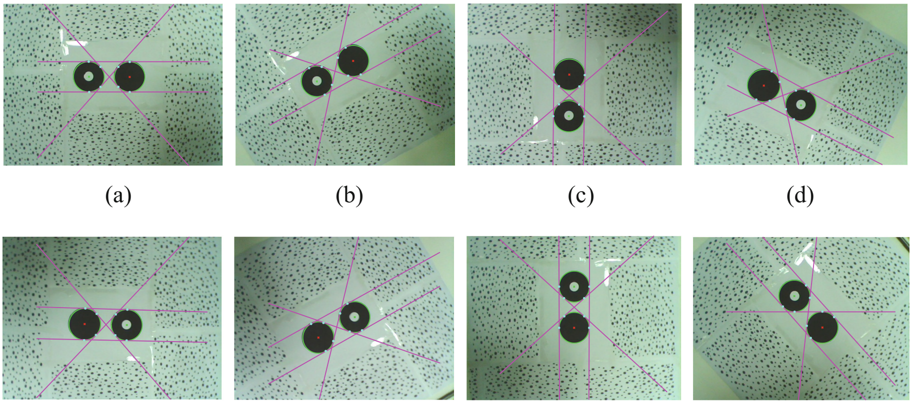


Fig. 6. The 4 tangent lines and 8 tangent points were marked in the image. The purple line represents tangent lines, and the cyan point represents tangent points. (Color figure online)



(a) (b) (c) (d)

(e) (f) (g) (h)

Fig. 7. The other 8 images and the corresponding processing results. Time consuming: (a) 0.062 s; (b) 0.048 s; (c) 0.065 s; (d) 0.046 s; (e) 0.068 s; (f) 0.051 s; (g) 0.047 s; (h) 0.058 s.

the image, as shown in Fig. 6. It can be seen that 4 common tangent lines and 8 tangent points was obtained in high precision.

In order to verify the stability of the rolling tangent line method, the other 8 images with ellipse targets were processed. The results are shown in Fig. 7. From the results shown in Fig. 7, we can see that regardless of how the ellipse object is distributed in the image coordinate system, the 4 common tangent lines and the 8 tangent points from the two ellipses can be stably and quickly obtained by using the rolling tangent method.

4 Conclusion

This paper presents a rolling tangent line method to obtain common tangent line equations and tangent point coordinates from two ellipses. The principle of this method is simple and it is easy to program on a computer. There is no iterative calculation, so this method can execute fast. The experimental results show that this method is stable and the common tangent line equations with the tangent point coordinates can be obtained in high precision. These advantages make the rolling tangent line method applicable for engineering practice.

References

1. Mateos, G.G.: A camera calibration technique using targets of circular features. In: 5th Ibero-America Symposium On Pattern Recognition (SIARP) (2000)
2. Guangjun, Z., Fuqiang, Z.: Position and orientation estimation method for landing of unmanned aerial vehicle with two circle based computer vision. Acta Aeronautica et Astronautica Sinica **26**(3), 344–348 (2005)

3. Xiaoxiang, Z.: Seek common tangent with the iterative method. Discipl. Explor. **5**, 28–29 (2015)
4. Hartley, R., Zisserman, A.: Multiple View Geometry in Computer Vision. Cambridge University Press, Cambridge (2003)
5. Otsu, N.: A threshold selection method from gray-level histograms. IEEE Trans. Syst. Man, Cybern. **9**(1), 62–66 (1979)
6. Cabaret, L., Lacassagne, L., Oudni, L.: A review of world's fastest connected component labeling algorithms: speed and energy estimation. In: 2014 Conference on IEEE Design and Architectures for Signal and Image Processing (DASIP), pp. 1–6 (2014)
7. Wu, K., Otoo, E., Shoshani, A.: Optimizing connected component labeling algorithms. In: Medical Imaging. International Society for Optics and Photonics, pp. 1965–1976 (2005)
8. Weiss, M.A.: Data Structures and Algorithm Analysis in C. Addison-Wesley, Reading (1997)
9. Amirfakhrian, M., Mafikandi, H.: Approximation of parametric curves by moving least squares method. Appl. Math. Comput. **283**, 290–298 (2016)
10. Apollonius of Perga: Treatise on Conic Sections. Cambridge University Press, Cambridge (2013)