Spatial Spectrum Estimation for Wideband Signals by Sparse Reconstruction in Continuous Domain

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Abstract. A novel spatial spectrum estimation method for two-dimensional wideband signals by sparse reconstruction in continuous domain is addressed in this paper. First, Discrete Fourier Transform (DFT) is employed for the data. Then the convex and corresponding dual problems of the data with most power are founded and solved. After that the sparse support sets are decided by semidefinite program and extracting roots. Finally, both of the direction of arrival (DOA) and the primary signals are determined. The proposed idea averts the off-grid effect based on grid partition, and some theoretical results are included to explain the effectiveness of the method.

Keywords: Direction of arrival · Sparse reconstruction · Wideband signals Continuous domain

1 Introduction

Spatial spectrum estimation through sparse reconstruction is a new kind of direction of arrival (DOA) method arisen in the past few decades [1–5]. Malioutov [6] transformed the DOA estimation into sparse recovery under redundant dictionary, optimized the solution by second-order cone programming. Tang [7] proposed a beam forming method based on sparse characteristic, then reconstruct the signals with orthogonal matching pursuit, but some false peaks exist when there are too many signals. Yin [8] presented the concept of space compression sampling matrix, the signals are sampled, and they are compressed at the same time, then calculated the initial signals and DOA through solving some optimization problems. Basis pursuit [9] and Matching pursuit [10] are both based on $L1$ penalty term. The former has a higher precision, but the

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computation is complex; the latter is the opposite. In 2013, Carlin [11] employed Bayesian learning for signal recovery, provided a new scheme according to the spatial of solution and timing structure.

Conventional sparse reconstruction technique has lowered the requirement of signal to noise ratio (SNR) and sampling number, but generally speaking, the actual DOAs are not at the grid point. Therefore, Candes and Fernandez [12, 13] studied the super-resolution from samples at the low end of the spectrum, as he reconstructed sources in continuous domain, which had improved the estimation precision to a great extent, but they did not studied how to estimate spatial spectrum for wideband signals according to the theory.

This paper presents a new spatial spectrum estimation algorithm, first, the sources are partitioned into some subbands, then the convex and corresponding dual problems of the data with most power are founded and solved. After that the sparse support sets are decided by semidefinite program and extracting roots. Finally, both of the direction of arrival (DOA) and the primary signals are determined. The proposed algorithm averts the error created by sparse reconstruction based on grid partition, and it has a preferable performance under the circumstance of low SNR and small samples.

2 Array Signal Model

As is shown in Fig. 1, assume that there is an arbitrary array with N sensors in X–Y plane, the origin O is defined as the reference, and the coordinate of these sensors are $(x_n, y_n)(n = 1, 2, \cdots, N)$. Suppose that there are K far-field wideband sources impinging on these sensors, DOAs are $(\phi_k, \theta_k)(k = 1, 2, \cdots, K)$, here ϕ_k and θ_k are the azimuth and elevation respectively, so output of the array is

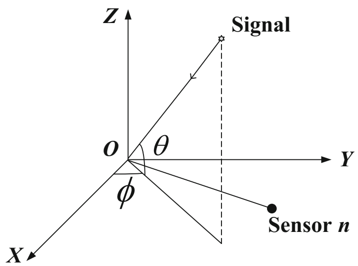

Fig. 1. Array signal model

$$\boldsymbol{y}(t) = [y_1(t), \cdots, y_N(t)]^{\mathrm{T}} = \left[\sum_{k=1}^{K} s_k(t - \tau_{1k}), \cdots, \sum_{k=1}^{K} s_k(t - \tau_{Nk}) \right]^{\mathrm{T}} + [b_1(t), \cdots, b_N(t)]^{\mathrm{T}} \quad (1)$$

where $y_n(t)\ (n = 1, 2, \cdots, N)$ is the output of the nth sensor, c is the speed of the source, $[b_1(t), \cdots, b_N(t)]$ is the additive Gaussian white noise vector, $b_n(t)$ is the corresponding noise of the nth sensor.

The frequency band is partitioned into G parts, perform discrete Fourier transform (DFT) on $\boldsymbol{y}(t)$, we have:

$$\boldsymbol{Y}(f_g) = \boldsymbol{A}(f_g)\boldsymbol{S}(f_g) + \boldsymbol{B}(f_g) \quad g = 1, 2, \cdots, G \tag{2}$$

Here, $\boldsymbol{A}(f_g)$ is the array manifold of f_g

$$\begin{aligned}\boldsymbol{A}(f_g) &= [\boldsymbol{a}(f_g, \phi_1, \theta_1), \cdots, \boldsymbol{a}(f_g, \phi_k, \theta_k), \cdots, \boldsymbol{a}(f_g, \phi_K, \theta_K)] \\ &= \begin{bmatrix} \mathrm{e}^{-\mathrm{j}2\pi f_g \tau_{11}} & \cdots \mathrm{e}^{-\mathrm{j}2\pi f_g \tau_{1k}} & \cdots \mathrm{e}^{-\mathrm{j}2\pi f_g \tau_{1K}} \\ \vdots & \vdots & \vdots \\ \mathrm{e}^{-\mathrm{j}2\pi f_g \tau_{n1}} & \cdots \mathrm{e}^{-\mathrm{j}2\pi f_g \tau_{nk}} & \cdots \mathrm{e}^{-\mathrm{j}2\pi f_g \tau_{nK}} \\ \vdots & \vdots & \vdots \\ \mathrm{e}^{-\mathrm{j}2\pi f_g \tau_{N1}} & \cdots \mathrm{e}^{-\mathrm{j}2\pi f_g \tau_{Nk}} & \cdots \mathrm{e}^{-\mathrm{j}2\pi f_g \tau_{NK}} \end{bmatrix}\end{aligned} \tag{3}$$

where $\boldsymbol{a}(f_g, \phi_k, \theta_k)$ is the steering vector of the source from $(\phi_k, \theta_k)(k = 1, 2, \cdots, K)$ at f_g, assume that f_0 is the frequency with the most power and $\boldsymbol{S}(f_g)$ is formed by some spikes [13], then we let

$$\varphi_k(f_0) = \frac{f_0}{c}[1 - (\cos\phi_k \cos\theta_k + \sin\phi_k \cos\theta_k)] \tag{4}$$

so the sparse source $\boldsymbol{S}(f_0)$ can be written

$$\boldsymbol{S}(f_0) = \begin{bmatrix} S_1(f_0) \\ \vdots \\ S_k(f_0) \\ \vdots \\ S_K(f_0) \end{bmatrix} = \begin{bmatrix} \upsilon_1(f_0)\delta_{\varphi_1(f_0)} \\ \vdots \\ \upsilon_k(f_0)\delta_{\varphi_k(f_0)} \\ \vdots \\ \upsilon_K(f_0)\delta_{\varphi_K(f_0)} \end{bmatrix} \tag{5}$$

where $\delta_{\varphi_k(f_0)}$ is the dirac measure at $\varphi_k(f_0)$, let $\{\varphi_1(f_0), \cdots, \varphi_K(f_0)\}$ be the support set of $\boldsymbol{S}(f_0)$, here $\varphi_k(f_0)$ contains DOA of the kth source, $\upsilon_k(f_0)$ is its amplitude.

3 Estimation Theory

Assume that the output $\boldsymbol{Y}(f_0)$ is infinite, given a measure $S(\varphi)$, the corresponding Fourier coefficients is

$$q(n, f_0) = \sum_{k=1}^{K} \exp(-\mathrm{j}2\pi n \varphi_k(f_0))\, \upsilon_k(f_0), \; n = 1, 2, \cdots, N \tag{6}$$

then we have

$$\boldsymbol{Q}(f_0) = \boldsymbol{F}(f_0)\boldsymbol{S}(f_0) \tag{7}$$

where

$$\boldsymbol{Q}(f_0) = [q(1,f_0), q(2,f_0), \cdots, q(N,f_0)]^{\mathrm{T}} \tag{8}$$

and

$$\boldsymbol{F}(f_0) = \begin{bmatrix} \exp(-\mathrm{j}2\pi\varphi_1(f_0)) & \cdots & \exp(-\mathrm{j}2\pi\varphi_K(f_0)) \\ \exp(-\mathrm{j}2\pi \times 2\varphi_1(f_0)) & \cdots & \exp(-\mathrm{j}2\pi \times 2\varphi_K(f_0)) \\ \vdots & \ddots & \vdots \\ \exp(-\mathrm{j}2\pi N\varphi_1(f_0)) & \cdots & \exp(-\mathrm{j}2\pi N\varphi_K(f_0)) \end{bmatrix} \tag{9}$$

We need to solve the following problem so as to recover the original wideband sources

$$\min_{\boldsymbol{S}(f_0)} \|\boldsymbol{S}(f_0)\|_{\mathrm{TV}}, \quad \text{s.t. } \boldsymbol{Q}(f_0) = \boldsymbol{F}(f_0)\boldsymbol{S}(f_0) \tag{10}$$

where $\|\boldsymbol{S}(f_0)\|_{\mathrm{TV}} = \sum_{k=1}^{K} S_k(f_0) = \sum_{k=1}^{K} \upsilon_k(f_0)$, thus we can reconstruct the source $\boldsymbol{S}(f_0)$ if the interval between $\varphi_\alpha(f_0)$ and $\varphi_\beta(f_0)$ is larger than $2/f_0$ for $1 \le \alpha, \beta \le N, \alpha \ne \beta$; $k = 1, \cdots, K$ [12].

Assume that sampling number at each frequency is Z, Eq. (2) is changed as

$$\bar{\boldsymbol{Y}}(f_0) = \boldsymbol{A}(f_0)\bar{\boldsymbol{S}}(f_0) + \bar{\boldsymbol{B}}(f_0) \tag{11}$$

that is

$$\bar{\boldsymbol{Y}}(f_0) = [\boldsymbol{Y}(f_0, 1), \cdots, \boldsymbol{Y}(f_0, z), \cdots, \boldsymbol{Y}(f_0, Z)] \tag{12}$$

$\boldsymbol{Y}(f_0, z)$ is the zth snapshots of f_0, $\bar{\boldsymbol{S}}(f_0)$ and $\bar{\boldsymbol{B}}(f_0)$ are respectively the source and noise matrix. It can be deduced from (11)

$$\bar{\boldsymbol{Y}}(f_0) - \bar{\boldsymbol{B}}(f_0) = \boldsymbol{A}(f_0)\bar{\boldsymbol{S}}(f_0) = \boldsymbol{A}(f_0)\boldsymbol{S}(f_0) + \boldsymbol{D}(f_0) \tag{13}$$

Obviously, $\boldsymbol{D}(f_0)$ is the corresponding perturbation, it reflects the error between infinite and finite received data. Combining (13), we can deduce the Fourier coefficients of finite samples

$$\begin{aligned}
&q(n,f_0)\\
&= \exp\left(-\mathrm{j}2\pi n\frac{f_0}{c}\right)\left(\bar{\mathbf{Y}}_n(f_0)-\bar{\boldsymbol{B}}_n(f_0)\right)\\
&= \exp\left(-\mathrm{j}2\pi n\frac{f_0}{c}\right)\left(\sum_{k=1}^{K}\mathrm{e}^{\mathrm{j}2\pi n\frac{f_0}{c}(\cos\phi_k\cos\theta_k+\sin\phi_k\cos\theta_k)}\upsilon_k(f_0)+\boldsymbol{D}(n,f_0)\right)\\
&= \sum_{k=1}^{K}\mathrm{e}^{-\mathrm{j}2\pi n\frac{f_0}{c}(1-(\cos\phi_k\cos\theta_k+\sin\phi_k\cos\theta_k))}\upsilon_k(f_0)+\exp\left(-\mathrm{j}2\pi n\frac{f_0}{c}\right)\boldsymbol{D}(n,f_0)\\
&= \sum_{k=1}^{K}\exp\left(-\mathrm{j}2\pi n\varphi_k(f_0)\right)\upsilon_k(f_0)+\boldsymbol{\omega}(n,f_0)
\end{aligned} \tag{14}$$

where $\boldsymbol{\omega}(n,f_0)=\exp\left(-\mathrm{j}2\pi n\frac{f_0}{c}\right)\boldsymbol{D}(n,f_0)$, so (14) can be modified as

$$\boldsymbol{Q}(f_0)=\boldsymbol{F}(f_0)\boldsymbol{S}(f_0)+\boldsymbol{\omega}(f_0) \tag{15}$$

here $\boldsymbol{\omega}(f_0)=[\boldsymbol{\omega}(1,f_0),\cdots,\boldsymbol{\omega}(N,f_0)]^{\mathrm{T}}$. Similarly, we can also solve the following problem so as to recover the original sources

$$\min_{\boldsymbol{S}(f_0)}\|\boldsymbol{S}(f_0)\|_{\mathrm{TV}}\ \text{s.t.}\ \|\boldsymbol{Q}(f_0)-\boldsymbol{F}(f_0)\boldsymbol{S}(f_0)\|_2\le|\varsigma(f_0)| \tag{16}$$

The question (16) is a multiple convex problem and difficult to be disposed, so we need to simplify it by corresponding dual problem [12]

$$\begin{aligned}
&\max_{\boldsymbol{\Phi}(f_0),\mathbf{U}}\left(\mathrm{Re}[\boldsymbol{Q}^*(f_0)\boldsymbol{\Phi}(f_0)]-\varsigma(f_0)\|\boldsymbol{\Phi}(f_0)\|_2\right)\ \text{s.t.}\\
&\begin{bmatrix}\boldsymbol{U} & \boldsymbol{\Phi}(f_0)\\ \boldsymbol{\Phi}^*(f_0) & 1\end{bmatrix}\succ,\ 0\ \|\boldsymbol{F}^*(f_0)\boldsymbol{\Phi}(f_0)\|_{L\infty}\le 1
\end{aligned} \tag{17}$$

here $\sum_{\alpha=1}^{N-\beta}\boldsymbol{Z}_{\alpha,\alpha+\beta}=\begin{cases}1, & \beta=0\\ 0, & \beta=1,2,\cdots,N-1\end{cases}$, $\boldsymbol{U}\in C^{N\times N}$ is a Hermitian matrix, and $\boldsymbol{\Phi}(f_0)$ is the corresponding Lagrangian multiplier for $\boldsymbol{Q}(f_0)=\boldsymbol{F}(f_0)\boldsymbol{S}(f_0)+\boldsymbol{\omega}(f_0)$, we can obtain the parameter according to the semidefinite program [14], which can be solved by the tool in [15].

The following lemma [13] can be used for describing the relation of (16) and (17)

$$\left(\hat{\boldsymbol{F}}^*\hat{\boldsymbol{\Phi}}\right)(f_0)=\mathrm{sign}\left(\left\|\hat{\boldsymbol{S}}(f_0)\right\|_{\mathrm{TV}}\right) \tag{18}$$

where $\left\|\hat{\boldsymbol{S}}(f_0)\right\|_{\mathrm{TV}}\neq 0$, $\hat{\boldsymbol{F}}(f_0)$, $\hat{\boldsymbol{\Phi}}(f_0)$ and $\hat{\boldsymbol{S}}(f_0)$ are respectively the estimated vector of $\boldsymbol{F}(f_0)$, $\boldsymbol{\Phi}(f_0)$ and $\boldsymbol{S}(f_0)$.

As $\left\|\hat{\boldsymbol{S}}(f_0)\right\|_{\mathrm{TV}} \neq 0$, we can solve absolute value of (18)

$$\left|\hat{\boldsymbol{F}}^*(f_0)\hat{\boldsymbol{\Phi}}(f_0)\right| = 1 \tag{19}$$

thus, the DOAs of the sources can be acquired by combining (4) and (9), then the sources will also be reconstructed by (5). The proposed sparse reconstruction method is implemented in continuous domain, so it can be abbreviated to SCD method.

4 Simulations

Next, several simulations is shown, the center frequency of the sources is 3 GHz, the sensors are places at (0, 0), (−0.15, 0.17), (−0.051, 0.079), (−0.18, 0.063), (−0.068, −0.041), (0.059, 0.21), (0.07, 0.31), (0.041, −0.039), unit is meter. Two-sided correlation transformation (TCT) [16], conventional sparse methods in discrete domain (SDD) [9] and SCD are compared for the simulations, $\varsigma(f_0)$ in SCD is taken as 2. The DOA grids of SDD and searching step size of TCT are both taken as 0.2°.

4.1 Normalization Spectrum

Assume that four far-field wideband sources impinge on the array with same power from $(20.5°, 80.5°)$, $(30.5°, 70.5°)$, $(40.5°, 60.5°)$,$(50.5°, 50.5°)$, SNR is 3 dB, sampling number at every frequency is 60, the width of the band is 20% of the center frequency, normalization spectrums of the three methods are given in Figs. 2, 3 and 4.

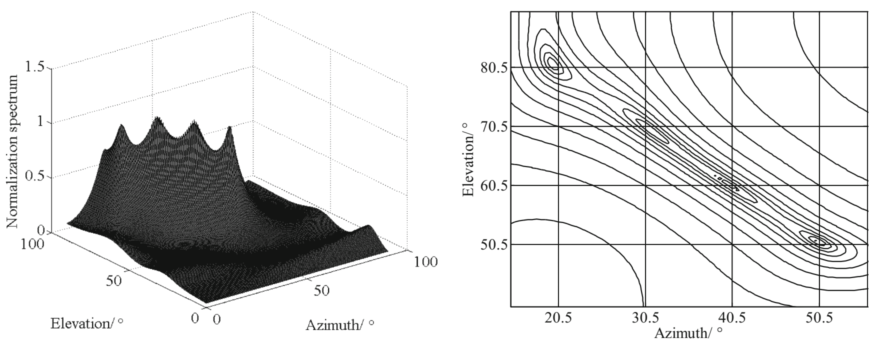

Fig. 2. Normalization spectrum of TCT

4.2 Estimation Error

Figure 5 has shown the estimation error versus SNR when sampling times of each frequency is 60, 400 Monte-Carlo simulations have run for each SNR, as is shown in Fig. 5, the estimation error of SCD is lower than the other two methods.

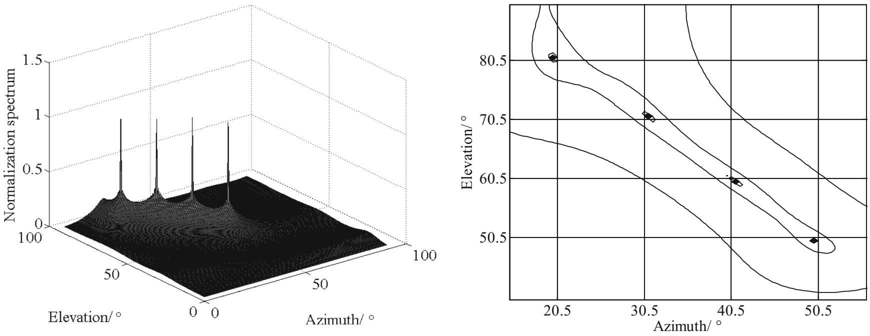

Fig. 3. Normalization spectrum of SDD

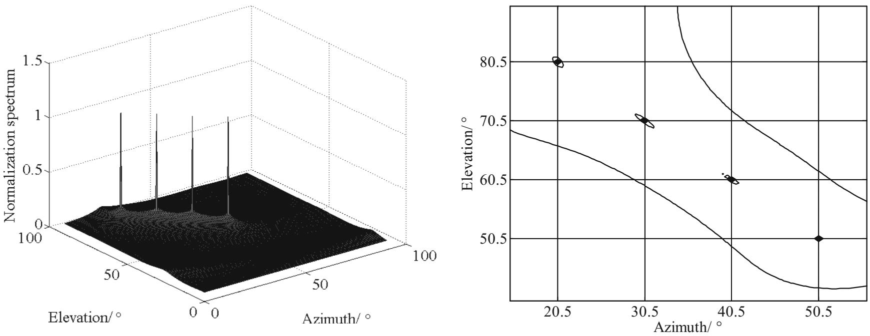

Fig. 4. Normalization spectrum of SCD method

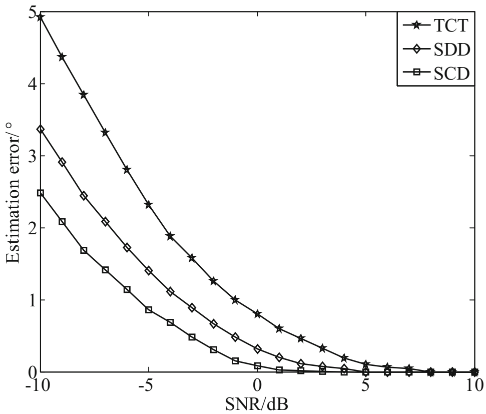

Fig. 5. Estimation error versus SNR

Figure 6 has shown the estimation error versus sampling times of each frequency when SNR is 2 dB, as is shown in Fig. 6, we can estimate the DOAs more accurately than TCT and SDD.

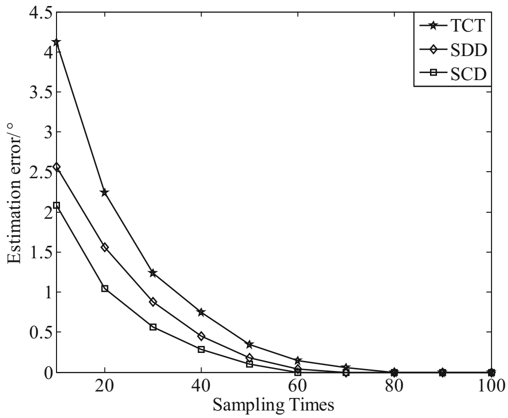

Fig. 6. Estimation error versus sampling times

5 Conclusion

This paper presents a new kind of spatial spectrum estimation for wideband sources by sparse reconstruction in continuous domain, the sources are partitioned into some subbands, then the convex and corresponding dual problems of the data with most power are founded and solved. The sparse support sets are decided by semidefinite program and extracting roots. Both of the DOA and the primary signals are determined. The proposed algorithm averts the error created by sparse reconstruction based on grid partition, and it has a preferable performance under the circumstance of low SNR and small samples. As the process of the optimization, we still have a great amount of computation, how to lower the calculation to improve the efficiency is worthy of going on researching.

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