# Reliability Evaluation of DCell Networks

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**Abstract.** Recently, the reliability evaluation of data center network (DCN) is important to the design and operation of DCNs. Extra connectivity determination and faulty networks structure analysis are two significant aspects for the reliability evaluation of DCNs. The DCell network is suitable for a massive data centers with high network capacity by only using cheap switches. A $k$-dimensional DCell built with $n$-port switches, denoted by $D_{k,n}$, is an $(n+k-1)$-regular graph. In this paper, we firstly prove that the extra-$h$ connectivity of $D_{k,n}$ for $n \geq 2$, $\kappa_h(D_{k,n}) = (k-1)(h+1)+n$ if $k \geq 2$ and $0 \leq h \leq n-1$, and $\kappa_h(D_{k,n}) = (k-1)(h+1)+2n-2$ if $k \geq n+1$ and $n \leq h \leq 2n-1$, respectively. What's more, for any faulty node set $F \subseteq V(D_{k,n})$ with $|F| \leq \kappa_h(D_{k,n}) - 1$, we obtain that there contains a large connected component in $D_{k,n} - F$, and the rest of small connected components have not more than $h$ nodes in total if $k \geq 2$ and $0 \leq h \leq n-1$ (resp. $k \geq n+1$ and $n \leq h \leq 2n-1$). Our result can provide a proper measure for the reliability evaluation of the DCell network when it is used to model the topological structure of a large-scale DCN.

**Keywords:** DCell network · Reliability · Extra connectivity
Data center network

## 1 Introduction

With the development of web applications such as email, online search, web game, cloud video, and productivity components such as Map reduce [1] and GFS [2], huge data center network (DCN) with millions of servers will become available in some day. Microsoft implied that Azure, Hotmail, Bing, and some other web services will be storaged by a million servers [3], for instance. With the remorselessly rising in the scale of DCN, the complexity of a DCN can disadvantageously impact its reliability. In order to design and operation of a DCN, proper measures of reliability ought to be sought out. A DCN can be modeled by a simple connected-graph $G = (V(G), E(G))$, where $V(G)$ denotes


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the node set with each node denotes a server, and $E(G)$ denotes the edge set with each edge denotes a link between servers, respectively. What's more, switches in a DCN can be identified as transparent devices of network [4]. Therefore, we can measure the reliability of a DCN (network for short) by using the graph parameters of its DCN.

The connectivity of a DCN as a traditional measure for the reliability of DCNs, is the minimum number of nodes eliminated to obtain the graph is disconnected or trivial, which is a worst case. In fact, this measure can accurately reflect the reliability of a small size DCN. However, some DCNs with large size have shown to can tolerate much more server failures while still keep connected. In other words, as one of the measures of reliability, the traditional connectivity would underestimates the ability of reliability of these large DCNs [5].

To counteract the weakness of the connectivity of a simple graph, Harary [6] proposed the definition of the restricted faulty nodes of a graph. Furthermore, Fabrega and Fiol [7,8] introduced the concept of extra connectivity and obtained the extra connectivity of graphs. Given a graph $G$ and a node cut $F \in V(G)$, if each connected component of $G - F$ has not less than $h + 1$ nodes, then $F$ is defined an extra-$h$ node cut. The extra-$h$ connectivity of $G$ is the minimum cardinality of all extra-$h$ node cuts (if exists), can be denoted by $\kappa_h(G)$. In a DCN, the status of the node has meaningless impact on the capability of the rest of graph when all the neighbors of a node are faulty. Thus, it is reasonable of the assumption that there is no isolated node on $G-F$, when we assume it is faulty. What's more, the structure study of an incomplete DCN with a large amount of faulty nodes is closely related to the extra connectivity of a DCN. The large connected component can be used to execute the operation of the DCN not have much capability degrade, when a disconnected DCN with massive faulty nodes contains a large connected component. Therefore, the extra connectivity is great important to the reliability of DCNs [5].

Since a complete graph $K_n$ is nonseparable, $\kappa_h(K_n)$ does not exist with $0 \leq h \leq n - 1$. Furthermore, if $G$ is not a complete graph, then $\kappa_0(G) = \kappa(G)$. Given a nonnegative integer $h$ and graph $G$, it is quite difficult to calculate $\kappa_h(G)$. As a matter of fact, the existence of $\kappa_h(G)$ is still an open problem so far when $h \geq 1$. Only a little research achievements have been obtained on $\kappa_h(G)$ in some particular graphs [5,9–16]. For example, Zhu et al. [9] and Gu and Hao [10] showed that $\kappa_2(Q_n^3) = 6n - 7$, $\kappa_3(Q_n^k) = 8n - 12$ for $n \geq 3$, where $Q_n^3$ is the 3-ary $n$-cube, respectively. Lin et al. obtained that for the $n$-dimensional alternating group graph $AG_n$, $\kappa_1(AG_n) = 4n - 11$, $\kappa_2(AG_n) = 6n - 19$, and $\kappa_3(AG_n) = 8n - 28$ for $n \geq 5$ [11]. For any integer $n \geq 6$, Chang et al. proved that the 3-extra connectivity of an $n$-dimensional folded hypercube is $4n-5$ [12]. For any integers $n \geq 4$ and $0 \leq h \leq n - 4$, Zhu et al. [5] showed that $\kappa_h(X_n) = n(h+1) - \frac{1}{2}h(h+3)$, where $X_n$ is the $n$-dimensional bijective connection network. Furthermore, Yang and Lin studied a sharp lower bound of extra-$h$ connectivity of $X_n$ which improves the result in [5] for $n \geq 4$ and $0 \leq h \leq 2n - 1$ [13].

Recently, Guo et al. introduced a server-centric DCN named DCell [4], which have many advantages over traditional tree-based DCN, such as fault-tolerance, scalability, reliability, low cost, and so on. What's more, DCell originated substitutive design considered the server-centric DCNs, and inspired a lot of novel DCN structures such as FiConn [17], BCube [18], and CamCube [19]. Some combinatorial properties of a $k$-dimensional DCell built from $n$-port switches, $D_{k,n}$, such as diameter [4], symmetry [20], broadcasting [4], connectivity [4], restricted connectivity [21], node disjoint paths [22], one to one disjoint path covers [23], and Hamiltonian properties [24] have recently been studied. Particulary, these measurement results indicate that a $D_{k,n}$ has excellent combinatorial properties.

In this paper, we have obtained the extra-$h$ connectivity of $D_{k,n}$ for $n \geq 2$, $\kappa_h(D_{k,n}) = (k-1)(h+1)+n$ when $k \geq 2$ and $0 \leq h \leq n-1$ (resp. $\kappa_h(D_{k,n}) = (k-1)(h+1)+2n-2$ when $k \geq n+1$ and $n \leq h \leq 2n-1$). What's more, we explore that there contains a large connected component in $D_{k,n} - F$, and the rest of small connected components have not more than $h$ nodes in total if $|F| < \kappa_h(D_{k,n})$ for any two integers $k \geq 2$ and $0 \leq h \leq n-1$ (resp. $k \geq n+1$ and $n \leq h \leq 2n-1$).

This paper is organized in this way: We provide some definitions and preliminaries in Sect. 2. In Sect. 3, the extra-$h$ connectivity of DCells are given. In the end, we conclude this paper in Sect. 4.

## 2 Preliminaries

We use $G$ to denote a DCN. The node number of $G$ is called the order of $G$. An edge of $G$ with two end nodes $u, v$ is denoted by $(u, v)$. For any node $v \in V(G)$, let $u$ be a neighbor of the node $v$ or $u$ is adjacent to the node $v$ if $(u, v) \in E(G)$. If $V' \subseteq V(G)$, let $G[V']$ denote the sub-graph of $G$ induced by a node subset $V' \in V(G)$ and let $G - V' = G[V(G) \setminus V']$. Then, let $N_G(V')$ denote the neighbor-set of $V'$ such that $N_G(V') \in V(G-V')$ and let $A_G(V') = V' \cup N_G(V')$.

For $k \geq 0$ and $n \geq 2$, let $D_{k,n}$ denote a $k$-dimensional DCell built on $n$-port switches. Then, we use $t_{k,n}$ to denote the order in $D_{k,n}$ with $t_{0,n} = n$ and $t_{i,n} = t_{i-1,n}(t_{i-1,n}+1)$ for $n \geq 2$, $k \geq 0$, and $i \in \{1, 2, \ldots, k\}$. Let $I_{0,n} = \{0, 1, \ldots, n-1\}$ and $I_{i,n} = \{0, 1, \ldots, t_{i-1,n}\}$ with $i \in \{1, 2, \ldots, k\}$. For any integer $1 \leq l \leq k$, let $V_{k,n}^l = \{u_k u_{k-1} \cdots u_l : u_i \in I_{i,n}$ and $i \in \{l, l+1, \ldots, k\}\}$. The definition of DCell $D_{k,n}$ is adopt from [4].

**Definition 1.** *$D_{k,n}$ is a regular graph with node set $V_{k,n}^0$, where a node $u = u_k u_{k-1} \cdots u_0$ is adjacent to a node $v = v_k v_{k-1} \cdots v_0$ if and only if there exists an integer $l$ with*

*(1) $u_k u_{k-1} \cdots u_l = v_k v_{k-1} \cdots v_l$,*

*(2) $u_{l-1} \neq v_{l-1}$,*

*(3) $u_{l-1} = v_0 + \sum_{j=1}^{l-2} (v_j \times t_{j-1,n})$ and $v_{l-1} = u_0 + \sum_{j=1}^{l-2} (u_j \times t_{j-1,n}) + 1$ with $l > 1$.*

Figure 1 shows the examples of $D_{k,n}$ with some small $n$ and $k$. It is clear that $D_{k,n}$ is a $(n+k-1)$-regular graph with $t_{k,n}$ nodes. When all the three conditions of Definition 1 hold, we define that two neighbor nodes $u, v$ have a differing bit

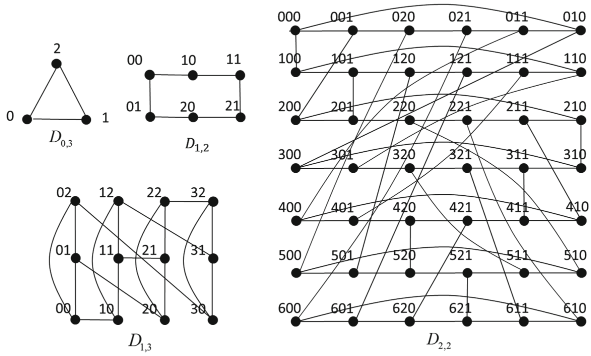

**Fig. 1.** Examples of $D_{k,n}$: $D_{0,3}$, $D_{1,3}$, $D_{1,2}$, and $D_{2,2}$.

of leftmost at position $l-1$, denoted by $d$, the $d$-neighbor of $u$ can be denoted by $(u)^d = v$ for $d \geq 1$. Usually, if $u = u_k u_{k-1} \cdots u_0$ is a node in $D_{k,n}$, let $(u)_i$ denote the $i$-th bit of $u$, and let $\sigma(u,i) = u_k u_{k-1} \cdots u_i$ for any $0 \leq i \leq k$. Clearly, $\sigma(u,0) = u$. For any $\alpha \in V^l_{k,n}$ with $1 \leq l \leq k$, let $D^\alpha_{l-1,n}$ denote the graph that attained by adding the prefix $\alpha$ to address of every node of one copy of $D_{l-1,n}$. Clearly, $D_{l-1,n} \cong D^\alpha_{l-1,n}$.

In this paper, a set of nodes to be deleted will be denoted as $F$. Define $F_i = F \cap V(D^i_{k-1,n})$ and $I = \{i : |F_i| \geq n+k-2\}$ for each $i \in I_{k,n}$. Furthermore, let $F_I = \bigcup_{i \in I} F_i$, $\bar{I} = I_{k,n} \setminus I$, $D^{\bar{I}}_{k-1,n} = D_{k,n}[\bigcup_{i \in \bar{I}} V(D^i_{k-1,n})]$, and $F_{\bar{I}} = \bigcup_{i \in \bar{I}} F_i$. These notations will be used throughout the paper.

The following studied results in DCells are helpful in our paper and thus showed as follows.

**Lemma 1** [4]. *The connectivity of $D_{k,n}$ is $\kappa(D_{k,n}) = n+k-1$.*

**Lemma 2** [4]. *The order of $D_{k,n}$ satisfies $t_{k,n} \geq (n+\frac{1}{2})^{2^k} - \frac{1}{2}$.*

**Lemma 3** [21]. *There exist $t_{k-1,n}$ node disjoint paths joining $D^i_{k-1,n}$ and $D^j_{k-1,n}$ with $i \neq j$.*

**Lemma 4** [21]. *Let $F \subset V(D_{k,n})$ denote a faulty node set with $|F| \leq (h+1)(k-1)+n$. For any three integers $n \geq 2$, $k \geq 2$, $0 \leq h \leq n-1$, $D^{\bar{I}}_{k-1,n} - F_{\bar{I}}$ is connected and $|I| \leq h+1$.*

**Lemma 5** [21]. *For any $n \geq 2$, $k \geq 2$, and any $H_0 \subseteq V(D^\alpha_{0,n})$ and $H_1 \subseteq V(D^\beta_{0,n})$ such that $\alpha, \beta \in V^1_{k,n}$ and $\alpha \neq \beta$, we have $|N_{D_{k,n}}(H_0) \cap H_1| \leq 1$.*

## 3 The Extra-$h$ Connectivity of DCells

In fact, the extra-$h$ connectivity for $h = 0$ on $D_{k,n}$ was gotten by Guo et al. [4] for any nonnegative integers $n \geq 2$ and $k \geq 0$. Nevertheless, the extra-$h$

connectivity for $h \geq 1$ of $D_{k,n}$ has not been obtained yet. In this section, for any integer $n \geq 2$, the extra-$h$ connectivity when $0 \leq h \leq n-1$ and $k \geq 2$, when $n \leq h \leq 2n-1$ and $k \geq n+1$ of $D_{k,n}$ will be studied, respectively.

**Lemma 6.** *Given an nonnegative integer* $n \geq 2$, *let* $f_n(m) = mn - \frac{m(m-1)}{2}$ *be a function of* $m$, $f_n(m) = mn - \frac{m(m-1)}{2}$ *is strictly monotonically increasing on* $m$ *if* $1 \leq m \leq n$.

*Proof.* If $1 \leq m \leq n$, we can verify that

$$\frac{df}{dm} = n - \frac{1}{2}(2m-1) = n - m + \frac{1}{2} > 0.$$

Thus, for any nonnegative integers $m'$ and $m$ such that $1 \leq m' < m \leq n$, we have $f_n(m') < f_n(m)$.

**Lemma 7.** *For any three integers* $k \geq 2$, $n \geq 2$, *and* $0 \leq h \leq n-1$, *let* $H \subseteq V(D_{k,n})$ *with* $|H| = h+1$. *Then, we have* $|N_{D_{k,n}}(H)| \geq (k-1)(h+1)+n$.

*Proof.* Let $S = \{\sigma(u,1) : u \in H\} = \{s_1, s_2, \ldots, s_m\}$ with $1 \leq m \leq h+1$. For any $1 \leq i \leq m$, let $H_i = V(D_{0,n}^{s_i}) \cap H$ and $h_i = |H_i|$. Obviously, $\sum_{i=1}^{m} h_i = |H| = h+1$. Definition 1 and Lemma 5 implies that any node in $H$ has exactly $k$ neighbor(s) in $D_{k,n} - V(D_{0,n}^{s_i})$, $H$ has at most $\frac{m(m-1)}{2}$ common neighbor(s) in $D_{k,n} - V(D_{0,n}^{s_i})$, and $H_i$ has exactly $n - h_i$ neighbor(s) in $D_{0,n}^{s_i}$ for any $1 \leq i \leq m$. Thus, we have

$$\begin{aligned} |N_{D_{k,n}}(H)| &\geq k(h+1) - \frac{m(m-1)}{2} + \sum_{1 \leq i \leq m} (n - h_i) \\ &= (k-1)(h+1) + mn - \frac{m(m-1)}{2}. \end{aligned}$$

For any $m$ with $1 \leq m \leq h+1 \leq n$, $mn - \frac{m(m-1)}{2} \geq n$ by Lemma 6. Then, we have

$$|N_{D_{k,n}}(H)| \geq (k-1)(h+1) + mn - \frac{m(m-1)}{2} \geq (k-1)(h+1) + n.$$

**Lemma 8.** *For any three integers* $k \geq 2$, $n \geq 2$, *and* $0 \leq h \leq n-1$, *and any node set* $F \subset V(D_{k,n})$ *with* $|F| \leq (k-1)(h+1)+n-1$, $D_{k,n} - F$ *contains a large connected component including not less than* $t_{k,n} - |F| - h$ *nodes.*

*Proof.* In this lemma, we will prove that by the induction on the integer $h$. If $h = 0$, $D_{k,n} - F$ is connected since $|F| \leq n+k-2 < n+k-1 = \kappa(D_{k,n})$, the result holds. Suppose that the result is correct when $h = \tau - 1$ with $n-1 \geq \tau \geq 1$. Then, we will show that it is correct for $h = \tau$ $(1 \leq \tau \leq n-1)$. Assume that $H_1, H_2, \ldots, H_m, H_{m+1}$ are total the components of $D_{k,n} - F$, and $|V(H_{m+1})|$ =max$\{|V(H_1)|, |V(H_2)|, \ldots, |V(H_{m+1})|\}$. By Lemma 4, $D_{k-1,n}^{\bar{I}} - F_{\bar{I}}$ is connected. So $V(D_{k-1,n}^{\bar{I}} - F_{\bar{I}}) \subseteq V(H_{m+1})$. Let $r = |I| \leq \tau + 1$ and

$I = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$. The lemma holds for $r = 0$ since $D_{k,n} - F = D^{\bar{I}}_{k-1,n} - F_{\bar{I}}$ is connected by Lemma 4. To complete the proof, if $1 \le r \le \tau + 1$, we consider the following three cases.

Case 1. $D^{\alpha_i}_{k-1,n} - F_{\alpha_i}$ is connected for any $1 \le i \le r$.

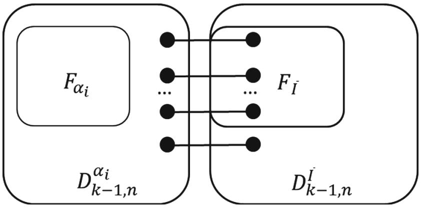


**Fig. 2.** An illustration for Case 1 in Lemma 8.

By Definition 1, each node in $D^{i}_{k-1,n}$ for $i \in I$ has accurately one neighbor in $D_{k,n} - V(D^{i}_{k-1,n})$. For any three integers $k \ge 2$, $n \ge 2$, $1 \le \tau \le n-1$, and $\alpha_i \in I$, we have

$$\begin{aligned}|N_{D_{k,n}-V(D^{\alpha_i}_{k-1,n})}(V(D^{\alpha_i}_{k-1,n}) \setminus F_{\alpha_i})| &= |V(D^{\alpha_i}_{k-1,n})| - |F_{\alpha_i}| = t_{k-1,n} - |F_{\alpha_i}| \\ &> (k-1)(\tau+1) + n - 1 - |F_{\alpha_i}| \ge |F| - |F_{\alpha_i}| \\ &= |F_{\bar{I}}| + (|F_I| - |F_{\alpha_i}|).\end{aligned}$$

Thus, we can verify that there exists at least one node of $D^{\alpha_i}_{k-1,n} - F_{\alpha_i}$ to be adjacent to a node in $D^{\bar{I}}_{k-1,n} - F_{\bar{I}}$ for any $\alpha_i \in I$ (see Fig. 2). As a result, $D_{k,n} - F$ is connected. Hence, $D_{k,n} - F$ has a connected component including at least $t_{k,n} - |F| - \tau$ nodes, when $1 \le \tau \le n-1$.

Case 2. Exactly one subgraph of $D^{\alpha_1}_{k-1,n} - F_{\alpha_1}, D^{\alpha_2}_{k-1,n} - F_{\alpha_2}, \ldots, D^{\alpha_r}_{k-1,n} - F_{\alpha_r}$ is disconnected.

Let $D^{\alpha_\lambda}_{k-1,n}$ be disconnected such that $1 \le \lambda \le r$. According to the Case 1, we can verify that $D_{k,n} - V(D^{\alpha_\lambda}_{k-1,n}) - (F \setminus F_{\alpha_\lambda})$ is connected. So, $V(D_{k,n} - V(D^{\alpha_\lambda}_{k-1,n}) - (F \setminus F_{\alpha_\lambda})) \subseteq V(H_{m+1})$. Then, we have $\bigcup_{i=1}^{m} V(H_i) \subseteq V(D^{\alpha_\lambda}_{k-1,n} - F_{\alpha_\lambda})$. What's more, we will show that the order in $\bigcup_{i=1}^{m} V(H_i)$ will not larger than $\tau - 1$. Suppose that the sum orders in $\bigcup_{i=1}^{m} V(H_i)$ is at least $\tau$. By Lemma 7, we obtain $|N_{D_{k,n}}(\bigcup_{i=1}^{m} V(H_i))| \ge (k-1)(\tau+1) + n > |F|$, a contraction. Thus, $|\bigcup_{i=1}^{m} V(H_i)| \le \tau - 1$. Hence, $D_{k,n} - F$ has a connected component including at least $t_{k,n} - |F| - \tau$ nodes, where $1 \le \tau \le n-1$.

Case 3. Exactly $r'$ subgraphs of $D^{\alpha_1}_{k-1,n} - F_{\alpha_1}, D^{\alpha_2}_{k-1,n} - F_{\alpha_2}, \ldots, D^{\alpha_r}_{k-1,n} - F_{\alpha_r}$ are disconnected, where $2 \le r' \le r$.

Let $\{q_1, q_2, \ldots, q_{r'}\} \subseteq \{1, 2, \ldots, r\}$ such that $D^{\alpha_{q_i}}_{k-1,n}$ is disconnected for any $1 \le i \le r'$. According to the proof of Case 2, we can verify that $D_{k,n} - \bigcup_{i=1}^{r'} V(D^{\alpha_{q_i}}_{k-1,n}) - (F \setminus \bigcup_{i=1}^{r'} F_{\alpha_{q_i}})$ is connected. So, $V(D_{k,n} - \bigcup_{i=1}^{r'} V(D^{\alpha_{q_i}}_{k-1,n}) - (F \setminus \bigcup_{i=1}^{r'} F_{\alpha_{q_i}})) \subseteq V(H_{m+1})$. For any integers $1 \le \tau \le n-1$ and $1 \le i \le r'$, we have

$$\begin{aligned}|F_{q_i}| \leq |F| - \sum_{1\leq j\leq r', j\neq i} |F_{q_j}| &\leq (k-1)(\tau+1)+n-1-(r'-1)(n+k-2) \\ &\leq (k-1)(\tau+1)+n-1-(k-1+\tau) \\ &= (k-2)\tau+n-1\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}|F \setminus \bigcup_{i=1}^{r'} F_{\alpha_{q_i}}| = |F| - \sum_{j=1}^{r'} |F_{q_j}| \\ &\leq (k-1)(\tau+1)+n-1-r'(n+k-2) \\ &\leq (k-1)(\tau-1)-(n-1).\end{aligned} \tag{3.2}$$

By the induction hypothesis, $D_{k-1,n}^{\alpha_{q_i}} - F_{\alpha_{q_i}}$ contains a large component $A_{\alpha_{q_i}}$ including not less than $t_{k-1,n} - |F_{\alpha_{q_i}}| - (\tau-1)$ nodes if $1 \leq i \leq r'$. For any four integers $n \geq 2$, $k \geq 2$, $1 \leq \tau \leq n-1$, and $1 \leq i \leq r'$, we can verify

$$\begin{aligned}|V(A_{\alpha_{q_i}})| &\geq t_{k-1,n} - |F_{\alpha_{q_i}}| - (\tau-1) \\ &\geq (n+\frac{1}{2})^{2^{k-1}} - \frac{1}{2} - ((k-2)\tau+n-1) - (\tau-1) \\ &\geq 2(k-1)n - (k-1)\tau - n \geq (k-1)n \\ &> (k-1)(\tau-1)-(n-1) \\ &\geq |F \setminus \bigcup_{i=1}^{r'} F_{\alpha_{q_i}}|\end{aligned}$$

by (3.1) and (3.2). Therefore, for any nonnegative integer $1 \leq i \leq r'$, $A_{\alpha_{q_i}}$ is connected to $D_{k,n} - \bigcup_{i=1}^{r'} V(D_{k-1,n}^{\alpha_{q_i}}) - (F \setminus \bigcup_{i=1}^{r'} F_{\alpha_\lambda})$ in $D_{k,n} - F$. That is, we have $V(A_{\alpha_{q_i}}) \subset V(H_{m+1})$. Let $V_{\alpha_{q_i}} = \bigcup_{i=1}^{m}(V(H_i) \cap V(D_{k-1,n}^{\alpha_{q_i}}))$ for any nonnegative integer $1 \leq i \leq r'$. Then, we will prove the order of $\bigcup_{i=1}^{r'} V_{\alpha_{q_i}}$ will not larger than $\tau - 1$. Furthermore, assume that the total order of $\bigcup_{i=1}^{r'} V_{\alpha_{q_i}}$ is at least $\tau$. By Lemma 7, we have $|N_{D_{k,n}}(\bigcup_{i=1}^{r'} V_{\alpha_{q_i}})| \geq (k-1)(\tau+1)+n > |F|$, a contraction. Thus, $|\bigcup_{i=1}^{r'} V_{\alpha_{q_i}}| \leq \tau - 1$. Hence, $D_{k,n} - F$ has a connected component including at least $t_{k,n} - |F| - \tau$ nodes, where $1 \leq \tau \leq n-1$.

To sum up, the lemma holds for $h = \tau$. So far, the discussion of the lemma is complete.

**Theorem 1.** *Given any three integers $n \geq 2$, $0 \leq h \leq n-1$, and $k \geq 2$, the extra-$h$ connectivity of $D_{k,n}$ is $\kappa_h(D_{k,n}) = (k-1)(h+1)+n$.*

*Proof.* Let $H$ be a induced subgraph of the number of nodes is $h+1$ in $D_{0,n}^{\alpha}$ with $\alpha \in V_{k,n}^1$. Let $V' = V(H)$ and $F = N_{D_{k,n}}(V')$, obviously, $D_{k,n} - F$ is disconnected. Definition 1 implies that any node in $V'$ has exactly $k$ neighbors in $D_{k,n} - V(D_{0,n}^{\alpha})$ and $V'$ has accurately $n-(h+1)$ neighbors in $D_{0,n}^{\alpha} - V'$.

Thus, we have $|F| = (h+1)k + n - (h+1) = (h+1)(k-1) + n$. Furthermore, we will prove that the node set $F$ is an extra-$h$ node cut on $D_{k,n}$. Let $u \in V'$ and $\beta = (u)_k$. By Lemma 4, we can verify that $D_{k,n} - (V(D^{\beta}_{k-1,n}) \cup F)$ is connected. By Definition 1, each node in $D^{\beta}_{k-1,n} - A_{D^{\beta}_{k-1,n}}(V')$ has accurately one neighbor in $D_{k,n} - (V(D^{\beta}_{k-1,n}) \cup F)$. Therefore, $D_{k,n} - F$ contains two components, one of the components is $D_{k,n} - A_{D_{k,n}}(V')$ and the other of the components is $H$. Accordingly, for any three integers $n \geq 2$, $k \geq 2$, and $0 \leq h \leq n-1$, we have

$$\begin{aligned}|V(D_{k,n} - A_{D_{k,n}}(V'))| &\geq t_{k,n} - (|V'| + |F|) \geq (n + \frac{1}{2})^{2^k} - \frac{1}{2} - ((h+1)k + n)\\ &\geq (k+2)n - ((h+1)k + n) \geq n\\ &\geq h+1.\end{aligned}$$

Then, $F$ is an extra-$h$ node cut of $D_{k,n}$, and thus $\kappa_h(D_{k,n}) \leq (k-1)(h+1) + n$ for three integers $k \geq 2$, $n \geq 2$, and $0 \leq h \leq n-1$.

Nevertheless, given three integers $n \geq 2$, $k \geq 2$, and $0 \leq h \leq n-1$, if the number of nodes of each component of $D_{k,n} - F$ is not more than $h+1$ with $F \subseteq V(D_{k,n})$, then $|F| \geq (k-1)(h+1) + n$ by Lemma 8. So, $\kappa_h(D_{k,n}) \geq (k-1)(h+1) + n$.

Hence, $\kappa_h(D_{k,n}) = (k-1)(h+1) + n$ for the three integers $n \geq 2$, $k \geq 2$, and $0 \leq h \leq n-1$.

**Lemma 9.** *For any three nonnegative integers $n \geq 2$, $n \leq h \leq 2n-1$, and $k \geq n+1$, and any node sub-set $F \subset V(D_{k,n})$, if $|F| \leq (k-1)(h+1) + 2n - 2$, then, $D^{\bar{I}}_{k-1,n} - F_{\bar{I}}$ is connected and $|I| \leq h+1$.*

*Proof.* In the beginning, we prove that $|I| \leq h+1$. Assume that $|I| \geq h+2$, according to definition of $I$, for any three nonnegative $n \geq 2$, $n \leq h \leq 2n-1$, and $k \geq n+1$, we have

$$\begin{aligned}|F| &\geq (h+2)(n+k-2) \geq (k-1)(h+1) + 3(n-1) + (n+k-2)\\ &> (k-1)(h+1) + 2n - 2.\end{aligned}$$

In the following, we will show that $D^{\bar{I}}_{k-1,n} - F_{\bar{I}}$ is connected. For any $i \in \bar{I}$, $D^{i}_{k-1,n} - F_i$ is connected since $\kappa(D^{i}_{k-1,n}) = n+k-2$ and $|F_i| \leq n+k-3$. For any two $D^{i}_{k-1,n}$ and $D^{j}_{k-1,n}$ with distinct $i, j \in \bar{I}$, $k \geq 2$, and $n \geq 2$, according to

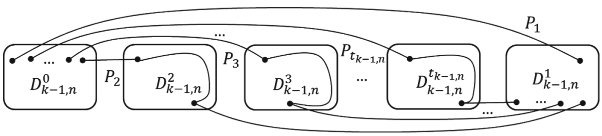


**Fig. 3.** An illustration of $t_{k-1,n}$ disjoint paths $P_1$, $P_2$, …. and $P_{t_{k-1,n}}$ joining $D^{i}_{k-1,n}$ and $D^{j}_{k-1,n}$ in Lemma 9.

Lemma 3, there exist $t_{k-1,n}$ disjoint paths $P_1$, $P_2$, ..., and $P_{t_{k-1,n}}$ joining $D^i_{k-1,n}$ and $D^j_{k-1,n}$ (see Fig. 3). Then, for any $n \geq 2$, $n \leq h \leq 2n-1$, and $k \geq n+1$, we have

$$\begin{aligned} t_{k-1,n} &\geq (n+\frac{1}{2})^{2^{k-1}} - \frac{1}{2} > 2kn \geq (k-1)2n + 2n \\ &> (k-1)(h+1) + 2n - 2. \end{aligned}$$

Thus, we can verify that there exists a path without any failure joining $D^i_{k-1,n}$ and $D^j_{k-1,n}$ in $D^{\bar{I}}_{k-1,n} - F_{\bar{I}}$ for any two distinct $i, j \in \bar{I}$. Then, $D^{\bar{I}}_{k-1,n} - F_{\bar{I}}$ is connected.

**Lemma 10.** *Given an integer $n \geq 2$, let $g_n(m) = 2mn - \frac{m(m+1)}{2} - n^2 + n$ be a function of $m$, $g_n(m) = 2mn - \frac{m(m+1)}{2} - n^2 + n$ is strictly monotonically increasing on $m$ if $n+1 \leq m \leq 2n-1$.*

*Proof.* When $n+1 \leq m \leq 2n-1$, we have $\frac{dg}{dm} = 2n - \frac{1}{2}(2m+1) = 2n - m - \frac{1}{2} > 0$. So, for any two positive integers $m'$ and $m$ such that $n+1 \leq m' < m \leq 2n-1$, we have $g_n(m') < g_n(m)$.

**Lemma 11.** *For any three three integers $n \geq 2$, $k \geq n+1$, and $n \leq h \leq 2n-1$, let $H \subseteq V(D_{k,n})$ and $|H| = h+1$. Then, we have $|N_{D_{k,n}}(H)| \geq (k-1)(h+1) + 2n - 1$.*

*Proof.* Let $S = \{\sigma(u,1) : u \in H\} = \{s_1, s_2, \ldots, s_m\}$ with $2 \leq m \leq h+1$. For any $1 \leq i \leq m$, let $H_i = V(D^{s_i}_{0,n}) \cap H$ and $h_i = |H_i|$. Obviously, $\sum_{i=1}^{m} h_i = |H| = h+1$. When $2 \leq m \leq n$, similar to the result of Lemma 7, we can verify

$$|N_{D_{k,n}}(H)| \geq (k-1)(h+1) + mn - \frac{m(m-1)}{2}.$$

When $n+1 \leq m \leq 2n$, let $T_1 = \bigcup_{i=1}^{n} H_i$ and $T_2 = \bigcup_{i=n+1}^{m} H_i$. Definition 1 and Lemma 5 implies that any node in $H$ has exactly $k$ neighbors in $D_{k,n} - \bigcup_{i=1}^{m} V(D^{s_i}_{0,n})$, $H_i$ has exactly $n - h_i$ neighbor(s) in $D^{s_i}_{0,n}$ for any $1 \leq i \leq m$, $T_1$ has not more than $\frac{n(n-1)}{2}$ common neighbor(s) in $\bigcup_{i=1}^{n} V(D^{s_i}_{0,n})$, and $T_2$ has not more than $\frac{(m-n)(m-n-1)}{2} + (m-n)$ common neighbors in $\bigcup_{i=1}^{m} V(D^{s_i}_{0,n})$. Thus, we have

$$\begin{aligned} |N_{D_{k,n}}(H)| &\geq k(h+1) + \sum_{i=1}^{m}(n-h_i) - \frac{n(n-1)}{2} - (\frac{(m-n)(m-n-1)}{2} + m - n) \\ &= (k-1)(h+1) + 2mn - \frac{m(m+1)}{2} - n^2 + n. \end{aligned}$$

For any $m$ with $2 \leq m \leq n$, by Lemma 6, we have

$$mn - \frac{m(m-1)}{2} \geq 2n - 2. \tag{3.3}$$

For any $m$ with $n+1 \leq m \leq 2n-1$, by Lemma 10, we have

$$2mn - \frac{m(m+1)}{2} - n^2 + n \geq \frac{n(n+3)}{2} - 1. \tag{3.4}$$

For $m = 2n$, we have

$$2mn - \frac{m(m+1)}{2} - n^2 + n = n^2. \tag{3.5}$$

Thus, by (3.3), (3.4), and (3.5), we have

$$\begin{aligned}|N_{D_{k,n}}(H)| &\geq (k-1)(h+1) + \min\{2n-2, \frac{n(n+3)}{2} - 1, n^2\} \\ &= (h+1)(k-1) + 2n - 2.\end{aligned}$$

So far, $|N_{D_{k,n}}(H)| \geq (k-1)(h+1) + 2n - 1$ for $n \geq 2$, $n \leq h \leq 2n-1$, and $k \geq n+1$.

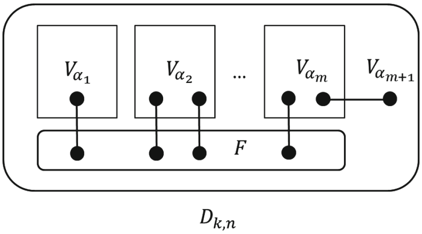


**Fig. 4.** An illustration of Lemma 12.

**Lemma 12.** *For any three integers $n \geq 2$, $n \leq h \leq 2n-1$, and $k \geq n+1$, and any node sub-set $F \subset V(D_{k,n})$ with $|F| \leq (k-1)(h+1) + 2n - 2$, $D_{k,n} - F$ contains a large component including not less than $t_{k,n} - |F| - h$ nodes.*

*Proof.* Assume that $H_1, H_2, \ldots, H_m, H_{m+1}$ are all the components of $D_{k,n} - F$, and the number of nodes of $H_{m+1}$ is the largest. By Lemma 9, $D_{k-1,n}^{\bar{I}} - F_{\bar{I}}$ is connected. So, $V(D_{k-1,n}^{\bar{I}} - F_{\bar{I}}) \subseteq V(H_{m+1})$. Let $r = |I| \leq h+1$ and $I = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$. Then, let $V_{\alpha_i} = \bigcup_{i=1}^{m} V(H_i) \cap V(D_{k-1,n}^{\alpha_i})$ for any $1 \leq i \leq r$. Furthermore, we will show the order in $\bigcup_{i=1}^{r} V_{\alpha_i}$ does not exceed $h$. We assume that the sum of orders in $\bigcup_{i=1}^{r} V_{\alpha_i}$ is at least $h+1$. By Lemma 11, we have $|N_{D_{k,n}}(\bigcup_{i=1}^{r} V_{\alpha_i})| \geq (k-1)(h+1) + 2n - 2 > |F|$, a contraction. Thus, we have $\bigcup_{i=1}^{r} |V_{\alpha_i}| \leq h$ (see Fig. 4). Then, for any $1 \leq i \leq r$, let $A_{\alpha_i} = V(D_{k-1,n}^{\alpha_i} - F_{\alpha_i} - V_{\alpha_i})$, we have

$$|A_{\alpha_i}| \geq t_{k-1,n} - |F| - h \geq (k-1)(h+1) + 2n - 2 > |F|.$$

Therefore, $A_{\alpha_i}$ is connected to $D_{k-1,n}^{\bar{I}} - F_{\bar{I}}$ in $D_{k,n} - F$ and thus $A_{\alpha_i} \subseteq V(H_{m+1})$ for any $1 \leq i \leq r$ . Hence, $D_{k,n} - F$ has a connected component including at least $t_{k,n} - |F| - h$ nodes, if $k \geq n+1$ and $n \leq h \leq 2n-1$.

**Theorem 2.** *For any three integers $n \geq 2$, $k \geq n+1$, and $n \leq h \leq 2n-1$, the extra-$h$ connectivity of $D_{k,n}$ is $\kappa_h(D_{k,n}) = (k-1)(h+1)+2n-2$.*

*Proof.* We use $H$ to denote a induced subgraph of order $h+1$ in $D_{k,n}$ with $V(H) = \{\alpha 00, \alpha 01, \ldots, \alpha 0(n-1), \alpha 10, \alpha 11, \ldots, \alpha 1(h-n)\}$, $E(H) = E(D_{k,n}[V(H)])$, and $\alpha \in V^2_{k,n}$. Letting $V' = V(H)$ and $F = N_{D_{k,n}}(V')$, obviously, $D_{k,n} - F$ is disconnected. Then, letting $T = V(D^{\alpha 0}_{0,n}) \cup V(D^{\alpha 1}_{0,n})$, Definition 1 implies that $\alpha 00$ has exactly $k-1$ neighbor(s) in $D_{k,n} - T$, any node in $H - \{\alpha 00\}$ has exactly $k$ neighbor(s) in $D_{k,n} - T$, and $V'$ has exactly $2n-h$ neighbor(s) in $D_{k,n}[T] - V'$. Thus, we have

$$|F| = hk + (k-1) + 2n - h = (k-1)(h+1) + 2n - 2.$$

Furthermore, we will prove that $F$ is an extra-$h$ node cut of $D_{k,n}$. Let $u \in V'$ and $\beta = (u)_k$. By Lemma 9, $D_{k,n} - (V(D^{\beta}_{k-1,n}) \cup F)$ is connected. By Definition 1, every node of $D^{\beta}_{k-1,n} - A_{D^{\beta}_{k-1,n}}(V')$ has accurate one neighbor in $D_{k,n} - (V(D^{\beta}_{k-1,n}) \cup F)$. Therefore, $D_{k,n} - F$ contains two distinct components, one is $D_{k,n} - A_{D_{k,n}}(V')$ and the other is $H$. Accordingly, for any two integers $n \geq 2$ and $k \geq 2$, we have

$$\begin{aligned} |V(D_{k,n} - A_{D_{k,n}}(V'))| &\geq t_{k,n} - |V'| \geq (n + \frac{1}{2})^{2^k} - \frac{1}{2} - 2n \\ &\geq 2kn - 2n \geq 2n^2 > 2n \\ &\geq h + 1. \end{aligned}$$

Furthermore, $F$ is an extra-$h$ node cut of $D_{k,n}$, and thus $\kappa_h(D_{k,n}) \leq (h+1)(k-1) + 2n - 2$ for $n \geq 2$, $n \leq h \leq 2n-1$, and $k \geq n+1$.

However, given three integers $n \geq 2$, $k \geq n+1$, and $n \leq h \leq 2n-1$, if the number of nodes of each connected component of $D_{k,n} - F$ is at least $h+1$ with $F \subseteq V(D_{k,n})$, then $|F| \geq (k-1)(h+1) + 2n - 2$ by Lemma 12. So, $\kappa_h(D_{k,n}) \geq (k-1)(h+1) + 2n - 2$.

Hence, $\kappa_h(D_{k,n}) = (k-1)(h+1) + 2n - 2$ when $n \geq 2$, $k \geq n+1$, and $n \leq h \leq 2n-1$.

The reliability a faulty DCN has close relations with its structure. We will determine extra-$h$ connectivity of DCell in the following theorem, use the above results on the structure of a faulty DCell network. The following theorem about the $\kappa_h(D_{k,n})$ follows Theorems 1 and 2.

**Theorem 3.** *For any positive integer $n \geq 2$,*

$$\kappa_h(D_{k,n}) = \begin{cases} (k-1)(h+1) + n & \text{if } 0 \leq h \leq n-1 \text{ and } k \geq 2, \\ (k-1)(h+1) + 2n - 2 & \text{if } n \leq h \leq 2n-1 \text{ and } k \geq n+1. \end{cases}$$

By Theorem 3, we will proposed the following theorem:

**Theorem 4.** *For any $n \geq 2$, $k \geq 2$, and $0 \leq h \leq n-1$ (resp. $n \geq 2$, $k \geq n+1$, and $n \leq h \leq 2n-1$), let $F \subset V(D_{k,n})$ with $|F| < \kappa_h(D_{k,n})$. Then, $D_{k,n} - F$ contains a large connected component and the rest of small connected components have not more than $h$ nodes in total.*

## 4 Conclusions

Our primary aim of this paper is to explore the boundary problem of node subsets in DCells. In this paper, we determine that the extra-$h$ connectivity of $D_{k,n}$ when $n \geq 2$, $\kappa_h(D_{k,n})$, as follows: (1) $\kappa_h(D_{k,n}) = (k-1)(h+1) + n$ if $k \geq 2$ and $0 \leq h \leq n-1$; (2) $\kappa_h(D_{k,n}) = (k-1)(h+1) + 2n - 2$ if $k \geq n+1$ and $n \leq h \leq 2n-1$. What's more, for any node sub-set $F \subseteq V(D_{k,n})$ with $|F| \leq \kappa_h(D_{k,n}) - 1$, we show that there has a large component in $D_{k,n} - F$, and the rest of small components contain not less than $h$ nodes in total with $0 \leq h \leq n-1$ and $k \geq 2$ (resp. $n \leq h \leq 2n-1$ and $k \geq n+1$). This approach studied in the paper may also be used to research the reliability of other DCNs such as BCube and Ficonn.

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