Research on Compressed Sensing Signal Reconstruction Algorithm Based on Smooth Graduation l_1 Norm

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Abstract. The compressed signal reconstruction of the sensing node has been a hot research topic for the mobile Internet. At present, some reconstruction algorithms finally adopt the minimum l_1 norm optimization algorithm. In order to solve the roughness, poor derivability and other defects of the minimum l_1 norm function, this paper constructs the smooth graduation algorithm based on l_1 norm, proves the monotonicity of the function and the sequence convergence of the optimal solution, and finally verifies the effectiveness of the function through examples. In the simulation experiment, the signal reconstruction algorithm and the classical OMP algorithm were compared, and the results show that it receives better reconstruction effects, small error and high precision.

Keywords: Compressed reconstruction · l_1 norm · Smooth graduation

1 Introduction

In the mobile Internet, the wireless sensor network is characterized by a large number of nodes and large data acquisition and transmission capacity. However, due to its small volume of its own node and the node energy restriction, how to reduce the energy consumption of nodes and prolong the network life cycle is a key challenge confronted by the development and application of the wireless sensor network technology [1, 2]. As a new sampling theory developed in recent years, compressed sensing (CS) [3–5] may achieve the data compression by using the redundancy of the wireless sensor network data, reduce the transmission of redundant information, and provide ideal solutions to reducing the energy consumption of network nodes.

In the compressed sensing theory, the signal reconstruction has become the key to obtaining accurate original signals and produced direct impacts on the measurements required by the reconstruction, i.e., obtaining the transmission data volume required by accurate obtaining of network data. Common signal reconstruction algorithms include the minimum convex optimization algorithm based on l_1 norm and the minimum greedy algorithm based on l_0 norm. The convex optimization algorithm is featured by large calculation and good reconstruction effects, which is represented by Basis Pursuit (BP) [6], Interior-point-iterative Algorithm, Gradient Projection For Sparse (GPSR) [7], Projection onto Convex sets (POCS) [8], Homotopy Algorithm [9] and Least Angle Regression (LARS) [10]. In spite of good reconstruction effects, the convex

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optimization algorithm are blocked easily when handling massive signal questions because of high computational complexity and slow computation speed. During the iteration each time, the greedy algorithm selects a local optimal solution to gradually get close to the original signals that are characterized by poor accuracy and high computation speed characteristics, which are represented by that Matching Pursuit (MP) [11], Orthogonal Matching Pursuit (OMP) [12] and Stagewise Orthogonal Matching Pursuit (StOmp) [13]. Literature [14] proposed to first use the arc-tangent function l_0 approximation norm, establish the noisy sparse reconstruction model approximate to l_0 norm, solve the model through quasi-Newton method, and analyze the convergence of the algorithm. Numerical simulations show that the proposed algorithm needs less measurements when reconstructing the sparse vectors and has high accuracy; Literature [15] proposed a new smooth function sequence approximation norm, solve by combining with the gradient projection method, improve the robustness of the algorithm by further proposing to adopt Singular value decomposition (SVD), and achieve the accurate reconstruction of the sparsity signal; Literature [16] put forward the fast smooth norm algorithm - FSL0 algorithm according to the characteristics of Gaussian smoothing function gradient and Hesse matrix as well as basic principles of Newton; Literature [17] utilized the signal sampling value, Laplace prior distribution and Gaussian likelihood model, and derived the signal posterior probability density estimate; finally, converted the MAP estimation process into a weighted iterative L1 norm minimization question, and the signal reconstruction performance had been improved significantly.

The ideal signal reconstruction is to adopt the signal reconstruction based on the minimum l_0 norm. However, this is an NP question, so it is converted into a solution to l_1 minimum norm. Since the minimum l_1 norm is not smooth, this paper constructs, presents and proves the minimum l_1 norm based on the smooth graduation. Simulation results show that the algorithm has better reconstruction effects than the traditional OMP algorithm.

2 Signal Reconstruction Based on Smooth Graduation l_1 Norm

The ideal signal reconstruction is obtained by solving the original reconstruction model or the model based on the minimum l_0 norm [23, 24]. However, this is an NP question, so it is converted to a solution to an l_1 minimum norm. Since this norm is not smooth, this paper constructs a smooth graduation algorithm based on the l_1 norm, describes the monotonicity of the function and the convergence of the optimal solution, and uses this function to perform the signal reconstruction.

2.1 Basic Knowledge

When solving the compressed sensing signal reconstruction, the l_0 norm solution is given as follows:

$$\begin{cases} \min \quad d(x) = \|x\|_0 \\ s.t.\ Ax = y \end{cases} \tag{1}$$

The l_0 norm is an NP question. It has been proved that signal reconstruction based on the minimum l_0 norm is equivalent to that based on solving the minimum l_1 norm [18, 19]. Therefore, signal-reconstruction questions are handled by solving the minimum l_1 norm with the following model:

$$\begin{cases} \min \quad d(x) = \|x\|_1 \\ s.t.\ Ax = y \end{cases} \tag{2}$$

2.2 The Improved l_1 Question Model

An algorithm based on the l_1 solution cannot be derived, so Eq. 2 cannot be solved by an algorithm based on massive derivation. Equation 2 is a convex programming question that can be converted to one of linear programming. However, the size of the original question is doubled and the computing space is increased. A solution involving large-scale data is characterized by slow computation speed and poor signal-reconstruction effects. This paper adopts smooth, gradual, and progressive ideas, constructs a smoothing function based on the l_1 norm, studies the monotonicity and optimal sequence, and finally solves Eq. 2.

Assuming Definition 1, when $x \in R^N,\ t > 0$, then

$$F(x) = \|x\|_1 = \sum_{i=1}^{N} |x_i| \quad F_t(x) = \sum_{i=1}^{N} \sqrt{x_i^2 + \frac{c}{t^2}} \tag{3}$$

Theorem 1:

$$\lim_{t \to \infty} F_t(x) = F(x),\ F_t(x) = \sum_{i=1}^{N} \sqrt{x_i^2 + \frac{c}{t^2}},\ x \in R^N \tag{4}$$

Proof:

$$\begin{aligned} F_t'(x) &= \sum_{i=1}^{N} \frac{1}{2\sqrt{x_i^2 + \frac{c}{t^2}}} \bullet \left(x_i^2 + \frac{c}{t^2}\right)' \\ &= \sum_{i=1}^{N} \frac{1}{2\sqrt{x_i^2 + \frac{c}{t^2}}} \bullet \left(\frac{-2c}{t^3}\right) \\ &= \sum_{i=1}^{N} \frac{-c}{t^3\sqrt{x_i^2 + \frac{c}{t^2}}} \\ &= \sum_{i=1}^{N} \frac{-c}{t^2\sqrt{(tx_i)^2 + c}} < 0 \end{aligned} \tag{5}$$

Then, $\{t_k\}$ is a monotonically decreasing integer sequence.
The following proves that $F_t(x)$ is bounded.
For any x and t,

$$\begin{aligned} F_t(x) - F(x) &= \sum_{i=1}^{N} \sqrt{x_i^2 + \frac{c}{t^2}} - \sum_{i=1}^{N} |x_i| \\ &= \sum_{i=1}^{N} \left(\sqrt{x_i^2 + \frac{c}{t^2}} - \sqrt{x_i^2}\right) \\ &= \sum_{i=1}^{N} \frac{\frac{c}{t^2}}{\sqrt{x_i^2 + \frac{c}{t^2}} + \sqrt{x_i^2}} \\ &\leq \sum_{i=1}^{N} \frac{\frac{c}{t^2}}{\frac{\sqrt{c}}{t}} \\ &= \frac{\sqrt{c}}{t} N \end{aligned} \tag{6}$$

Therefore $F(x) \leq F_t(x) \leq F(x) + \frac{\sqrt{c}}{t} N$, Because:

$$\begin{aligned} \lim_{t\to\infty} [F(x) + \frac{\sqrt{c}}{t} N] &= \lim_{t\to\infty} F(x) + \lim_{t\to\infty} \frac{\sqrt{c}}{t} N \\ &= F(x) + 0 \\ &= F(x) \end{aligned} \tag{7}$$

This is simplified to

$$0 \leq F_t(x) - F(x) \leq \frac{\sqrt{c}N}{t} \tag{8}$$

Take the limit toward both sides and obtain $\lim_{t\to\infty} F_t(x) = F(x)$. The proof ends.
According to Theorem 1, question 1 can be rewritten as

$$\begin{aligned} &\min F_t(x) \\ &s.t.\ Ax = y(t \to +\infty) \end{aligned} \tag{9}$$

Given that there is a continuous real number t, it is very difficult to solve Eq. 9. Through discretization of t, we can obtain

$$\begin{aligned} &\min F_t(x) \\ &s.t.\ Ax = y(t_k \to +\infty) \end{aligned} \tag{10}$$

where $\{t_k\}$ is a monotonically increasing integer sequence.

Theorem 2: The existence set $S = \{x \mid F_t(x) \leq F_k(x)\}$ has certain limits. The optimal solution to problem 5 is $x^*(t_k)$, i.e., $t = t_k$, so x^* is the optimal solution of Eq. 1, and $\{x^*(t_k)\}$ existence sub-column converges to x^*.

Proof: Since $F_t(x^*(t_k)) \geq F_t(x^*(t_{k+1})) \geq F_{t+1}(x^*(t_{k+1}))$ and $F_\infty(x) = F(x) \leq F_t(x)$, the combination set S has limits and $\{x^*(t_k)\}$ has a certain limit, so there is a converged sub-sequence $\{x^*(t_k)\}$. When the variable i approaches infinity, $\{x^*(t_k)\} \to \bar{x}$, and it is proved that $\bar{x} = x^*$. Proof by contradiction is as follows:

Assume $\bar{x} \neq x^*, F(x^*) - F(\bar{x}) < 0$.

Taking $\varepsilon_0 > 0$ and assuming $F(x^*) - F(\bar{x}) = -\varepsilon_0$, and $\lim_{t \to \infty} F_t(x) = F(x)$, then

$$\exists I_1 > 0,\ \forall_i \geq I_1,\ F_t(x^*) - F(x^*) < \frac{\varepsilon_0}{2} \tag{11}$$

Therefore,

$$F_t(x^*) - F(\bar{x}) = F_t(x^*) - F(x^*) + F(x^*) - F(\bar{x}) < -\frac{\varepsilon_0}{2} \tag{12}$$

Because

$$\lim_{t \to \infty} F_t(x^*(t_k)) = F_\infty(\bar{x}) = F(\bar{x}) \tag{13}$$

Therefore,

$$\exists I_2 > 0, \forall_i \geq I_2, F_t(\bar{x}) - F(x^*(t_k)) < \frac{\varepsilon_0}{2} \tag{14}$$

Therefore, $\forall i \geq \max\{I_1, I_2\}$, the following equation can be obtained:

$$F_t(x^*) - F(x^*(t_k)) = F_t(x^*) - F(\bar{x}) + F(\bar{x}) - F(x^*(t_k)) < 0 \tag{15}$$

$F_t(x^*) < F_t(x^*(t_k))$ and $x^*(t_k)$ are obtained as the optimal solution of problem 5, so $\bar{x} = x^*$.

Theorem 3: Problem 6 is a convex programming problem.

Proof: Suppose set $D = \{x \mid Ax = y\}$.

Where A is the matrix of $N \times M$, $x \in R^N, y \in R^M$

For $\forall x^{(1)}, x^{(2)} \in D$ and $\forall \lambda \in [0, 1]$

$$\begin{aligned} & A[\lambda x^{(1)} + (1-\lambda)x^{(2)}] \\ & = \lambda A x^{(1)} + (1-\lambda)Ax^{(2)} \\ & = \lambda y + (1-\lambda)y \\ & = y \end{aligned} \tag{16}$$

So $\lambda x^{(1)} + (1+\lambda)x^{(2)} \in D$. And therefore D is Convex set.

The following proves that the objective function $F_t(x) = \sum_{i=1}^{N} \sqrt{x_i^2 + \frac{c}{t^2}},\ x \in R^N$ x is a strictly convex function on the set D.

$$\begin{aligned} F_t(x+\Delta x) &= \sum_{i=1}^{N} \sqrt{(x_i+\Delta x_i)^2 + \frac{c}{t_k^2}} \\ &\geq \sum_{i=1}^{N} \sqrt{x_i^2 + \frac{c}{t_k^2}} + \sum_{t=1}^{N} \frac{x_i \Delta x_i'}{\sqrt{x_i^2 + \frac{c}{t_k^2}}} \\ &= F_t(x) + \nabla F_t(x)^T \Delta x \end{aligned} \tag{17}$$

So $F_t(x)$ is a convex function on D.

In fact $\nabla^2 F_t(x) = \begin{bmatrix} \frac{t_k}{\sqrt{[(x_1 t_k) + c]^3}} & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & & \frac{t_k}{\sqrt{[(x_w t_k) + c]^3}} \end{bmatrix} \in R^{N \times M}$ is Positive-definite matrix.

So $F_t(x)$ is a strictly convex function.

Theorem 4: Suppose $x^*(t_k)$ is the optimal solution of $t = t_k$ for Eq. 10, and x^* is the global optimal solution to problem 2. Therefore, for any $t_k > 0$ and $k \to +\infty$, then

$$\|x^* - x^*(t_k)\| \leq \sqrt{\max\{\frac{2\sqrt{[(x_i t_k)^2 + c]^3}}{t_k^2}\}\sqrt{c}N} \tag{18}$$

Proof: Select the target function $F_t(x)$. Its $x = x^*(t_k)$ Taylor expansion is

$$\begin{aligned} F_t(x) = {} & F_t(x^*(t_k)) + \nabla F_t(x^*(t_k))^T (x - x^*(t_k)) + \nabla F_t(x^*(t_k))(x - x^*(t_k)) \\ & + o(x - x^*(t_k))^T (x - x^*(t_k)) \end{aligned} \tag{19}$$

Given $x = x^*$ and the necessary conditions of the first-order derivative, the following equation can be obtained:

$$F_t(x) - F_t(x^*(t_k)) = \sum_{i=1}^{N} \frac{1}{2} \frac{t_k}{\sqrt{[(x_i t_k)^2 + c]^3}} (x^* - x^*(t_k))^2 + o(x - x^*(t_k))^T (x - x^*(t_k)) \tag{20}$$

Because $\nabla^2 F_t(x)$ is a diagonal matrix, the following equation can be obtained:

$$F_t(x) - F_t(x^*(t_k)) \geq \min\{\frac{t_k}{2\sqrt{[(x_i t_k)^2 + c]^3}}\} \|x^* - x^*(t_k)\|_2^2 \tag{21}$$

Because $F_t(x)$ is monotonously decreasing over t, we can obtain the inequality $F_t(x^*(t_k)) - F_{t+1}(x^*(t_k)) < 0$. x^* is the global optimal solution of problem 2, and the following equation can be obtained:

$$F_t(x) - F_t(x^*(t_k)) < 0 \tag{22}$$

$$\begin{aligned}
\|x^* - x^*(t_k)\|_2^2 &\leq \max\{\frac{2\sqrt{[(x_i t_k)^2 + c]^3}}{t_k}\}(F_t(x^*) - F(x^*(t_k))) \\
&= \max\{\frac{2\sqrt{[(x_i t_k)^2 + c]^3}}{t_k}\}(F_t(x^*) - F(x^*) + F(x^*) - F(x^*(t_k)) \\
&+ F(x^*(t_k) - F_t(x^*(t_k))) \\
&\leq \max\{\frac{2\sqrt{[(x_i t_k)^2 + c]^3}}{t_k}\}(F_t(x^*) - F(x^*))
\end{aligned} \tag{23}$$

Substitute Eq. 8 into the above equation to obtain $0 \leq F_t(x^*) - F(x^*) \leq \frac{\sqrt{c}N}{t}$, i.e.,

$$\|x^* - x^*(t_k)\| \leq \sqrt{\max\{\frac{2\sqrt{[(x_i t_k)^2 + c]^3}}{t_k^2}\}\sqrt{c}N} \tag{24}$$

The proof ends.

Therefore, according to Theorem 2, the algorithm of Eq. 25 is as follows.

Algorithm 1 steps:

Step 1: Enter the matrix A and t_0, the measured value y, the threshold ε, and the step h

Step 2: Given $k = 0, x_0^*(t_0) = A'y$

Step 3: Given $t_k = t_0 + kh$, obtain the optimal solution $x^*(t_k)$ of Eq. 25.

Step 4: Given $\|x^*(t_k) - x^*(t_{k-1})\| > \varepsilon$, set $k = k + 1$, return to step 3, and otherwise output $x^*(t_k)$

2.3 Algorithm Examples

Suppose we have $A = \begin{bmatrix} 1 & 0 & 3 & 4 & 5 & 8 & 2 & 3 & -1 & 5 \\ 0 & -10 & 4 & 1 & 2 & 3 & 4 & 7 & 8 & 3 \\ -9 & 15 & 4 & 3 & 8 & 6 & 4 & 7 & 2 & 4 \\ 2 & 5 & 1 & 7 & 6 & 3 & -5 & 0 & 9 & 7 \end{bmatrix}$, $y = \begin{pmatrix} 1 \\ 4 \\ 2 \\ 5 \end{pmatrix}$; assume $t_k = 10 + 200k$, $F_t(x^*(t_k))$, $\|x^*(t_k) - x^*(t_{k-1})\|_2$, and the computational results of $x^*(t_k)$ can be obtained according to the algorithm solution process, as shown in Tables 1 and 2:

Table 1. Numerical results

k	$F_t(x^*(t_k))$	$\|\|x^*(t_k) - x^*(t_{k-1})\|\|_2$
1	1.07768	0.55466
2	1.05858	0.04529
3	1.05232	0.00919
4	1.0492	0.00859
6	1.04569	0.00379
8	1.04352	0.00291
10	1.0432	0.00121

Table 2. Numerical results

k	$x^*(t_k)$
1	(−0.02517 −0.07485 0.00241 −0.00669 −0.01147 0.72901 0.06089 0.00701 −0.01242)
2	(−0.02519 −0.07131 0.00129 −0.00361 −0.00621 0.71601 0.08321 0.00401 −0.00653)
3	(−0.02521 −0.07129 0.00091 −0.00312 −0.00521 0.71982 0.08643 0.00261 −0.00489)
4	(−0.02521 −0.07129 0.00091 −0.00312 −0.00521 0.71982 0.08643 0.00261 −0.00489)
6	(−0.02497 −0.06751 0.00041 −0.00125 −0.00219 0.69758 0.09999 0.00239 −0.00231)
8	(−0.02492 −0.06763 0.00038 −0.00101 −0.00149 0.59501 0.10152 0.00099 −0.00108)
10	(−0.02516 −0.06731 0.00029 −0.00103 −0.00129 0.69371 0.10371 0.00081 −0.00128)

According to Tables 1 and 2, it is feasible to discrete the question, proving that Eq. 10 has effects and demonstrating that the algorithm 1 can achieve signal reconstruction.

3 Reconstructed Signal Algorithm Based on Smooth Approximation Norm l_p

In the previous chapter, we improved the signal reconstruction based on norm l_1 by constructing a smooth approximation function. However, the pseudo-norm $\|x\|_p$ $(0 \le p \le 1)$ rather than $\|x\|_1$ is a better approximation of the norm $\|x\|_0$ in the original problem. This chapter adopts the maximum entropy function smooth approximation l_p, proposes the MEFM algorithm, and validates using a one-dimensional signal-reconstruction example.

3.1 Preliminary Knowledge

Restoration signals will sometimes receive better effects by adopting $l_p(0 < p < 1)$ norm optimization than by adopting the l_1 norm optimization method.

The pseudo-norm $\|x\|_p(0 \le p \le 1)$ rather than $\|x\|_1$ is more approximate to the norm $\|x\|_0$ in the original problem. The problem model is as follows:

$$\begin{cases} \min d(x) = \|x\|_p \\ s.t.\ Ax = y \end{cases} (0<p<1) \tag{25}$$

When choosing the sparse vector to be the global $l_p(0<p<1)$ minimum solution of Ax = y, fewer y observations are required than with the l_1 norm optimization method [20]. In addition, existing proved sufficient conditions for lower signal reconstruction requirements than the norm l_1 [21–23]. This paper constructs a smooth approximation function of the norm $l_p(0<p<1)$ by the maximum entropy function, thus realizing the signal reconstruction.

In formula 25 for the $l_p(0<p<1)$ norm minimization, the objective function can be expressed as

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} = \left[\sum_{i=1}^{n} (\max\{x_i, -x_i\})^p\right]^{\frac{1}{p}} \tag{26}$$

So, formula 25 can be written as

$$\begin{cases} \min \varphi(x) = \left[\sum_{i=1}^{n} (\max\{x_j - x_i\})^p\right]^{\frac{1}{p}} (0<p<1) \\ s.t.\ Ax = y \end{cases} \tag{27}$$

where $\varphi(x)$ is the objective function.

Since the objective function in formula 8 is not derivable, the smoothing constraint algorithm cannot be employed. This paper constructs a smoothing function to approximate formula 27 and transforms the problem into a constrained smoothing problem that can be solved using the smooth constraint algorithm.

3.2 Smooth Approximation of the Norm l_p in the Algorithm

The maximum entropy function ρ^{-1} In$[\exp(\rho\, t) + \exp(-\rho\, t)]$ is a smooth approximation of the maximum function max{t, −t} [24], where $\rho > 0$ and t is a variable. By substituting the function $\max\{x_i, -x_i\}$ for ρ^{-1} In$[\exp(\rho\, x_i) + \exp(-\rho\, x_i)]$ in formula 27, we obtain the following smoothing problem:

$$\begin{cases} \min \Gamma(x,\rho) = \left[\sum_{i=1}^{n} (\rho^{-1} \ln[\exp(\rho x_i) + \exp(-\rho x_i)]^p\right]^{\frac{1}{p}} (0<p<1) \\ s.t.\ Ax = y \end{cases} \tag{28}$$

Lemma 1 [23]. $\forall p \in \{1,2,\ldots,k\},\ k \in N,\ g_p(x) : R^n \to R$ is assumed to be a continuously differentiable function,

$$h(x) = \max_{1\le p\le k}[g_p(x)], H(x,p) = \frac{1}{p}\ln\left[\sum^{1} \exp(\rho g_p(x))\right] \tag{29}$$

Then, the function $H(x,p)$ has the following properties:

(1) $\forall x \in R^n$ and $0 < \rho_1 < \rho_2$, there exists $\mathrm{H}(x, \rho_1) \geq H(x, \rho_2)$
(2) $\forall x \in R^n$ *and* $\rho > 0$, there exists $h(x) \leq H(x, \rho) \leq h(x) + \frac{\ln k}{\rho}$
(3) $\forall x \in R^n$ *and* $\rho > 0$, there exists $\lim_{\rho \to \infty} H(x, \rho) = h(x)$.

Lemma 2: The function $\Gamma(x,p) = ||\frac{\ln(e^{\rho x} + e^{-\rho x})}{\rho}||_p = [\sum_{i=1}^{n} (\frac{\ln(e^{\rho x} + e^{-\rho x})}{\rho})^p]^{\frac{1}{p}}$ $\forall x \in R^n$ *and* $\rho > 0$, there exists

$$\varphi(x) \leq \Gamma(x, \rho) \leq \varphi(x) + \frac{\sqrt[\rho]{n}\, ln\, 2}{\rho} \tag{30}$$

where $\varphi(x) = \|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$.

Proof: According to Lemma 1, $\forall x \in R^n$, $\rho > 0$, there exists the relation:

$$0 \leq \frac{1}{\rho} \ln[\exp(\rho x_i) + \exp(-\rho x_i)] - \max\{x_i, -x_i\} \leq \frac{\ln 2}{\rho} \tag{31}$$

According to the definitions of $\Gamma(x, \rho)$ *and* $\varphi(x)$, this paper concludes that

$$\begin{aligned} \Gamma(x,p) - \varphi(x) &= ||\frac{\ln(e^{\rho x} + e^{-\rho x})}{\rho}||_p - ||x||_p \\ &\leq ||\frac{\ln(e^{\rho x} - e^{-\rho x})}{\rho} - x||_p \\ &= \left[\sum_{i=1}^{n} (\frac{\ln(e^{\rho xi} + e^{-pxi})}{\rho}) - |x_i|^p\right]^{\frac{1}{p}} \end{aligned} \tag{32}$$

According to formula 9, we conclude that

$$0 \leq \Gamma(x, \rho) - \varphi(x) \leq \frac{\sqrt[p]{n} \ln 2}{\rho} \tag{33}$$

The proof ends.

Lemma 3: $\forall x \in R^n, \rho > 0$, there exists

$$\varphi(x) \leq \Gamma(x, \rho) \leq \varphi(x) + \left[\sum_{i=1}^{n} (\frac{\ln 2}{\rho})\right]^{\frac{1}{p}} = \varphi(x) + \frac{\sqrt[p]{n} \ln 2}{\rho} \tag{34}$$

And

$$\lim_{\rho \to \infty} \Gamma(x, \rho) = \varphi(x) = \|x\|_p \tag{35}$$

The following hypotheses are assumed for problem 1.

The rank of the matrix A is m, and the variable $x = [x^B, x^N]^T$, where $x^B = (x_1^B, x_2^B, \ldots, x_m^B)$, is the vector corresponding to the basic vector; $x^N = (x_{m+1}^N, x_{m+2}^N, \ldots, x_n^N)^T$ is the vector corresponding to the non-basic variable; B is the m-th column of matrix A corresponding to the basic vector; N is the (n–m)-th column corresponding to the non-basic variable; the matrix $A = [B, N]$; and the feasible region $\{x \mid Ax = y\}$ is not empty.

In the above hypotheses, the constraint condition $Ax = y$ in (1) can be written as

$$Bx^B + Nx^N = y \tag{36}$$

$x^B = B^{-1}(y - Nx^N)$ is available. Substitute x^B into the objective function of problem 3 to transform formula 3 into the following unconstrained problem:

$$\min_{x \in R} M(x^N, \rho) = \Gamma(B^{-1}(y - Nx^N), x^N, \rho) \tag{37}$$

This paper obtains the respective optimal solutions of problems 1 and 2 by using x^* and $x(\rho^*)$.

Algorithm Steps

Step 1: *Take* $(x^N)^0 \in R^{n-m}$ and the error $1 > \beta > 0, \delta \in (0, 1)$

Step 2: $\rho^* = \dfrac{\sqrt[p]{n}\ln 2}{\beta(1-\delta)}$

Step 3: Solve formula 10, i.e., solve the following problem:

$$\min_{x \in R} M(x^N, \rho^*) = \Gamma(B^{-1}(y - Nx^N), x^N, \rho^*) \tag{38}$$

where $M(x^N, \rho^*)$ is the objective function.

The algorithm has the following properties:

Theorem 5: Assume that the optimal solution of problem 12 is $(x^N(\rho^*))^*$ and the point generated by the algorithm satisfies the condition

$$M\left(\left(x^N\right)^s, \rho^*\right) - M\left(\left(x^N(\rho^*)\right)^*, \rho^*\right) \leq \delta\beta \tag{39}$$

Then $\varphi(x^s) - \varphi(x^*) \leq \beta$, where, s is the number of iterations.

Proof: $x^*(\rho^*)$ is the optimal solution of formula 3, i.e.,

$$\Gamma(x^*(\rho^*); \rho^*) \leq \Gamma(x^*; \rho^*) \tag{40}$$

From formula 30, we obtain

$$\varphi(x^*) \leq \Gamma(x^*; \rho^*) \leq \varphi(x^*) + \frac{\sqrt[p]{n}\ln 2}{\rho^*} \varphi(x^*(\rho^*)) \leq \Gamma(x^*(\rho^*); \rho^*) \leq \varphi(x^*(\rho^*)) + \frac{\sqrt[p]{n}\ln 2}{\rho^*} \tag{41}$$

So, we obtain

$$\Gamma(x^*(\rho^*);\rho^*) \leq \Gamma(x^*;\rho^*) \leq \varphi(x^*) + \frac{\sqrt[p]{n}\ln 2}{\rho^*} \tag{42}$$

and

From formula 30, we obtain

$$\varphi(x^s) \leq \Gamma(x^s;\rho^*) \tag{43}$$

By adding formulas 42 and 43, we obtain

$$\varphi(x^s) - \varphi(x^*) \leq \Gamma(x^s;\rho^*) - \Gamma(x^*(\rho^*);\rho^*) + \frac{\sqrt[p]{n}\ln 2}{\rho^*} \tag{44}$$

and from formula 38, we obtain

$$\left.\begin{array}{l} M\left((x^N(\rho^*))^*;\rho^*\right) = \Gamma(x^*(\rho^*);\rho^*) \\ M\left((x^N)^s;\rho^*\right) = \Gamma(x^s;\rho^*) \end{array}\right\} \tag{45}$$

From formulas 39 and 45, we obtain

$$\Gamma(x^s;\rho^*) - \Gamma(x^*(\rho^*);\rho^*) \leq \delta\beta \tag{46}$$

From formulas 45 and 46, we obtain

$$\varphi(x^s) - \varphi(x^*) \leq \delta\beta + \frac{\sqrt[p]{n}\ln 2}{\rho^*} \tag{47}$$

Substituting $\rho^* = \dfrac{\sqrt[p]{n}\ln 2}{\beta(1-\delta)}$ into formula 47, we obtain

$$\varphi(x^s) - \varphi(x^*) \leq \beta \tag{48}$$

The proof ends.

Theorem 6: Assume that $x(\rho) = (x_1(\rho),\ x_2(\rho), \cdots x_n(\rho))$ is the optimal solution of formula 39. If $\lim\limits_{p\to+\infty} x(\rho)$ exists, we assume $\lim\limits_{p\to+\infty} x(\rho) = x^* = [x_1^*, x_2^*, \cdots, x_n^*]^T$. If there is $x_k^* = 0 (1 \leq k \leq n)$ in $x_1^*, x_2^*, \cdots, x_n^*$, we obtain

$$\lim_{p\to+\infty} \rho x_k(\rho) = 0 \tag{49}$$

Then x^* is the optimal solution to formula 25.

Proof: $\forall x \in R^n$, $\forall \rho > 0$, and we obtain the equation below by Lemma 3:

$$\varphi(x) \leq \Gamma(x;\rho) \leq \varphi(x) + \frac{\sqrt[p]{n}\ln 2}{\rho^*} \tag{50}$$

In formula 30, assuming $\rho \to +\infty$, we can obtain the following equation according to the conditions in Theorem 2:

$$\lim_{p\to+\infty} \Gamma(x(\rho);\rho) = \lim_{p\to+\infty} \Gamma(x^*;\rho) = \lim_{p\to+\infty} \varphi(x(\rho)) = \varphi(x^*) = \|x^*\|_p \tag{51}$$

Since $x(\rho)$ is the optimal solution of formula 3, for the point x satisfying $Ax = y$, we can obtain $\Gamma(x(\rho);\rho) \leq \varphi(x;\rho)$.

So,

$$\lim_{p\to+\infty} \Gamma(x;\rho) = \varphi(x) = \|x\|_p \geq \|x^*\|_p \tag{52}$$

The following proves to be the feasible solution of problem 3. Since $x(\rho)$ is the optimal solution of formula 9, $Ax(\rho) = y$.

Assume $\lim_{p\to+\infty} (x;\rho) = x^*$. Then we obtain $Ax^* = y$, i.e., x^* is the optimal solution of formula 30.

4 Experimental Description

The one-dimensional signal is reconstructed by adopting the proposed algorithm and OMP algorithm. Where, the reconstruction signal sparsity is 6, the signal length is 256, the signal observation M is 64, $f_1 = 50$, $f_2 = 100$, $f_3 = 200$, $f_4 = 400$, $f_s = 800$ and $t_s = 1/f_s$.

$x = 0.3 \sin(2\pi * 50 * t_s * t_s) + 0.6 \sin(2\pi * 100 * t_s * t_s) + 0.1 \sin(2\pi * 200 * t_s * t_s) + 0.9 \sin(2\pi * 400 * t_s * t_s)$, the test results are shown in Fig. 1:

Since the measured values must satisfy $M \geq K \bullet \log(\frac{n}{k})$, when the value of M is 64, the sampling rate M/n is 0.25. Taking taken p as 0.25, β as 0.5 and δ as 0.02, Figs. 2 and 3 shows that these two algorithms receive excellent reconstruction effects. According to Table 3, at the same sampling rate, the signal reconstruction effects of the proposed algorithm are superior to OMP algorithm with small reconstruction errors. Besides, OMP algorithm needs to obtain the known signal sparsity K at running time and the proposed algorithm does not need it, so the proposed algorithm has more convenient computation and high efficiency.

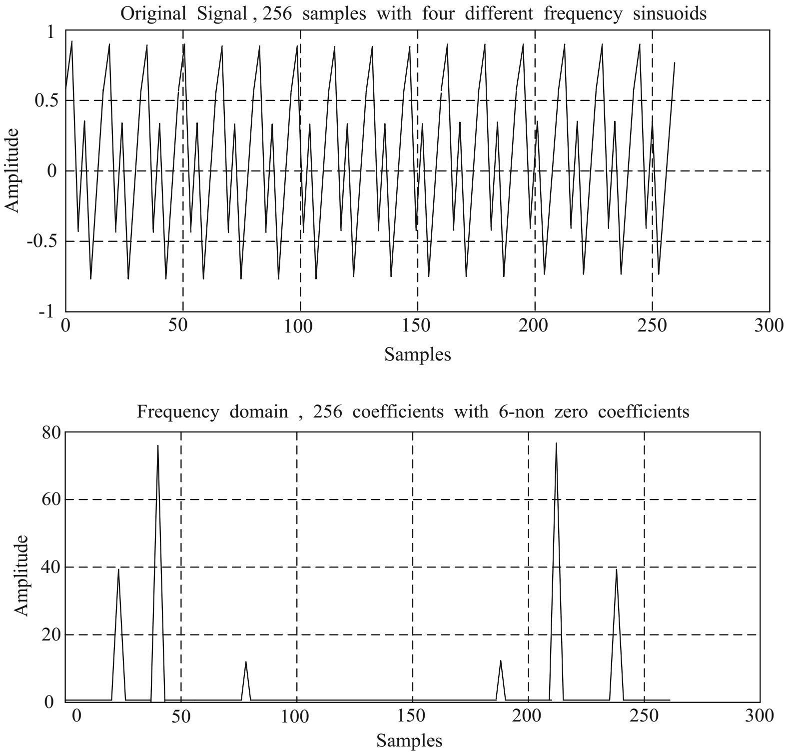

Fig. 1. Original signal and frequency-domain signal

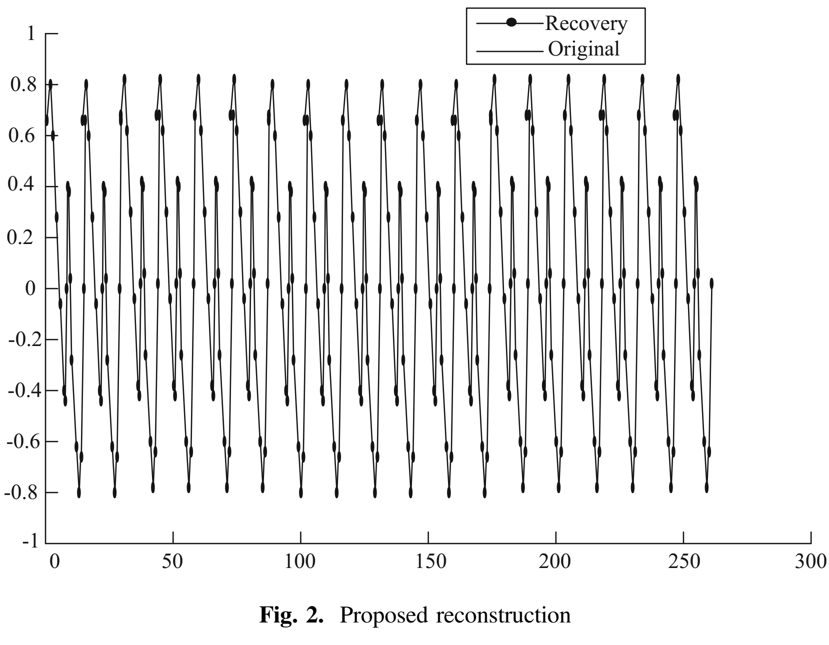

Fig. 2. Proposed reconstruction

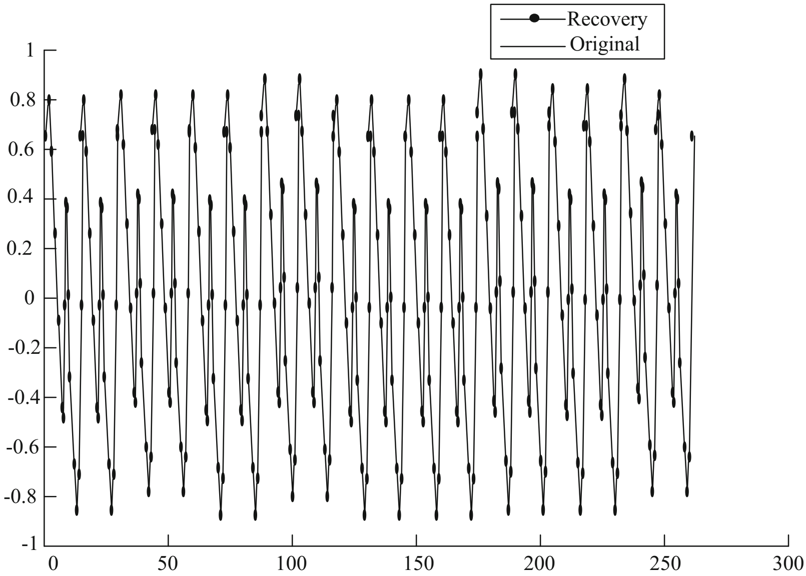

Fig. 3. OMP algorithm reconstruction

Table 3. Reconstruction error

Reconstruction algorithm	Sampling rate	Reconstruction error
OMP	0.25	7.1997e−004
The proposed algorithm	0.25	2.9821e−024

5 The References Section

The signal reconstruction is an important part of the compressed sensing. This paper constructs the l_1 norm function based on the smooth graduation. By proving that the function has asymptotic monotonicity and sequence convergence of the optimal solution, this paper illustrates that the proposed algorithm can improve the signal reconstruction effects, further shows that the proposed algorithm has better results in the reconstruction through the simulation, and the reconstruction errors are reduced.

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