# **Noncoherent Joint Multiple Symbol Differential Detection and Channel Decoding in Massive MIMO System**

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**Abstract.** A noncoherent joint multiple symbol differential detection (MSDD) and channel decoding framework is proposed for massive multiple-input multiple-output (M-MIMO) system. The proposed framework bears the potential to solve the high channel estimation overhead for conventional coherent M-MIMO systems. Employing the autocorrelation receiver (AcR) and the belief propagation (BP) message passing algorithm, the proposed soft-input soft-output (SISO) MSDD can be easily integrated with advanced channel coding. Simulation results show that the BER performance can be significantly improved within a few iterations of the proposed scheme.

**Keywords:** MSDD · Massive MIMO · SISO · BP

## **1 Introduction**

Massive multiple-input multiple-output (M-MIMO) system has attracted much attention recently. Equipped with the massive antenna arrays, M-MIMO system can achieve very high spectral efficiency and energy efficiency [\[1](#page-9-0)[–3\]](#page-9-1). One of the challenges of coherent M-MIMO is the acquisition of channel state information (CSI). As the number of antennas at the base station grows large, the system overhead and the complexity associated with channel estimation will become too high and unaffordable. When considering low-complexity and low-overhead M-MIMO system, some noncoherent transmission schemes may be more favorable.

One of the typical noncoherent detectors is the differential detection (DD) with the autocorrelation receiver (AcR). Noting the performance of the simple DD may suffer from noise enhancement, multiple symbol differential detection (MSDD) is proposed to suppress the noise in the reference signals [\[4\]](#page-9-2). MSDD

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jointly detects multiple consecutive symbols, and it has been shown to achieve comparable performance of the coherent counterpart in many interesting scenarios [\[5](#page-9-3)[–7\]](#page-9-4). In particular, the authors of [\[8\]](#page-9-5) introduce the MSDD to M-MIMO system by noting the similarity between channel responses of the impulse radio ultra-wide band  $(IR-UWB)$  system  $[9]$  and the single-user M-MIMO system. Recently, MSDD has been combined with advanced channel coding techniques to further improve the system performance for IR-UWB [\[10](#page-9-7)], which is built upon an novel soft-input soft-output (SISO) framework. Regarding the M-MIMO system, the hard-output decision metric in [\[8\]](#page-9-5) is not suitable for SISO channel decoding [\[11\]](#page-9-8). To this end, a soft-output decision metric and a corresponding SISO framework is needed to further improve the performance of [\[8\]](#page-9-5).

In this paper, we propose a SISO framework jointly employing MSDD and channel decoding for single user M-MIMO system. The proposed framework does not need any knowledge of CSI, which reduces the system overhead and the intensive computational cost of acquiring CSI. In particular, we develop the SISO MSDD scheme by belief propagation (BP) message passing on a factor graph [\[12\]](#page-9-9). Since BP algorithm is also employed by advanced channel codes, such as LDPC decoding, we unify the treatment for SISO MSDD and LDPC channel decoding with a single factor graph and simple message flow scheduling. To be more specific, the soft outputs of SISO MSDD are considered as the inputs of the channel decoding, and the soft outputs of channel decoding are fed back to SISO MSDD, which turns conventional soft-output MSDD into an iterative manner. We also propose a simple blind method to estimate a key parameter needed for detection. Simulations show the good performance of estimated parameter in detection and that the bit error rate (BER) performance of the system can be significantly enhanced by a few iterations with our proposed scheme.

### **2 System Model**

We consider the *uplink* transmission scenario, where a signal-antenna user transmit to a  $N_R$ -antenna base station, and  $N_R$  is very large. In each discrete time k, the user transmits a symbol  $b_k$  drawn from an M-ary PSK constellation.

<span id="page-1-0"></span>
$$
\mathbf{r}_k = \mathbf{h}_k b_k + \mathbf{n}_k, \tag{1}
$$

where the channel vector is denoted as  $\mathbf{h}_k = [h_{k,1}, h_{k,2}, ..., h_{k,N_R}]^T$  and the noise vector is denoted as  $\mathbf{n}_k = [n_{k,1}, n_{k,2}, ..., n_{k,N_R}]^T$ . We consider a rich scat-<br>terms environment so the spannel sofficient by from the user to the m<sub>th</sub> tering environment, so the channel coefficient  $h_{k,m}$  from the user to the m-th receive antenna and the i.i.d. AWGN components are modeled as a zero-mean circularly-symmetric complex Gaussian random variables, i.e.,  $h_{k,m} \sim \mathcal{CN}(0, \sigma_h^2)$ <br>and  $n_k \sim \mathcal{CN}(0, \sigma_h^2)$ , which are independent over receive antennas. We conand  $n_{k,m} \sim \mathcal{CN}(0, \sigma_n^2)$ , which are independent over receive antennas. We consider quasi static channel i.e. the channel coefficients remain stationary for a sider quasi-static channel, i.e., the channel coefficients remain stationary for a sufficient long transmission  $[8]$  $[8]$ . Therefore, the time subscript k of the channel vector  $\mathbf{h}_k$  in [\(1\)](#page-1-0) is omitted for simplicity hereafter.

# **3 Noncoherent Detection in Massive MIMO System**

#### **3.1 Noncoherent Autocorrelation Receiver**

The performance of nonherent DD receiver can be improved by MSDD [\[4](#page-9-2)]. The differential encoding is needed at the transmitter and 2-DPSK is used to facilitate the noncoherent detection. To avoid explicit CSI, the detector is built on the autocorrelation receiver (AcR) which calculates the correlations between received differential signal vectors. Each transmission burst consists of  $K$  data symbols  $[a_1, a_2, \ldots, a_K]$ , i.e.,  $(K+1)$  DPSK modulated differential symbols  $[b_0, b_1, \ldots, b_K]$ . Correspondingly, the received differential signal vectors during the whole transmission burst can be stacked into a matrix  $R = [\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_K]^T$ . In MSDD, R is divided into  $S = K/M$  blocks, where M is the block size. The s-th block,  $s = 1, ..., S$ , includes M data symbol vectors. The length of the observation window of the s-th block is  $M+1$  and the window slides down M differential signal vectors after they have been processed jointly, as shown in Fig. [1.](#page-2-0) The adjacent observation windows overlap one differential signal vector, and different blocks are processed independently.



The observation window of the  $(s+1)$ -th block

<span id="page-2-0"></span>**Fig. 1.** The illustration of the observation window of the *s*-th and the (*s*+ 1)-th block. The solid lines denote the correlation operation between the received differential signal vectors in a block.

Using the result of differential modulation  $b_k^* b_l = (\prod_{z=l+1}^k a_z)$ , we drive the relation operation between the k-th and the *l*-th received differential signal correlation operation between the  $k$ -th and the  $l$ -th received differential signal vectors in the s-th block. So the correlation coefficient is calculated as

<span id="page-2-1"></span>
$$
z_{k,l} = \mathbf{r}_k^{\mathrm{H}} \mathbf{r}_l
$$
  
=  $||\mathbf{h}||^2 \cdot b_k^* b_l + n_{k,l},$   
=  $E_g \cdot (\prod_{z=l+1}^k a_z) + n_{k,l},$  (2)

<span id="page-3-1"></span>
$$
n_{k,l} = b_k^* \cdot \mathbf{h}^{\mathrm{H}} \mathbf{n}_l + \mathbf{n}_k^{\mathrm{H}} \mathbf{h} \cdot b_l + \mathbf{n}_k^{\mathrm{H}} \mathbf{n}_l. \tag{3}
$$

We stack all  $z_{k,l}$  of the s-th block into a vector  $\mathbf{z}_s = [z_{k,l}], k = (s-1)M +$  $1, ..., sM$ ,  $l = (s-1)M, ..., k-1$ , which contains  $M(M+1)/2$  correlation coefficients.  $|| \cdot ||$  denotes 2-norm of a vector, and  $E_q = ||\mathbf{h}||^2$  represents the captured energy of the signal vector, whose estimation is discussed in Sect. [4.](#page-3-0)

#### **3.2 Multiple Symbol Differential Detection**

With the knowledge of  $z_s$ ,  $[8]$  $[8]$  adopts the hard-output decision based on maximum-likelihood estimate of the symbols in s-th block jointly, and the MSDD decision metric is

$$
\mathbf{b}_s = \underset{\bar{\mathbf{b}}_s \in \{\pm 1\}^{M+1}, \bar{\mathbf{b}}_0 = 1}{\arg \max} \bar{\mathbf{b}}_s \mathbf{Z}_s \bar{\mathbf{b}}_s^H,
$$
(4)

where the diagonal weighting matrix  $\mathbf{Z}_s = \textbf{diag}(\mathbf{z}_s)$  and  $\bar{\mathbf{b}}_s \in {\{\pm 1\}}^{M+1}$  represents the differential candidate symbols.

# <span id="page-3-0"></span>**4 Noncoherent Joint Detection and Channel Decoding in Massive MIMO System**

In order to further improve the system performance, we incorporate the channel codes. The decoding of powerful channel codes, such as LDPC and Turbo code, relies on soft-outputs of detector to realize iterative algorithm within channel decoding [\[11](#page-9-8)]. Obviously, the hard-output decision metric in [\[8\]](#page-9-5) is not suitable for iterative decoding, so a new soft-output decision metric for the massive MIMO system is needed to study. In this section, we firstly investigate a new softoutput decision metric, and then propose the joint MSDD and channel decoding framework which enables novel additional iterative processing as shown in Fig. [2.](#page-4-0)

#### **4.1 The Soft-Output Decision Metric for SISO MSDD**

<span id="page-3-2"></span>In order to enable the iterative algorithm, we need to calculate the probability distribution of correlation coefficient  $z_{k,l}$ .

**Theorem 1.** *For massive MIMO system with AcR, the correlation coefficient*  $z_{k,l}$  can be approximated as a Gaussian random variable with the mean  $\mu =$  $(\prod_{z=l+1}^{k} a_z) \cdot E_g$  and the variance  $\sigma^2 = 2E_g \cdot \sigma_n^2 + N_R \cdot \sigma_n^4$ .

$$
p(z_{k,l}|\mathbf{x}_s) = \frac{1}{\pi\sigma^2} \exp\left[-\frac{1}{\sigma^2}|z_{k,l} - \mu|^2\right],\tag{5}
$$

*where*  $\mathbf{x}_s = [a_{(s-1)M+1}, a_{(s-2)M+2}, ..., a_{sM}]^T$  *is the candidate data symbols in the* s*-th block.*



<span id="page-4-0"></span>**Fig. 2.** The block diagram of noncoherent joint MSDD and channel decoding in massive MIMO system.

*Proof.* It is noted that the noise component  $n_{k,l}$  in [\(3\)](#page-3-1) contains two linear terms and a noise-by-noise product term. Given the assumed channel model, clearly the first and second term are Gaussian random variables, and the third one can be decomposed into a sum of  $N_R$  independent random variables. According to the central limit theorem, the third noise term can be approximated as a Gaussian random variable with zero mean and the variance  $N_R \cdot \sigma_n^4$ . The approximation<br>is very good when  $N_D$  is large in massive MIMO scenario. For DPSK  $|h_1| = 1$ is very good when  $N_R$  is large in massive MIMO scenario. For DPSK,  $|b_k| = 1$ . The conditional variance of  $n_{k,l}$  is

$$
Var\left[n_{k,l} \mid \mathbf{h}\right] = Var\left[b_k^* \cdot \mathbf{h}^{\mathrm{H}} \mathbf{n}_l \mid \mathbf{h}\right] + Var\left[\mathbf{n}_k^{\mathrm{H}} \mathbf{h} \cdot b_l \mid \mathbf{h}\right] + Var\left[\mathbf{n}_k^{\mathrm{H}} \mathbf{n}_l \mid \mathbf{h}\right] \\
= 2E_g \cdot \sigma_n^2 + N_R \cdot \sigma_n^4.
$$

So  $Var[z_{k,l} | \mathbf{h}] = Var[n_{k,l} | \mathbf{h}] = 2E_g \cdot \sigma_n^2 + N_R \cdot \sigma_n^4$ , and the mean  $E[z_{k,l} | \mathbf{h}] = E[\sigma_n^k]$  $E_g \cdot (\prod_{z=l+1}^{\kappa} a_z).$ 

Given the observation  $z_s$ , the SISO MSDD scheme aims to calculate the *a posteriori* probability (APP) of data symbol  $a_k$ :

<span id="page-4-1"></span>
$$
p(a_k|\mathbf{z}_s) \propto \sum_{\mathbf{x}_s : \sim a_k} p(\mathbf{z}_s|\mathbf{x}_s) p(\mathbf{x}_s),
$$
\n(6)

for  $k = (s-1)M + 1, ..., sM$ ,  $s = 1, 2, ..., S$ , and the notation  $\sum_{\mathbf{x}_s : \sim a_k}$  means the summation over all data symbols in  $\mathbf{x}_s$  except  $a_t$ . summation over all data symbols in  $\mathbf{x}_s$  except  $a_k$ .

#### **4.2 Parameter Estimation**

It can be concluded from [\(2\)](#page-2-1) that the correlation coefficient depends on the data symbols and the captured energy  $E_q$ . To obtain the knowledge of the parameter  $E<sub>g</sub>$ , our receiver employs an energy estimation method

$$
\hat{E}_g = \frac{1}{J} \sum_{s=1}^{S} \sum_{k=(s-1)M+1}^{sM} \sum_{l=(s-1)M}^{k-1} |z_{k,l}|, \tag{7}
$$

where  $J$  is the number of the elements in the set of the correlation coefficients  ${\bf z}_s$   ${\bf z}_s$  = 1, 2, ..., S. In total  $J = SM(M+1)/2$ . With the estimation  $E_q$ , we substitute it into the signal model [\(2\)](#page-2-1). We compare the performances of the perfect  $E_q$  and the estimated  $\hat{E}_q$  in our proposed algorithm in the simulation.



<span id="page-5-0"></span>**Fig. 3.** The factor graph of joint MSDD and channel decoding with the block size *M*.  $a_k$  is the data symbol.  $\hat{d}_k$  is the decoding result of the information bit  $d_k$ . The red arrows denote extrinsic information from SISO MSDD to the channel decoder in the *i*-th iteration, and the blue arrows denote the extrinsic information from the channel decoder to SISO MSDD in the  $(i - 1)$ -th iteration. (Color figure online)

#### **4.3 Joint MSDD and Channel Decoding Framework**

In this subsection, we propose a SISO MSDD scheme using BP message passing algorithm. For the calculation of  $(6)$ , we apply a factor graph to represent the probabilistic model of the system. Then the BP massages can pass on the factor graph. To realize this, we factorize the global probability function  $p(\mathbf{z}_s|\mathbf{x}_s)p(\mathbf{x}_s)$ in [\(6\)](#page-4-1) into many small local functions. Firstly, the channel evidence information function  $p(\mathbf{z}_s|\mathbf{x}_s)$ , or channel transition probability, is obtained by the Gaussian approximation on the discrete noise components according to the Theorem [1](#page-3-2)

<span id="page-6-0"></span>
$$
p(\mathbf{z}_s|\mathbf{x}_s) = \prod_{l=(s-1)M+1}^{sM} \prod_{k=(s-1)M}^{l-1} p(z_{k,l}|\mathbf{x}_s)
$$
  
 
$$
\propto \prod_{l=(s-1)M+1}^{sM} \prod_{k=(s-1)M}^{l-1} \prod_{k=(s-1)M}^{k}
$$
  
 
$$
\exp(-\frac{1}{\sigma^2}|z_{k,l} - (\prod_{z=l+1}^{k} a_z) \cdot \hat{E}_g|^2).
$$
 (8)

Then we factorize  $p(\mathbf{x}_s)$ , though in coded system, data symbols can be seen approximately independent by interleaving operation, so

<span id="page-6-1"></span>
$$
p(\mathbf{x}_s) = \prod_{k=(s-1)M+1}^{sM} p(a_k).
$$
 (9)

Substituting  $(8)$ ,  $(9)$  into  $(6)$  leads to

<span id="page-6-2"></span>
$$
\gamma(a_k) \propto \sum_{\mathbf{x}_s : \sim a_k} p(\mathbf{z}_s | \mathbf{x}_s) \prod_{((l \in I_s) \cap (l \neq k))} \delta(a_l). \tag{10}
$$

 $I_s = \{(s-1)M + 1, ..., sM\}$ , and  $\gamma(a_k)$  and  $\delta(a_l)$  denote the APP and the *priori* information of  $a_k$ , respectively.

According to [\(10\)](#page-6-2), we can now calculate  $p(a_k|\mathbf{z}_s)$  using a BP message passing algorithm for all  $a_k$ . Figure [3](#page-5-0) shows the framework for joint MSDD and channel decoding. The framework can be divided into two parts: the upper one in dash line box illustrates the SISO MSDD processing in one of the blocks of the size  $M$ ; the under one shows the processing of deinterleaver and channel decoder. The massages exchanged between the MSDD and the channel decoding are known as extrinsic information. The total length of data symbols of a transmission burst are divided into S blocks, they are processed simultaneously.

The algorithm of joint MSDD and channel decoding is briefly described as follows:

- *Initialization*: Since no extrinsic information from channel decoder, the *priori* information  $\delta^{(0)}(a_l)$  is assumed equiprobable.
- $-$  *Step*1: For the s-th block in the *i*-th iteration as shown in the Fig. [3,](#page-5-0) with the knowledge of  $p(\mathbf{z}_s|\mathbf{x}_s)$  and the *priori* information  $\delta^{(i-1)}(a_l)$ , compute  $\gamma^{(i)}(a_k)$ in the probability domain using [\(10\)](#page-6-2).  $\gamma^{(i)}(a_k)$  is new APP of data symbol  $a_l$ , which is considered as the extrinsic information from MSDD and sent into channel decoder.
- *Step 2*: The extrinsic information  $\gamma^{(i)}(a_k)$  run several rounds of within the channel decoder, and the channel decoder updates  $\delta^{(i-1)}(a_l)$ . Updated  $\delta^{(i-1)}(a_l)$  is new *priori* information of data symbol  $a_l$ , which is treated as the extrinsic information from channel decoder and sent into MSDD.
- *Step 3* : Steps 1 and 2 are repeated after a certain number of iterations, and the final channel decoding results of information bits are obtained as results.

### **5 Simulation Results**

The performance of the proposed scheme is validated by numerical simulations. There is an uniform linear array with  $N_R = 100$  antennas at the receiver. Block fading channel changes after a transmission burst of the length  $K = 300$ , and the results is averaged over 10000 channel realizations. The signal-to-noise (SNR) is defined as  $E_s/\sigma_n^2$ .  $E_s$  is the transmitted energy per PSK symbol. The LDPC<br>code [13] with coding rate  $R = 1/2$  and 10 iterations within the LDPC channel code [\[13](#page-9-10)] with coding rate  $R = 1/2$  and 10 iterations within the LDPC channel decoder is applied.

Figure [4](#page-7-0) presents the BER comparisons of the coded MSDD with different number of iterations between detection and decoding. The block size is  $M = 3$ , using estimated  $E<sub>q</sub>$ . It is clear to see that the BER performance of coded MSDD scheme outperforms the uncoded one with enormous gains by iterative processing with powerful channel code. When the iteration between MSDD and channel decoding is one, the SISO MSDD degrades to the soft-output MSDD since no extrinsic information from LDPC decoding is sent into MSDD, which is the case of MSDD in [\[8\]](#page-9-5) that we modify with channel encoding. In addition, it is also shown that six iterations brings about 0.5 dB gain compared to single iteration. The gain benefits from that more accurate *priori* information of data symbols is sent into MSDD and more accurate *posteriori* information of data symbols is sent into decoder, at the cost of extra iterations between SISO MSDD and SISO



<span id="page-7-0"></span>**Fig. 4.** BER comparisons of the coded MSDD with different number of iterations between detection and decoding. Block size *M* = 3.



<span id="page-8-0"></span>**Fig. 5.** The BER comparisons of the coded MSDD with different block size.

channel decoder. Moreover, the BER result of five iterations is near to that of six iterations, which denotes six iterations between SISO MSDD and SISO LDPC decoder can reach converged error performance.

Figure [5](#page-8-0) depicts the BER comparisons of the coded MSDD with different block size. We can observe that when the block size increases, the BER performance is also improved (1.1, 2.3, 3.5 dB at the BER of  $10^{-6}$  for  $M = 2, 3, 5$ ). It also shows the impact of the estimated  $E_q$  on the BER performance. One can see that the estimated  $\hat{E}_q$  almost has the same performance as that of the perfect  $E_q$ , which inspires us that we proposed straightforward way of energy estimation is sufficient for the implementation of our algorithm.

### **6 Conclusions**

In this paper, we propose a noncoherent joint MSDD and channel decoding framework in M-MIMO system with BP message passing algorithm. We have integrated BP for SISO MSDD and BP for channel decoding under the massage passing framework, which enables MSDD for computing the *posteriori* of the data symbols with updated information from channel decoder. Simulations indicate that the proposed joint detection and decoding scheme has significantly improved the performance over the hard-output and the soft-output MSDD, with the price of extra iterations between detector and channel decoder. A noncoherent joint MSDD and channel decoding framework adapted for multi-user in massive MIMO system shall be for future work.

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