On the Finite Population Evolutionary Stable Strategy Equilibrium for Perfect Information Extensive Form Games

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Abstract. This study presents an adaptation of finite population evolutionary stable strategy definition by Schaffer in [1,2] to perfect information extensive form games. In this adaptation, players reach a finite population evolutionary stable strategy equilibrium by using finite population evolutionary stable strategies which ensure that the game ends up with equal payoffs. We studied the fpESS equilibria of some famous two-player bargaining games such as the ultimatum game, the dictatorship game and a dollar auction game. Not all Perfect Information Extensive form games have an fpESS equilibrium. However, when there exist an fpESS equilibrium in these games, the outcome is a perfectly fair one; that is, all players get equal payoffs.

Keywords: Perfect information extensive form game · Ultimatum game · Fairness · Finite population evolutionary stable strategy

1 Introduction

Perfect Information Extensive Form games are very important in game theory. As one of them, the ultimatum game is a widely researched problem. There is some amount of money that the first player is asked to divide between himself and the second player. If the second player does not accept his share, he rejects it and both players take nothing. If he accepts the offer, both players take the amounts that they hold. The dictatorship game is also widely researched in game theory. The first player determines the shares again, but the second player cannot reject the offer.

When we look at the experiments on the ultimatum game, the results that we encounter are very different than the theory predicts. Second players often reject the offers less than half of the money and first players are willing to offer much more than the least that they can offer [3,4]. However, when we look at the experimental results of the dictatorship game, first players are more selfish and they offer much less compared to the offers in the ultimatum game.

The first and the second player consider each other's actions in the perfect information extensive form games. Even if it does not seem rational, it is important to get greater or equal payoff for players. Evolutionary Stable Strategies

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(ESS) take this point into account. Therefore, we decided to study fpESS (finite population ESS) approach to analyze the perfect information extensive form games and especially the ultimatum and the dictatorship games. We show that for some perfect information extensive form games, there are some finite population evolutionary stable strategies that a player can quarantee a payoff at least as large as any opponent's payoff. When players pick one of these strategies, they can prevent to be beaten by their opponents (receiving a smaller payoff than any of the opponent's).

In the rest of the paper we first give, in Sect. 2, the relevant background and definitions for the application of fpESS to Extensive Form games followed by, in Sects. 2.1 and 2.2, the work to find the fpESS equilibria for three well known game instances: the ultimatum game, the dictatorship game and the dollar auction game.

2 Adaptation of FpESS to Extensive Form Games in Induced Form

The fpESS concept is introduced in [5] and restated as the following definition.

Definition 1. *Let S be the strategy set of a symmetric normal form game. An fpESS s is a strategy in which $\forall s^* \in S, u(s, s^*) \geq u(s^*, s)$ (by Schaffer [1, 2]).*

In the fpESS concept, the game is symmetric and the players have the same strategy sets. We applied this approach to the induced normal forms of extensive form games. It is obvious that an induced normal form does not have to be symmetric. However, we can apply this definition to the induced normal form of extensive form games.

Definition 2. *Let S_1, S_2, ..., S_n be the strategy sets in the induced normal form of a perfect information extensive form game with n players. A strategy $s_i \in S_i$ is an fpESS if $\forall j, u_i(s_i, s_{-i}) \geq u_j(s_j, s_{-j})$ for all $s_{-i} \in S_{-i}$ and $s_{-j} \in S_{-j}$ where $S_{-i} = (S_1 \times ...S_{i-1} \times S_{i+1}... \times S_n)$ and $S_{-j} = (S_1 \times ...S_{j-1} \times S_{j+1}... \times S_n)$.*

Definition 2 implies that a strategy s_i is an fpESS for player i if there is no strategy available to any opponent that returns a greater payoff than that of the i^{th} player's payoff.

Definition 3. *If $\forall i$ s_i^* is an fpESS, then (s_1^*,..., s_n^*) is an fpESS equilibrium.*

Example 1. Consider a two player game with the payoffs $z_1 = (10, 10), z_2 = (30, -30), z_3 = (-20, 20)$. This extensive form game tree is shown in Fig. 1(a). When the induced normal form of this game is obtained as shown in Table 1, we see from the row labeled s_1, i.e., $[(10, 10), (30, -30)]$, that the first players payoff is always greater than or equal to the second player. In other words strategy s_1 is an fpESS for the first player. Similarly from the column s_3, which is $[(10, 10), (-20, 20)]$, under this strategy the second player's payoff is at least as big as its opponent, and thus s_3 is an fpESS for the second player. Thus, (s_1, s_3) is an fpESS equilibrium.

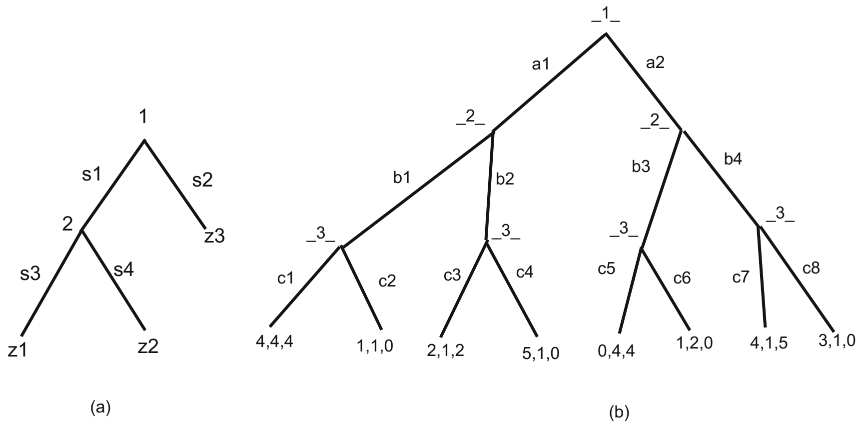

Fig. 1. Two examples of perfect information extensive form games (a) A two player game, (b) A three player game

Table 1. Induced normal form of the game in Fig. 1(a) with the payoffs $z_1 = (10, 10), z_2 = (30, -30), z_3 = (-20, 20)$

	s_3	s_4
s_1	$(10, 10)$	$(30, -30)$
s_2	$(-20, 20)$	$(-20, 20)$

In a perfect information extensive form game, there does not have to be an fpESS equilibrium.

Example 2. Assume that in Fig. 1(a), $z_1 = (10, -10), z_2 = (-20, -30), z_3 = (0, 0)$. Although the first player has two fpESSs (s_1 and s_2), the second player has no fpESS. Therefore, there is no fpESS equilibrium for this game. This shows that a player may not have an fpESS and a game may not have an fpESS equilibrium.

Example 3. We can analyze some famous cases. Pyrrhic victory is one of them in which the result is so devastating that the victor loses everything except the victory. This devastating lost is equal to defeat. We applied this result to the game tree in Fig. 1(a). The terminal nodes are $z_1 = (a_1, b_1), z_2 = (a_2, b_2), z_3 = (a_3, b_3)$.

Let S_1 be going to war, S_2 be not going to war as the strategies of the king Pyrrhus of Epirus and S_3 be accepting the challenge and going to war against Pyrrhus, S_4 be surrendering without battling as the strategies of the Romans at Heraclea in 280 BC.

In this game, if we assume that $a_1 = b_1$, $a_2 > b_2$,$a_3 \leq b_3$, we see that *(going to war as Pyrrhus, going to war as the Romans)* is an fpESS equilibrium. Here our assumption is that when both sides go to war, they will lose everything equally, when Pyrrhus choose not to go to war, the Romans' payoff is greater than or equal to Pyrrhus, and finally when Pyrrhus goes to war and the Romans surrender, the Pyrrhus's payoff is greater than the Romans' payoff. This victory can be modeled in different ways. In this model, we have an fpESS equilibrium. We interpret this as both sides can do anything in order not to be beaten by the opponent in war.

Example 4. Consider the perfect information extensive form game with three players in Fig. 1(b). Here, there are 8 outputs whose paths are (a_1, b_1, c_1), (a_1, b_1, c_2), (a_1, b_2, c_3), (a_1, b_2, c_4), (a_2, b_3, c_5), (a_2, b_3, c_6), (a_2, b_4, c_7), (a_2, b_4, c_8) in this game. $[a_1, (b_1, b_3), (c_1, c_3, c_5, c_7)]$ is an fpESS equilibrium.

In Tables 2 and 3, the second player's payoff is greater than or equal to the first and the third players' payoffs in (a_1, b_1, c_1), (a_1, b_1, c_2), (a_2, b_3, c_5) , (a_2, b_3, c_6).

The third player's payoff is greater than or equal to the first and the second player' payoffs in (a_1, b_1, c_1), (a_1, b_2, c_3), (a_2, b_3, c_5), (a_2, b_4, c_7).

The first player's payoff is greater than or equal to the second and third players' payoffs in (a_1, b_1, c_1), (a_1, b_1, c_2), (a_1, b_2, c_3), (a_1, b_2, c_4).

It may be hard to find a game with an fpESS equilibrium in which all players participate. However, if some of the players have fpESS strategies, they can choose them to play mutually.

Table 2. When the first player selects a_1

	(c_1, c_3)	(c_1, c_4)	(c_2, c_3)	(c_2, c_4)
b_1	(a_1, b_1, c_1)	(a_1, b_1, c_1)	(a_1, b_1, c_2)	(a_1, b_1, c_2)
b_2	(a_1, b_2, c_3)	(a_1, b_2, c_4)	(a_1, b_2, c_3)	(a_1, b_2, c_4)

Table 3. When the first player selects a_2

	(c_5, c_7)	(c_5, c_8)	(c_6, c_7)	(c_6, c_8)
b_3	(a_2, b_3, c_5)	(a_2, b_3, c_5)	(a_2, b_3, c_6)	(a_2, b_3, c_6)
b_4	(a_2, b_4, c_7)	(a_2, b_4, c_8)	(a_2, b_4, c_7)	(a_2, b_4, c_8)

2.1 FpESS Equilibria of Ultimatum and Dictatorship Games

Ultimatum game is a widely researched bargaining problem. We used [6] to express the equilibria in the ultimatum game. In this game, one of the players must divide A dollar as $(A - x, x)$. He takes $A - x$ to himself, gives x to the other player. If the second player accepts this sharing, he takes x and the first player takes $A - x$. If he does not accept the sharing, all the players take *zero* as their payoffs.

Payoffs do not have to be integers. For any value of x, there is a subgame for the second player. In this case, we can analyze the second player's action for each x value. x can be *zero* or greater than *zero*. When x is *zero*, to say *yes* or *no* is indifferent for the second player. When x is greater than *zero*, the second player says *yes* because $x > 0$. By this, we have two optimal strategies for the second player. The first is to say *yes* for $x \geq 0$. The second is to answer *yes* for $x > 0$ and *no* for $x = 0$.

Assume that offered payoffs are not integers. For the second player's first optimal strategy, the first player must offer *zero*. For the second player's second optimal strategy, the first player must offer any value greater than *zero*. Thus, first player's optimal strategy (considering the optimal strategies for player 1) is to offer the smallest $x > 0$. However, if the offers are made in real numbers, there is no such smallest $x > 0$. However, if offered values are integer, for Example 1 cent as the least value, the first player must offer 1 cent to the second player.

In this game, the subgame perfect equilibrium is that the first player offers *zero* and the second player accepts this. When the offered values are integers, we have one more subgame perfect equilibrium so that the first player offers the least value $x > 0$ to the second player, the second player accept this. However, when we look at the experiments, we do not see the theoretical predictions (that is, the first player offers the least positive amount and the second player accepts this minimum offer) are realized. Instead, we encounter more fair outcomes where offers considerably higher than minimum are typical and minimum offers usually rejected.

When we investigate the ultimatum game for any amount N, we see the remarkable feature of fpESS's is that a player does not offer more than half in the first position and a player does not accept less than half. When two players pick fpESS strategies to play, game ends fairly.

Proposition 1. *In an ultimatum game, let $N \in \boldsymbol{R}$ be the total payoff to share. $\forall N \in \boldsymbol{R}$ a strategy $s \in S_1$ in the induced normal form is an fpESS if and only if it includes always to offer less than or equal to $N/2$ as first player's strategy.*

Proof. If $\forall N \in \mathbf{R}$ a strategy $s \in S_1$ in the induced normal form is an fpESS, then it includes always to offer less than or equal to $N/2$ as first player's strategy. Assume that s is an fpESS, but it does not include to offer less than or equal to $N/2$. There exists a cell in the induced matrix row in which the first player offers more than half. The first players payoff becomes less than the second players payoff. This is in contrast with the definition of fpESS. Our assumption is invalid.

If $\forall N \in \mathbf{R}$ a strategy $s \in S_1$ in the induced normal form includes always to offer less than or equal to $N/2$ as first player's strategy, then it is an fpESS. Assume that s includes always to offer less than or equal to $N/2$, but it is not an fpESS. There is a cell where s resides in which the first player's payoff is less than the second player's payoff. This is a contradiction; our assumption is invalid. □

Proposition 2. *In an ultimatum game, $\forall N \in \boldsymbol{R}$ a strategy $s \in S_2$ in the induced normal form is an fpESS if and only if it includes always not to accept any offer less than $N/2$ as second player's strategy.* □

Proof. If $\forall N \in \mathbf{R}$ a strategy $s \in S_2$ in the induced normal form is an fpESS, then it includes always not to accept any offer less than $N/2$ as second player's strategy. Assume that s is an fpESS, but it includes to accept an offer less than $N/2$ as second player's strategy. There exists a cell in the induced matrix column in which the second player accepts less than half. The second players payoff becomes less than the first players payoff. This is contrast with the definition of fpESS. Our assumption is invalid.

If $\forall N \in \mathbf{R}$ a strategy $s \in S_2$ in the induced normal form includes always not to accept any offer less than $N/2$ as second player's strategy, then it is an fpESS. Assume that s includes always not to accept any offer less than $N/2$, but it is not an fpESS. There is a cell where s resides in which the second player's payoff is less than the first player's payoff. However, we accept that s includes always not to accept any offer less than $N/2$ as second player's strategy. This is contradiction. Our assumption is invalid. □

Proposition 3. *In an ultimatum game, $\forall N \in \boldsymbol{R}$, an fpESS equilibrium is an outcome in which $s_1 \in S_1$ and $s_2 \in S_2$ are fpESSs. The payoffs in the fpESS equilibrium are equal.* □

Proof. If s_1 and s_2 are fpESSs, then there are two possible solutions for the game. If the first player offers $N/2$ to the second player, he accepts and the game ends $(N/2, N/2)$ which the payoffs which are equal. If the first player offers less than $N/2$ to the second player, he does not accept and the game ends with the payoffs with equal payoffs $(0, 0)$. □

fpESS equilibrium finalizes the game so that players can't gain advantage over each other. However, when we cannot do anything in a game in order not to be beaten by our opponent, we do not have an fpESS and there is not an fpESS equilibrium in the game. An example of this type of game is dictatorship game where the second player has no power to affect the outcome of the game. Dictatorship game does not include an fpESS for the second player and does not have an fpESS equilibrium.

2.2 FPESS Equilibrium of an Instance of the Dollar Auction Game

The Dollar auction game is a sequential game designed by Martin Shubik [7] to show that players are led to make irrational decisions in a perfect information game. In the game, the winner and the second highest bidder pay the last dollar amount that they bid. The game starts with a randomly selected player. When the first player says 5 cents, the second player can escalade the number by saying 10 cents or he can give up the game. If he says 10 cents, the first player can escalade the number by saying 15 cents, or he can also give up the game. If the first player gives up the game at this point, he pays 5 cents and he wins nothing. The second player pays 10 cents and he wins a dollar.

However, one bids 1.05$, another may bid 1.10$. They escalade the number above 1$ in order not to risk losing the game because they must pay the last dollar amount they bid even if they lose. At this point, bidding above 1$ is not rational to win a dollar as prize.

An example of this game is presented in [8]. There is 3$ as the prize and the maximum amount that a player can bid is 4$. The original game does not have an upper limit, but here there is.

Example 5. We adapted above game so that 2$ is the prize and the maximum amount that a player can bid is 3$. When a player says 3$, he wins the game but loses 1$. The game tree is given in Fig. 2.

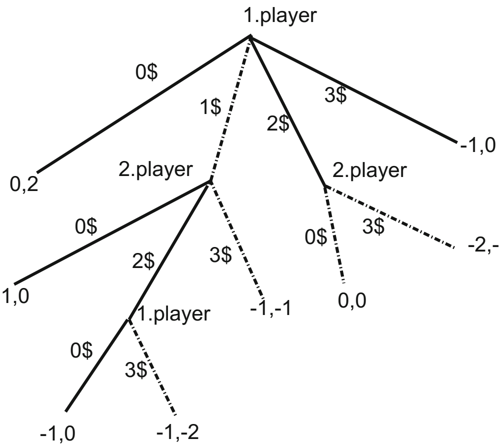

Fig. 2. A dollar auction game example with 2$ prize

We benefited from [9] to analyze the subgame perfect equilibrium for this game. The only difference between the game in Fig. 2 and the game in [9] is one of the first moves of first player. In our game, when the first player bids 0$, the game ends with the payoffs $(0, 2)$. On the other hand, in the game given in [9] if the first player bids 0$, the game ends with the payoffs (0,0). The subgame perfect equilibria are [(1$, 0$), (0$, 0$)], [(2$, 0$), (2$, 0$)], [(0$, 0$), (2$, 0$)], [(1$, 3$), (0$, 0$)]. These two games have the same subgame perfect equilibria because none of the first player's first moves in our game changes.

For example, the strategy that the first player will play in [(1$, 3$), (0$, 0$)] is to play 1$ for the first move and to play 3$ for the last move in the tree. The strategy that the second player will play is to move 0$ for the subtree tied to 1$, to move 0$ for the subtree tied to 2$.

In this game, the first player has one fpESS. At the beginning, his move is 1$. If the second player plays 2$, he plays 3$. (1$, 3$) is the first player's fpESS. The second player has two fpESS's. These are (3$, 0$) and (3$, 3$). If the first player plays 1$, the second player plays 3$. If the first player plays 2$, the second player plays 0$ or 3$. The game ends up with $(-1, -1)$. We can interpret the second player's behavior as risking himself and the other player in order not to be beaten in the game. He does not want to gain less than or lose more than the opponent.

3 Conclusion

When we determine a strategy to play in a perfect information extensive form game, we consider our opponents' possible strategies. If we do not want to be beaten by our opponent at any cost, we use finite population evolutionary stable strategies if exist. It is hard to say that a player who uses these strategies is rational, but we frequently encounter this attitude in real life.

In an fpESS equilibrium, the players select the strategies so that they don't get less than their opponent. It guarantees that when we use an fpESS, the game will end up with a tie. This may bring a new understanding for player attitudes and their positions in the perfect information extensive form games.

In ultimatum game, the theoretical solutions do not explain experimental results thoroughly. In ultimatum games, when the second player picks a strategy that brings him to an fpESS equilibrium, he can't be worse off than the first player. When the first player knows this, he does not offer any unacceptable amount to the second player. He knows that any unfair (less than half the total amount) offer will be rejected. Any fpESS strategy which brings the players to an fpESS equilibrium ensures the game ends up fairly.

In dictatorship game, there is nothing to do for the second player when he does not want to gain less than the first player, thus there does not exist any fpESS and fpESS equilibrium in the dictatorship game.

In the instance of dollar auction game that we have analyzed, the fpESS equilibrium can be interpreted as any bidder can risk losing more and more by continuing with higher bids. A player can prefer staying in the game and losing equally (with the opponent) to withdrawing from the game earlier with relatively more loss.

We suppose there may be an intersection between subgame perfect equilibrium solution concept and fpESS equilibrium, it is a new research topic.

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