Capacity Region of the Dirty Two-Way Relay Channel to Within Constant Bits

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Abstract. In this paper, we consider a dirty Gaussian two-way relay channel, where the two user nodes exchange messages with the help of a relay node. The three nodes experience two independent additive interferences which are assumed to be known at some nodes. We consider two cases: (1) the two user nodes know each of the two interferences respectively; (2) the relay node knows both the interferences. With nested lattice coding and compute-and-forward relaying, we derive achievable rate regions for the above two cases. The achievable rate regions are shown to be within constant bits from the cut-set outer bound for all channel parameters regardless of the interferences. Comparing the two achievable rate regions of the above two cases, it is shown that more information about the interferences the relay node knows more interferences will be canceled, larger achievable rate region can be achieved.

Keywords: Two-way relay channel · Dirty paper channel · Nested lattice coding · Compute-and-forward

1 Introduction

The channel with channel state information causally known at the source node was first considered by Shannon [1]. For the channel with state information non-causally known at the encoder, Gel'fand and Pinsker [2] derived the capacity for general discrete memoryless channels and Costa [3] derived the capacity of the Gaussian dirty channel with dirty-paper coding.

The two-way relay channel in which two users wish to exchange messages with the help of a third relay node is a practical channel model for wireless communication systems. A number of coding schemes and relaying strategies have been proposed for the two-way relay channels. Lattice coding which has been shown to be good for almost everything [4] was applied to two-way relay channels during recent years and was demonstrated to be capacity-approaching [5–9]. Nam *et al.* [6] proposed a scheme based on nested lattice codes formed from a lattice chain. They exploited the structural gain of computation coding and derived the capacity region for the two-way relay channel to with 1/2 bit. In [7], a lattice-based achievability strategy was proposed to derive a symmetric rate which is within $\frac{1}{2}\log(3)$ bits of the capacity for the two-way two-relay channel.

Q. Chen et al. (Eds.): ChinaCom 2016, Part I, LNICST 209, pp. 305–315, 2018.
DOI: 10.1007/978-3-319-66625-9_30

For some certain classes of channels with side information, Zamir *et al.* [9] used lattice codes to derive the capacity of these channels. The nested approach of [9] for the dirty-paper channels is extended to multi-user dirty-paper channels, e.g. the authors in [10] showed that lattice-based binning seemed to be necessary to achieve capacity of the dirty multiple access channel (MAC). In [11], Song *et al.* studied the two-hop Gaussian relay channel with a source, a relay and a destination. The destination experienced an additive interference which is known to the source non-causally. They proposed a new achievable scheme based on nested lattice code and decode-and-forward (DF) relaying. This strategy used the structure provided by nested lattice codes to cancel part of the interference at the source and achieved a rate to within 1/2 bit of the clean channel.

In this paper, we consider a Gaussian two-way relay channel with channel state information. The two user nodes (node 1 and node 2) exchange messages with the help of a relay node (node 3). The three nodes experience two independent additive Gaussian interferences S_1 and S_2 which can be viewed as the signals transmitted from the primary users in cognitive radio systems. We assume that only part of the nodes in the two-way relay channel are capable of acquiring some knowledge about the interferences S_1 and S_2. Thus, the state-dependent two-way relay channel studied in this paper can be viewed as a secondary relay communication with some cognitive nodes. The nodes knowing the interference S_1 or S_2 may adapt their coding schemes to mitigate the interferences caused by the primary communication. We consider the following two cases: (1) user node 1 and user node 2 know the interferences S_1 and S_2 respectively; (2) the relay node knows both interferences S_1 and S_2. We generalize the lattice coding schemes used in [6,11], and derive corresponding inner bounds for the capacity regions of the above two cases based on compute-and-forward relaying at the relay node and nested lattice coding with interferences pre-cancellation at the nodes which know the interference. With these achievable schemes, the achievable rate regions derived for the above two cases are within constant bits from the cut-set outer bounds for all channel parameters regardless of the interferences.

2 Channel Model and Lattice Coding Preliminaries

2.1 Channel Model

As shown in Fig. 1, we consider a two-way relay channel with interferences. We assume that there is no direct path between the two user nodes. The channel is corrupted by two independent additive Gaussian interferences S_1 and S_2 known non-causally at some of the nodes. The message $w_i \in \{1, 2, ..., 2^{nR_i}\}$ is uniformly distributed over the message set $\mathcal{W}_i = \left\{1, 2, ..., 2^{nR_i}\right\}$, where $i \in \{1, 2\}$, n is the number of channel uses, and R_i denotes the rate of message w_i. The messages w_1 and w_2 are independent of each other. We let $\mathbf{X}_i^n = (X_i(1), X_i(2), ..., X_i(n))$ where $X_i(k)$ denotes the input from node i at channel use k (and similarly for the channel outputs $\mathbf{Y}_i^n$ of node i) where $i \in \{1, 2, 3\}$. Node i transmits $X_i(k)$ at time k to the relay through the channel specified by

$$Y_3(k) = X_1(k) + X_2(k) + S_1(k) + S_2(k) + Z_3(k) \quad (1)$$

where $Z_3(k)$ is an independent identically distributed (i.i.d.) Gaussian random variable with zero mean and variance 1, S_1 and S_2 are the two additive i.i.d. Gaussian interferences with zero mean and variance Q_1 and Q_2 respectively.

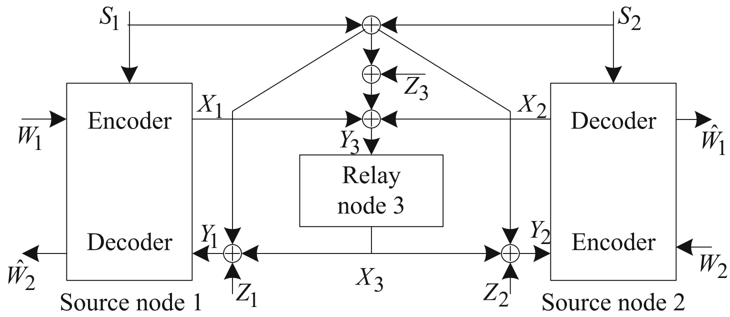

Fig. 1. Dirty two-way relay channel

The relay node 3 transmits $X_i(k)$ to user node 1 and user node 2 through the channel specified by

$$Y_i(k) = X_3(k) + S_1(k) + S_2(k) + Z_i(k) \tag{2}$$

where $Z_i(k)$ is an additive white Gaussian noise with zero mean and variance 1.

For the first case when user node 1 and user node 2 know the channel state information non-causally, a $(2^{nR_1}, 2^{nR_2}, n)$-code consists of message w_i uniformly distributed over the message set $\mathcal{W}_i = \left\{1, 2, ..., 2^{nR_i}\right\}$; two encoding functions at node 1 and node 2

$$f_i^n : \{1, 2, ..., 2^{nR_i}\} \times \mathcal{S}_i^n \to \mathbb{R}^n \tag{3}$$

such that $\frac{1}{n}\sum_{k=1} E(x_{i,k}^2) \leq P_i$, where $i = 1, 2$; a series of encoding functions $\{f_3^{(k)}\}_{k=1}^n$ at the relay node 3 such that $X_3(k) = f_3^{(k)}\left(\mathbf{Y}_3^{k-1}\right)$ and $\frac{1}{n}\sum_{k=1}^n \left(X_3(k)\right)^2 \leq P_3$; decoding functions at node 1 and node 2

$$g_i^n : \mathcal{Y}_i^n \times \mathcal{S}_i^n \times \mathcal{W}_i \to \{1, 2, ..., 2^{nR_{\bar{i}}}\} \tag{4}$$

where, $i, \bar{i} \in \{1, 2\}, i \neq \bar{i}$.

The definition of $(2^{nR_1}, 2^{nR_2}, n)$-code for the second case when the relay node knows both S_1 and S_2 is similar to that for the first case, except that the encoding functions and decoding functions at node 1 and node 2 should be replaced by $f_i^n : \{1, 2, ..., 2^{nR_i}\} \to \mathbb{R}^n$ and $g_i^n : \mathcal{Y}_i^n \times \mathcal{W}_i \to \{1, 2, ..., 2^{nR_{\bar{i}}}\}$, $i, \bar{i} \in \{1, 2\}, i \neq \bar{i}$, respectively, and encoding functions at the relay node should be replaced by $X_3(k) = f_3^{(k)}\left(\mathbf{Y}_3^{k-1}, \mathbf{S}_1^n, \mathbf{S}_2^n\right)$.

The decoding error probability is defined as

$$P_{n,e} = \frac{1}{2^{n(R_1+R_2)}} \sum\nolimits_{w_1 \in \mathcal{W}_1, w_2 \in \mathcal{W}_2} \Pr\left((w_1, w_2) \neq (\hat{w}_1, \hat{w}_2) \,|\, (w_1, w_2) \text{ was sent}\right)$$

For any $\varepsilon > 0$ and sufficiently large n, if there exists $(2^{nR_1}, 2^{nR_2}, n)$-code such that $P_{n,e} < \varepsilon$, the rate pair (R_1, R_2) is said to be achievable. The capacity region for the dirty two-way relay channel is defined as the supremum of the set of all achievable rate pairs.

2.2 Lattice Coding Preliminaries

We briefly outline the notations and definitions for nested lattice codes. For details of the lattice coding, please refer to [4,12] and the references therein.

An n-dimensional lattice Λ is a discrete group in the Euclidian space $\mathbb{R}^n$ which is closed with respect to the addition and reflection operations. The lattice is specified by $\Lambda = \{\lambda = G \cdot \mathbf{i} : \mathbf{i} \in \mathbb{Z}^n\}$, where G is a $n \times n$ real valued matrix.

The nearest neighbor quantizer $Q_\Lambda(\cdot)$ is defined by $Q_\Lambda(\mathbf{x}) = \arg\min_{\lambda \in \Lambda} \|\mathbf{x} - \lambda\|$, where $\|\cdot\|$ denotes Euclidian norm. The fundamental Voronoi region of Λ is defined as $\mathcal{V}(\Lambda) = \{\mathbf{x} \in \mathbb{R}^n : Q_\Lambda(\mathbf{x}) = 0\}$. The modulo lattice operation with respect to Λ is defined as $\mathbf{x} \bmod \Lambda = \mathbf{x} - Q_\Lambda(\mathbf{x})$. The second moment of a lattice Λ is given by $\sigma^2(\Lambda) = \frac{1}{V} \cdot \frac{1}{n} \int_{\mathcal{V}} \|\mathbf{x}\|^2 d\mathbf{x}$, and the normalized second moment of lattice Λ is given by $G(\Lambda) = \frac{\sigma^2(\Lambda)}{V^{\frac{2}{n}}}$, where V is the volume of the Voronoi region.

A lattice Λ is said to be Rogers-good if $\lim_{n\to\infty} G(\Lambda) = \frac{1}{2\pi e}$ and Poltyrev-good if $\Pr\{Z \notin \mathcal{V}\} \leqslant e^{-nE_p(\mu)}$ for any $Z \in \mathcal{N}(0, \sigma^2\mathbf{I})$, where $\mu = \frac{(\mathrm{Vol}(\mathcal{V}))^{2/n}}{2\pi e \sigma^2}$ is the volume-noise ratio.

A nested lattice code is defined in terms of two n-dimensional lattices Λ and Λ_c such that $\Lambda \subseteq \Lambda_c$ with fundamental regions $\mathcal{V}$, $\mathcal{V}_c$ of volumes $V := \mathrm{Vol}(\mathcal{V})$, $V_c := \mathrm{Vol}(\mathcal{V}_c)$ respectively. Lattice Λ is called the coarse lattice which is a sublattice of the fine lattice Λ_c. The set $\mathcal{C}_{\Lambda_c,\Lambda} = \{\Lambda_c \cap \mathcal{V}\}$ can be employed as the codebook using Λ_c as codewords and the Voronoi region $\mathcal{V}$ of Λ as a shaping region. The coding rate R is defined as $R = \frac{1}{n} \log |\mathcal{C}_{\Lambda_c,\Lambda}| = \frac{1}{n} \log \frac{V}{V_c}$.

3 Main Results

3.1 Achievable Rate Region When the Two User Nodes Know S_1 and S_2 Respectively

Since the two user nodes 1 and 2 know only part of the interferences, each user node pre-cancels part of the interferences according to their own knowledge about the interferences using the nested lattice codes. The relay node decodes and forwards the function of the codewords transmitted from the two user nodes exploiting the structure property of the nested lattice codes. The two user nodes decode the message from the other node using their own messages as side information. Combining the structured interference pre-cancelation technique in [11] and the compute-and-forward relaying used to derive the achievable rate region for the two-way relay channel by Nam in [6], an achievable rate region is derived for the dirty two-way relay channel studied in this paper as shown in the following theorem.

Theorem 1. *For the dirty Gaussian two-way relay channel with partial channel state information known at the user nodes as shown in Fig. 1, the rate region which is the closure of the set of all points* (R_1, R_2) *satisfying*

$$\begin{aligned} R_1 &< \min\left\{\left[\frac{1}{2}\log\left(\frac{P_1}{\frac{P_1+P_2}{P_1+P_2+1}+\frac{P_2}{P_2+1}}\right)\right]^+, \left[\frac{1}{2}\log\left(\frac{P_3}{\frac{2P_3}{2P_3+1}+\frac{P_3}{P_3+1}}\right)\right]^+\right\} \\ R_2 &< \min\left\{\left[\frac{1}{2}\log\left(\frac{P_2}{\frac{P_1+P_2}{P_1+P_2+1}+\frac{P_2}{P_2+1}}\right)\right]^+, \left[\frac{1}{2}\log\left(\frac{P_3}{\frac{2P_3}{2P_3+1}+\frac{P_3}{P_3+1}}\right)\right]^+\right\} \end{aligned} \quad (5)$$

is achievable and is within 1 bit from the cut-set outer bound, where $[x]^+ \triangleq \max\{x, 0\}$.

Remark 1. Compared with the achievable rate region of the two-way relay channel with no interference considered in [6] which is shown to be within 1/2 bit from the outer bound, the achievable rate region of the two-way relay channel with two additional interferences derived in this paper is within 1 bit from the outer bound. This 1/2 bit larger gap is due to the fact that the two user nodes know only part of the interference.

Proof. Without loss of generality, we assume $P_1 \geq P_2$. We will prove the theorem in two cases: (1) $P_3 \geq P_2$; (2) $P_2 \geq P_3$.

(1) the first case: $P_3 \geq P_2$
For the first case, let us consider a good nested lattice chain $\Lambda_1 \subset \Lambda_2 \subset \Lambda_c \subset \Lambda_q$ as in Sect. 2.2, where the second moment of each lattice is constrained to be $\sigma^2(\Lambda_1) = \min(P_1, P_3)$, $\sigma^2(\Lambda_2) = P_2$, $\sigma^2(\Lambda_c) = \sigma_c^2$ and $\sigma^2(\Lambda_q) = \sigma_q^2$. The lattices Λ_1, Λ_2 and Λ_q are both Rogers good and Poltyrev good, while Λ_c is Poltyrev good. The proof in [13] ensures the existence of such lattice chain. The Voronoi regions of the lattices Λ_1, Λ_2, Λ_c, Λ_q are denoted by $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_c$, $\mathcal{V}_q$ of volumes V_1, V_2, V_c, V_q respectively.

Encoding at the User Nodes. We use $\mathcal{C}_1 = \{\Lambda_c \cap \mathcal{V}_1\}$ for node 1, $\mathcal{C}_2 = \{\Lambda_c \cap \mathcal{V}_2\}$ for node 2. For node i, each message $w_i \in \{1, 2, ..., 2^{nR_i}\}$ is one-to-one mapped to the lattice point $\mathbf{t}_i \in \mathcal{C}_i$, where $R_i = \frac{1}{n}\log(\frac{V_i}{V_c})$. We also define two sets $\mathcal{C}_{q,1} = \{\Lambda_q \cap \mathcal{V}_1\}$ and $\mathcal{C}_{q,2} = \{\Lambda_q \cap \mathcal{V}_2\}$ for quantizing the interferences at node 1 and node 2 with quantization rates $R_{q,1} = \frac{1}{n}\log(\frac{V_1}{V_q})$ and $R_{q,2} = \frac{1}{n}\log(\frac{V_2}{V_q})$ respectively. To transmit a message w_i, node i chooses $\mathbf{t}_i$ associated with the message and sends

$$\mathbf{X}_i = (\mathbf{T}_i - \alpha\mathbf{S}_i + \mathbf{U}_i) \bmod \Lambda_i$$

where $\mathbf{T}_i = (\mathbf{t}_i - Q_{\Lambda_q}(\alpha_i\mathbf{S}_i + \mathbf{U}_{q_i})) \bmod \Lambda_i \in \mathcal{C}_{q,i}$, $\mathbf{U}_i$ is the channel coding dither uniformly distributed over $\mathcal{V}_i$ and is known to the relay node, $\mathbf{U}_{q_i}$ is the quantization dither uniformly distributed over $\mathcal{V}_q$. From the dithered quantization property, $\mathbf{X}_i$ is uniformly distributed over $\mathcal{V}_i$ and is independent of $\mathbf{T}_i$.

Encoding at the Relay Nodes. The relay receives $\mathbf{Y}_3 = \mathbf{X}_1+\mathbf{X}_2+\mathbf{S}_1+\mathbf{S}_2+\mathbf{Z}_3$ and computes

$$\begin{aligned}\tilde{\mathbf{Y}}_3 &= (\alpha\mathbf{Y}_3 - \mathbf{U}_1 - \mathbf{U}_2) \bmod \Lambda_1 \\ &= [\mathbf{T}_1 + \mathbf{T}_2 - Q_{\Lambda_2}(\mathbf{T}_2 - \alpha\mathbf{S}_2 + \mathbf{U}_2) - (1-\alpha)(\mathbf{X}_1 + \mathbf{X}_2) + \alpha\mathbf{Z}_3] \bmod \Lambda_1 \\ &= [\mathbf{T}_3 - (1-\alpha)(\mathbf{X}_1 + \mathbf{X}_2) + \alpha\mathbf{Z}_3] \bmod \Lambda_1\end{aligned}$$

where $\mathbf{T}_3 = (\mathbf{T}_1+\mathbf{T}_2-Q_{\Lambda_2}(\mathbf{T}_2-\alpha\mathbf{S}_2+\mathbf{U}_2)) \bmod \Lambda_1$. Since $\Lambda_1 \subset \Lambda_2 \subset \Lambda_c \subset \Lambda_q$, it follows that $\mathbf{T}_3 \in \mathcal{C}_{q1}$. Using the crypto-lemma, $\mathbf{T}_3$ is uniformly distributed over $\mathcal{C}_{q1}$ and independent of $-(1-\alpha)(\mathbf{X}_1 + \mathbf{X}_2)$ and $\alpha\mathbf{Z}_3$ which can be seen as two independent noise terms. According to Theorem 3 in [6], choosing $\alpha = \frac{\min(P_1,P_3)+P_2}{\min(P_1,P_3)+P_2+1}$, the relay can decode $\hat{\mathbf{T}}_3$ successfully with the error probability $\Pr\{\hat{\mathbf{T}}_3 \neq \mathbf{T}_3\}$ vanishes as $n \to \infty$ if

$$\begin{aligned}R_{q1} &< \left[\tfrac{1}{2}\log\left(\min(P_1,P_3)/(\min(P_1,P_3)+P_2) + \min(P_1,P_3)\right)\right]^+ \\ R_{q2} &< \left[\tfrac{1}{2}\log\left(P_2/(\min(P_1,P_3)+P_2) + P_2\right)\right]^+\end{aligned} \tag{6}$$

Following that $R_{qi} = \frac{\sigma^2(\Lambda_i)}{\sigma^2(\Lambda_q)}$, the following inequality must be satisfied.

$$\sigma_q^2 = \sigma^2(\Lambda_q) > \min(P_1,P_3) + P_2/(\min(P_1,P_3)+P_2+1) \tag{7}$$

The relay transmits $\mathbf{X}_3 = (\hat{\mathbf{T}}_3 + \mathbf{U}_3) \bmod \Lambda_1$ where $\mathbf{U}_3$ is the channel coding dither known at the user nodes 1 and 2. Again, according to the crypto-lemma, $\mathbf{X}_3$ is uniformly distributed over $\mathcal{V}_1$ and independent of $\hat{\mathbf{T}}_3$.

Decoding at the User Node 1. The user node 1 estimates the message $\hat{w}_2$ by its received vector $\mathbf{Y}_1 = \mathbf{X}_3 + \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{Z}_1$.

Since the user node 1 has known the interference $\mathbf{S}_1$ in advance, it subtracts it from $\mathbf{Y}_1$ to derive $\mathbf{Y'}_1 = \mathbf{X}_3 + \mathbf{S}_2 + \mathbf{Z}_1$. It then computes

$$\begin{aligned}\tilde{\mathbf{Y}}_1 &= (\alpha_2\mathbf{Y'}_1 + \mathbf{U}_{q_2} - \mathbf{U}_3 - \mathbf{T}_1) \bmod \Lambda_2 \\ &= [(\mathbf{T}_3 + \mathbf{U}_3) \bmod \Lambda_1 - (1-\alpha_2)\mathbf{X}_3 \\ &\quad + \alpha_2(\mathbf{S}_2 + \mathbf{Z}_1) + \mathbf{U}_{q2} - \mathbf{U}_3 - \mathbf{T}_1] \bmod \Lambda_2 \\ &\overset{(a)}{=} [\mathbf{T}_2 - (1-\alpha_2)\mathbf{X}_3 + \alpha_2\mathbf{S}_2 + \alpha_2\mathbf{Z}_1 + \mathbf{U}_{q_2}] \bmod \Lambda_2 \\ &= \mathbf{t}_2 + (\alpha_2\mathbf{S}_2 + \mathbf{U}_{q_2}) \bmod \Lambda_q - (1-\alpha_2)\mathbf{X}_3 + \alpha_2\mathbf{Z}_1 \bmod \Lambda_2\end{aligned} \tag{8}$$

where (a) follows from the facts that $\mathbf{T}_3 = (\mathbf{T}_1 + \mathbf{T}_2 - Q_{\Lambda_2}(\mathbf{T}_2 - \alpha\mathbf{S}_2 + \mathbf{U}_2)) \bmod \Lambda_1$ and the mod-lattice operation $(\mathbf{x} \bmod \Lambda_1) \bmod \Lambda_2 = \mathbf{x} \bmod \Lambda_2$ resulting from the fact $\Lambda_1 \subset \Lambda_2$.

From (8), it is easy to find that $(\alpha_2\mathbf{S}_2 + \mathbf{U}_{q_2}) \bmod \Lambda_q$ is a random variable uniformly distributed over $\mathcal{V}_q$ and independent of all the others. Thus, $(\alpha_2\mathbf{S}_2 + \mathbf{U}_{q_2}) \bmod \Lambda_q$, $-(1-\alpha_2)\mathbf{X}_3$ and $\alpha_2\mathbf{Z}_1$ can be seen as three independent noise terms with variance σ_q^2, $(1-\alpha)^2\min(P_1,P_3)$ and α_2^2 respectively. Choosing $\alpha_2 = \frac{\min(P_1,P_3)}{\min(P_1,P_3)+1}$, node 1 decodes $\mathbf{t}_2$ by lattice decoding $\hat{\mathbf{t}}_2 = Q_{\Lambda_c}(\tilde{\mathbf{Y}}_1)$ where

$Q_{\Lambda_c}(\cdot)$ denotes the nearest neighbor lattice quantizer associated with Λ_c. The error probability $\Pr\left\{\hat{\mathbf{t}}_2 \neq \mathbf{t}_2\right\}$ vanishes as $n \to \infty$, if

$$R_2 < \frac{1}{2}\log\left(P_2 \Big/ \left(\sigma_q^2 + \frac{\min(P_1, P_3)}{\min(P_1, P_3) + 1}\right)\right) \tag{9}$$

Considering the inequality (7), we have

$$R_2 < \frac{1}{2}\log\left(P_2 \Big/ \left(\frac{\min(P_1, P_3) + P_2}{\min(P_1, P_3) + P_2 + 1} + \frac{\min(P_1, P_3)}{\min(P_1, P_3) + 1}\right)\right) \tag{10}$$

Decoding at the User Node 2. The user node 2 estimates the message $\hat{w}_1$ by its received vector $\mathbf{Y}_2$. Taking similar steps as decoding at the user node 1, we can derive the following achievable rate of the message w_1

$$R_1 < \frac{1}{2}\log\left(\min(P_1, P_3) \Big/ \left(\frac{\min(P_1, P_3) + P_2}{\min(P_1, P_3) + P_2 + 1} + \frac{\min(P_1, P_3)}{\min(P_1, P_3) + 1}\right)\right) \tag{11}$$

(2) the second case $P_2 \geq P_3$,
For the case $P_2 \geq P_3$, we let $\Lambda_1 = \Lambda_2$ of the nested lattice chain $\Lambda_1 \subset \Lambda_2 \subset \Lambda_c \subset \Lambda_q$ used in the first case. The second moments of both the two lattices Λ_1 and Λ_2 are restricted to be P_3. In this case, all the three nodes will transmit with the same average power P_3. Taking the same encoding and decoding steps as in the first case, we can derive the following achievable rate pair (R_1, R_2).

$$R_1 = R_2 < \frac{1}{2}\log\left(P_3 \Big/ \left(\frac{2P_3}{2P_3 + 1} + \frac{P_3}{P_3 + 1}\right)\right) \tag{12}$$

Therefore, according to Eqs. (10)–(12), we conclude that the following rate pair (R_1, R_2) is achievable

$$\begin{aligned} R_1 &< \min\left\{\left[\tfrac{1}{2}\log\left(\tfrac{P_1}{\frac{P_1+P_2}{P_1+P_2+1} + \frac{P_2}{P_2+1}}\right)\right]^+, \left[\tfrac{1}{2}\log\left(\tfrac{P_3}{\frac{2P_3}{2P_3+1} + \frac{P_3}{P_3+1}}\right)\right]^+\right\} \\ R_2 &< \min\left\{\left[\tfrac{1}{2}\log\left(\tfrac{P_2}{\frac{P_1+P_2}{P_1+P_2+1} + \frac{P_1}{P_1+1}}\right)\right]^+, \left[\tfrac{1}{2}\log\left(\tfrac{P_3}{\frac{2P_3}{2P_3+1} + \frac{P_3}{P_3+1}}\right)\right]^+\right\} \end{aligned} \tag{13}$$

Next, we will show that the above achievable rate pair is within 1 bit from the cut-set outer bound which is given by

$$\begin{aligned} R_1 &\leq \min\left\{\tfrac{1}{2}\log(1 + P_1), \tfrac{1}{2}\log(1 + P_3)\right\} \\ R_2 &\leq \min\left\{\tfrac{1}{2}\log(1 + P_2), \tfrac{1}{2}\log(1 + P_3)\right\} \end{aligned} \tag{14}$$

Considering (13) and (14), we can conclude that

$$\begin{aligned} \tfrac{1}{2}\log\left(\tfrac{P_1}{\frac{P_1+P_2}{P_1+P_2+1} + \frac{P_2}{P_2+1}}\right) &> \tfrac{1}{2}\log\left(\tfrac{P_1}{\frac{P_1+P_2}{P_1+P_2+1} + \frac{P_1+P_2}{P_1+P_2+1}}\right) \\ &= \tfrac{1}{2}\log\left(\tfrac{P_1}{P_1+P_2} + P_1\right) - \tfrac{1}{2} \\ &> \tfrac{1}{2}\log(1 + P_1) - 1 \end{aligned} \tag{15}$$

where the last inequality is due to the fact $\frac{1}{2}\log\left(\frac{P_1}{P_1+P_2}+P_1\right) > \frac{1}{2}\log(1+P_1)-\frac{1}{2}$ which has been shown in [6]. Similarly, we have

$$\frac{1}{2}\log\left(\frac{P_2}{\frac{P_1+P_2}{P_1+P_2+1}+\frac{P_1}{P_1+1}}\right) > \frac{1}{2}\log(1+P_2)-1 \tag{16}$$

$$\frac{1}{2}\log\left(\frac{P_3}{\frac{2P_3}{2P_3+1}+\frac{P_3}{P_3+1}}\right) > \frac{1}{2}\log(1+P_3)-1 \tag{17}$$

The three inequalities (15)–(17) show that the achievable rate region derived in this paper is within 1 bit from the cut-set outer bound.

3.2 Achievable Rate Region When Relay Knows Both S_1 and S_2

When the relay node knows both interferences S_1 and S_2, it can subtract the two interferences before decoding since the interferences are additive. Therefore, the uplink (the channel from the two user nodes to the relay node) can be viewed as a clean channel. The downlink (the channel from the relay node to the two user nodes), however, is dirty. The relay node helps to eliminate the interferences with dirty paper coding.

Theorem 2. *For the dirty Gaussian two-way relay channel, when relay node 3 knows both the interferences S_1 and S_2, the rate region which is the closure of the set of all points (R_1, R_2) satisfying*

$$\begin{aligned} R_1 &< \min\left\{\left[\frac{1}{2}\log\left(\frac{P_1}{P_1+P_2}+P_1\right)\right]^+, \left[\frac{1}{2}\log(1+P_3)\right]^+\right\} \\ R_2 &< \min\left\{\left[\frac{1}{2}\log\left(\frac{P_2}{P_1+P_2}+P_2\right)\right]^+, \left[\frac{1}{2}\log(1+P_3)\right]^+\right\} \end{aligned} \tag{18}$$

is achievable and is within 1/2 bit from the cut-set outer bound, where $[x]^+ \triangleq \max\{x, 0\}$.

Remark 2. The achievable rate region derived in Theorem 3 is the same as that derived in the work of Nam [6] in which the two-way relay channel is not corrupted by extra interferences. This means that the interferences can be eliminated completely by the relay node with dirty paper coding when the relay node knows all the interferences in advance.

Proof. Again, without loss of generality, we assume $P_1 \geq P_2$ and construct a good nested lattice chain $\Lambda_1 \subset \Lambda_2 \subset \Lambda_c \subset \Lambda_q$ where the second moment of each lattice is constrained to be $\sigma^2(\Lambda_1) = \min(P_1, P_3)$, $\sigma^2(\Lambda_2) = P_2$, $\sigma^2(\Lambda_c) = \sigma_c^2$ and $\sigma^2(\Lambda_q) = \sigma_q^2$, Λ_1, Λ_2 and Λ_q are both Rogers good and Poltyrev good while Λ_c is Poltyrev good. Two codebooks $\mathcal{C}_1 = \{\Lambda_c \cap V(\Lambda_1)\}$ and $\mathcal{C}_2 = \{\Lambda_c \cap V(\Lambda_2)\}$ are generated.

Since the interferences S_1 and S_2 are additive and known to the relay node, they can be subtracted from the signals received at the relay node. Therefore, the uplink channel can be viewed as a clean channel. Following the same steps in [6], the relay node can estimate $\hat{\mathbf{T}}_3 = \mathbf{T}_3 = (\mathbf{t}_1 + \mathbf{t}_2 - Q_{\Lambda_2}(\mathbf{t}_2 + \mathbf{U}_2)) \bmod \Lambda_1$ with error probability approaching 0 if the rate pair (R_1, R_2) satisfies (19), where $\mathbf{t}_1 \in \mathcal{C}_1$ and $\mathbf{t}_2 \in \mathcal{C}_2$ are the lattice points associated with the messages transmitted by user node 1 and user node 2 respectively.

$$\begin{aligned} R_1 &< \tfrac{1}{2}\log\left(\tfrac{P_1}{P_1+P_2} + P_1\right) \\ R_2 &< \tfrac{1}{2}\log\left(\tfrac{P_2}{P_1+P_2} + P_2\right) \end{aligned} \tag{19}$$

Having successfully decoded $\hat{\mathbf{T}}_3$, the relay node sends $\hat{\mathbf{T}}_3$ to user node 1 and user node 2 using Gaussian codebooks. Again, we assume $P_1 \geq P_2$, thus $R_1 \geq R_2$ for the uplink. Fix a measure P_{U,S_1,S_2}. Generate $2^{n(R_1+R_s)}$ i.i.d. codewords $\{\mathbf{u}^n(t_3, t_s)\}$ each with i.i.d. components drawn according to P_U. Randomly and uniformly distribute $2^{n(R_1+R_s)}$ codewords $\{\mathbf{u}^n(t_3, t_s)\}$ into 2^{nR_1} bins each with 2^{nR_s} codewords. We assume one-to-one correspondence between $\hat{\mathbf{T}}_3 \in \mathcal{C}_1$ and the bin index t_3 and denote it as $t_3(\hat{\mathbf{T}}_3)$. It is easy to verify $t_3(\hat{\mathbf{T}}_3)$ is uniformly distributed over $\{1, 2, ..., 2^{nR_1}\}$ since $\hat{\mathbf{T}}_3$ is uniformly distributed over $\mathcal{C}_1$. Knowing the interferences non-causally, the relay node searches the smallest $\tilde{t}_s \in \{1, 2, ..., 2^{nR_s}\}$ from the bin indexed by $t_3(\hat{\mathbf{T}}_3)$ such that $\mathbf{u}^n(t_3(\hat{\mathbf{T}}_3), \tilde{t}_s)$ is jointly typical with $\mathbf{s}_1^n$ and $\mathbf{s}_2^n$. By the covering lemma [14], if $R_s > I(U; S_1, S_2)$, there exists at least one such codeword. The relay node then transmits $\mathbf{x}_3^n = \mathbf{u}^n(t_3(\hat{\mathbf{T}}_3), \tilde{t}_s) - \alpha_r(\mathbf{s}_1^n + \mathbf{s}_2^n)$.

User node 1 estimates $\hat{\mathbf{T}}_3$ according to its received vector $\mathbf{Y}_1$. It chooses one unique codeword $\mathbf{u}^n(t_3(\hat{\mathbf{T}}_{3,1}), \tilde{t}_s) \in \mathcal{C}_{r,1}$ such that $\mathbf{u}^n(t_3(\hat{\mathbf{T}}_{3,1}), \tilde{t}_s)$ and $\mathbf{Y}_1$ are jointly typical, where

$$\mathcal{C}_{r,1} = \{\mathbf{u}^n(t_3(\mathbf{T}), t_s) : \mathbf{T} = \mathbf{t}_1 + \mathbf{t}'_2 - Q_{\Lambda_2}(\mathbf{t}_2 + \mathbf{U}_2), \mathbf{t}'_2 \in \mathcal{C}_2, t_s \in [1 : 2^{nR_s}]\}$$

$\mathbf{t}_1$ is the lattice point associated with the message transmitted by itself. Using $\mathbf{t}_1$ as side information, user node 1 estimates the message of user node 2 as

$$\hat{\mathbf{t}}_2 = (\hat{\mathbf{T}}_{3,1} - \mathbf{t}_1) \bmod \Lambda_2 \tag{20}$$

If and only if $\hat{\mathbf{T}}_{3,1} = \hat{\mathbf{T}}_3$, the probability that user node 1 successfully estimates the message $\hat{\mathbf{t}}_2 = \mathbf{t}_2$ from user node 2 tends to 1. Notice that the cardinality of $\mathcal{C}_{r,1}$ is $2^{n(R_2+R_s)}$. Thus, from the argument of GP-coding [2], the probability that $\hat{\mathbf{T}}_{3,1} \neq \hat{\mathbf{T}}_3$ tends to 0 as $n \to \infty$ if

$$R_2 < I(U; Y_1) - I(U; S_1, S_2) \tag{21}$$

By dirty paper coding, we choose $\alpha_r = \frac{P_3}{P_3+1}$, the decoding error probability at user node 1 approaches 0 if

$$R_2 < \frac{1}{2}\log(1 + P_3) \tag{22}$$

Taking similar steps, user node 2 can successfully decode message of user node 1 with decoding error probability approaching 0 if

$$R_1 < \frac{1}{2}\log(1 + P_3) \tag{23}$$

Following (19), (22) and (23), the following rate pair is achievable

$$\begin{aligned} R_1 &< \min\left\{\frac{1}{2}\log\left(\frac{P_1}{P_1+P_2}+P_1\right)^+, \frac{1}{2}\log(1+P_3)\right\} \\ R_2 &< \min\left\{\frac{1}{2}\log\left(\frac{P_2}{P_1+P_2}+P_2\right)^+, \frac{1}{2}\log(1+P_3)\right\} \end{aligned} \tag{24}$$

The achievable rate region derived in (24) is the same as that derived in [6] in which the two-way relay channel is not corrupted by extra interferences. Therefore, the rate region derived by (24) is within 1/2 bit from the cut-set outer bound.

4 Conclusions

In this paper, we consider a Gaussian dirty two-way relay channel with additive interferences known partially at some of the nodes. Achievable rate regions are derived using nested lattice coding and compute-and-forward relaying for two cases. The nodes which know the interference in advance make use of the structure of the nested lattice codes to pre-cancel part of the interferences. At the relay node, structural gain of computation coding is exploited using nested lattice codes. With the schemes used in this paper, we show that the achievable rate regions are within constant bits from the cut-set outer bound regardless of the interferences.

Acknowledgments. This research was supported by National Natural Science Foundation of China (Nos. 61501171, 61271232, 61401146, 61401147).

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