Relative Localization for Small Wireless Sensor Networks

Yifeng Zhou[1(✉)] and Franklin Wong[2]

[1] Communications Research Centre Canada, Nepean, Canada
yifeng.zhou@canada.ca
[2] Defence Research and Development Canada, Ottawa, Canada
franklin.wong@drdc-rddc.gc.ca

Abstract. In this paper, we investigate relative localization techniques based on internode distance measurements for small wireless networks. High precision ranging is assumed, which is achieved by using technologies such as ultra-wide band (UWB) ranging. A number of approaches are formulated and compared for relative location estimation, which include the Linear Least Squares (LLS) approach, the Maximum Likelihood Estimation (MLE) approach, the Map Registration Approach (MAP), the Multidimensional Scaling (MDS) approach and the enhanced MDS approaches. Finally, computer simulations are used to compare the performances and effectiveness of these techniques, and conclusions are drawn on the suitability of the relative localization techniques for small networks.

Keywords: Wireless sensor networks · Localization · Ranging · Ultra-wide band (UWB) · Least squares (LS) · Maximum likelihood method (MLE) · Multidimensional scaling (MDS)

1 Introduction

Localization refers to the process of estimating the locations of objects based on various types of measurements and the use of a number of anchors. Anchors are simply objects that know their coordinates *a priori*. Localization is a prerequisite for many military operations where location information must be known *a priori* in order to monitor the environment, gather data measurements, track objects to make right decisions. Although GPS can be used for providing coordinates, it requires line-of-sight (LOS) conditions to satellites, and does not work reliably in urban and indoor environments. In addition, GPS is subject to jamming. In the last two decades, many localization techniques have been developed for wireless sensor network applications [1,2]. In general, localization can be relative or global. Relative localization provides relative coordinates that are defined without reference to an external coordinate system while global localization provides coordinates that are defined in the form of specific geographic coordinates such as latitude and longitude. Relative coordinates can be derived

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from corresponding global coordinates. Relative coordinates are not unique, and are arbitrary rigid transformations of their global coordinates.

In this study, a number of relative localization approaches are formulated and discussed, which include the Linear Least Squares (LLS) approach, the Maximum Likelihood Estimation (MLE) approach, the Map Registration Approach (MAP), the Multidimensional Scaling (MDS) approach and the enhanced MDS approaches. All approaches are based on internode distance measurements that are assumed to be provided by the ultra-wide band (UWB) ranging technology. UWB radios employ very short pulse waveforms with energy spread over a wide swath of the frequency spectrum. Due to the inherently fine temporal resolution of UWB, arriving multi-path components can be sharply timed at a receiver to provide accurate time of arrival estimates, and thus the internode distance measurements. The LLS and the MLE method are based on the multilateration technique, which is seen to be one of the most popular localization techniques [1,3]. The MDS and MAP approaches are based on the approaches in [4–6], respectively. They use internode distance measurements to provide relative coordinates. MDS requires the full knowledge of the Euclidean distance matrix of the nodes, which is usually not available in practice due to the limited ranging capability. Unavailable distance measurements need to be approximated, which may introduce large localization errors. The MAP approach is a more elaborated approach that is proposed to counter this difficulty by dividing the network into many small sub-groups with adjacent groups sharing common nodes, constructing local maps for the all sub-group, and merging them into a global map. The MAP approach is able to alleviate the problems due to using the shortest path distances for remote sensor nodes.

The rest of the paper is organized as follows. In Sect. 2, the LLS and MLE methods are formulated. The procedures for determining the anchors are discussed in detail in this section. In Sect. 3, the MDS and the MAP method are discussed and formulated in the context of relative localization. In Sect. 4, the performance of various approaches are evaluated using computer simulations. A number of different application scenarios are simulated, which include fully and partially connected networks. Finally, conclusions are drawn on the suitability of various approaches for relative localization for small networks.

2 The MLE and LLS Methods

The linear least squares (LLS) method and the maximum likelihood estimation (MLE) method, in general, have two steps. The first step is to estimate three node locations. These nodes will be used as anchors. The second step is to iteratively estimate the locations of the rest of the nodes.

First, an arbitrary node is selected, denoted by s_1, as the first anchor and define it as the origin of the coordinate system. Secondly, a neighbour node of s_1, denoted by s_2, is selected as the second anchor. Define the line connecting s_1 and s_2 as the x-axis. The coordinates of s_2 are given by $(d_{12}, 0)$, where d_{12} is the measured distance between s_1 and s_2. Select a third node, s_3, that is a neighbour

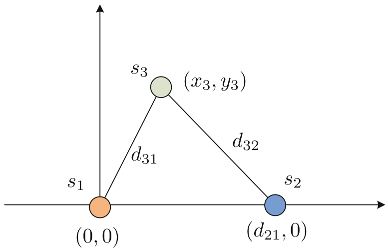

Fig. 1. Geometry of the first three anchor nodes.

to both s_1 and s_2 with distances d_{31} and d_{32}, respectively. The geometry of the first three anchor nodes are shown in Fig. 1. If the distances d_{21}, d_{31} and d_{32} satisfy the triangle inequality relationship, the coordinates of s_3 can be obtained as the intersections of two circles with centers at s_1 and s_2 and radii of d_{31} and d_{32}, respectively. They are given as [7]

$$x_3 = \frac{d_{21}^2 + d_{31}^2 - d_{32}^2}{2d_{21}}, \quad y_3 = \pm\sqrt{d_{31}^2 - x_3^2}. \tag{1}$$

In (1), the positive root is selected for y_3. Note the selection is arbitrary and will not affect the performance of relative localization. When the triangle inequality is not satisfied due to distance measurement errors, the two circles will not intersect. In this case, the coordinates of s_3 can be estimated using the following nonlinear least squares solution

$$\min_{\{x_3, y_3\}} (\sqrt{x_3^2 + y_3^2} - d_{31})^2 + (\sqrt{(x_3 - d_{21})^2 + y_3^2} - d_{32})^2. \tag{2}$$

Note that (2) is a nonlinear optimization problem and an analytical solution does not exist. Numerical techniques are required to solve for minimizing x_3 and y_3.

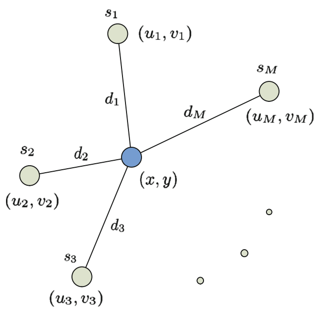

Fig. 2. Geometric of anchor nodes and the node to be localized.

2.1 Formulation of the MLE and LLS Methods

The second step of MLE and LLS is to estimate the locations of the rest of the nodes based on the nodes with known locations. Figure 2 shows the geometric configuration of anchors and the node to be localized. In the figure, M anchors are used, and their coordinates and measured distances to the node to be localized are denoted by $\{u_m, v_m, d_m\}$, for $m = 1, 2, \ldots, M$, respectively. The multilateration approach is to estimate the coordinates (x, y) given $\{u_m, v_m, d_m; m = 1, 2, \ldots, M\}$. The maximum likelihood method minimizes the following sum of squared errors between the measured distances and hypothetical ones based on the unknown sensor node location

$$\min_{x,y} \sum_m \left[\sqrt{(x - u_m)^2 + (y - v_m)^2} - d_m \right]^2 . \tag{3}$$

Under the assumption that $\{d_m; m = 1, 2, \ldots, M\}$ contain additive measurement errors that are an independent, identically distributed (*i.i.d.*) Gaussian process with zero mean, (3) can be shown to be equivalent to the maximum likelihood estimator [8]. We refer to the formulation (3) as the maximum likelihood estimator (MLE). Since (3) is a nonlinear minimization problem, a closed-form solution does not exist, and numerical techniques are typically the resort. As mentioned before, numerical optimization techniques are subject to convergence difficulties and always suffer from the local minimum problem.

In practice, the least squares problem is often formulated in the squared distance domain to simplify the solution

$$\min_{x,y} \sum_m \left[(x - u_m)^2 + (y - v_m)^2 - d_m^2 \right]^2 . \tag{4}$$

It can be shown that (4) is equivalent to solving the following least squares problem

$$B\underline{z} - \frac{r_x^2}{2} \cdot \underline{\mathbf{1}} = \underline{\eta}, \tag{5}$$

where $\underline{z} = [x, y]^T$, $\underline{\mathbf{1}}$ denotes an all one vector of length M,

$$\underline{\eta} = -\frac{1}{2} \begin{bmatrix} d_1^2 - r_1^2 \\ d_2^2 - r_2^2 \\ \vdots \\ d_M^2 - r_M^2 \end{bmatrix}, \quad B = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ \vdots & \vdots \\ u_M & v_M \end{bmatrix} . \tag{6}$$

and $r_x^2 = x^2 + y^2$ and $r_m^2 = u_m^2 + v_m^2$. The nonlinear term r_x^2 can be eliminated from the equation by the use of projection operations. Define $P_{\mathbf{1}}^{\perp}$ as the orthogonal projection onto the null subspace of $\underline{\mathbf{1}}$. By multiplying both sides of (5), we can obtain the following equation

$$A\underline{z} = \underline{b}, \tag{7}$$

where $A = P_{\mathbf{1}}^{\perp} B$ and $\underline{b} = P_{\mathbf{1}}^{\perp} \underline{\eta}$. Equation (7) is linear in $\underline{z}$ and has a closed-form least squares (LS) solution [9].

3 The MDS and MAP Methods

The MDS method was first proposed for solving the problem of sensor localization by Shang *et al.* [4,10], where either connectivity information or distance measurements between neighbor nodes were used for localization. It is based on the application of the popular multidimensional scaling (MDS) technique in statistics. It is a data analysis technique that can be used to represent a set of data as a configuration of points in some Euclidean spaces based on their similarity measures. The distances of the resulting configuration of points resemble the original similarities. There are many types of MDS techniques, including metric MDS and nonmetric MDS, replicated MDS, weighted MDS, deterministic and probabilistic MDS [11]. The classical MDS method is more attractive than the others because it has analytical solutions that can be obtained via eigendecomposition of a transform of the Euclidean distance matrix. In [12], the authors proposed an iterative MDS algorithm that uses a multivariate optimization for location estimation. The iterative MDS is similar to the least squares refinement step in [10]. The iterative MDS approach is less tractable than the classical MDS solution because it involves complex computations and suffers from global convergence problems. In general, the MDS technique is relatively resilient to distance errors due to the over-determined nature of the solution. However, MDS requires full knowledge of the Euclidean distance matrix of the sensor nodes, which is usually not available in practice due to the limited transmission range of beacons or ranging modules on each sensor node. A commonly used approach is to approximate the distances between nodes that are separated further than the transmission range by their shortest path distances. The shortest path distances can be computed using shortest path algorithms such as Dijkstra's [13] or Floyd's [14]. The approximation of the Euclidean distance matrix introduces sensor localization errors, especially when the shortest paths do not correspond well with the Euclidean distance in sparse networks or networks of irregular topology. Refer to [4,10] for the details of the MDS method.

3.1 The MAP Approach

The MAP approach refers to the map registration approach proposed by Zhou *et al.* [5,15]. It is known that the MDS approach requires that the Euclidean distance matrix for all nodes be known, which may not be always available in practice due to the limited ranging distance of the nodes. When two nodes are out of their transmission range, the distance between them cannot be directly obtained, and needs to be estimated. In MDS, the unavailable internode distances are typically approximated by its shortest path distance. A shortest path distance corresponds well to the corresponding Euclidean distance in a network of regular topology or a densely distributed network of nodes. In a sparse network or a network of nodes of irregular topology, however, a shortest path distance may not match its Euclidean distance and the use of the approximated distance matrix will result in degraded localization performance [4,5]. A more elaborate approach is to divide the network into many small sub-groups of nodes, where adjacent groups share common nodes. For each sub-group of nodes, a local map with relative coordinates of the nodes, is built using some localization techniques (*e.g.*, MDS). The local maps are then merged into a global map based on the common nodes. In [4], an incremental greedy algorithm was proposed for merging the local maps in a sequential manner. Each time a local map that has the maximal number of common nodes with the core map is selected and merged with the core map. The incremental greedy approach is locally optimal since it only explores the commonalities of the shared nodes in two maps. In practice, the common nodes are often shared by more than two local maps. In some cases, adjacent local maps may not have a sufficient number of common nodes.

The MAP approach was introduced to counter the problems of the sequential approach. Instead of using a sequential pairwise approach for merging local maps, the MAP approach constructs the global map at a global level. An affine transformation is defined for each local map to transform it to a global map. The set of optimal affine transformations are determined simultaneously by considering all available nodes that are shared by various local maps. The discrepancy is represented by the sum of the squared distances of all nodes to their respective geometric centers in the global map. Assume that a network consists of N nodes. For each node, the local map is assumed to contain its neighbor nodes within k-hops. Define a neighbor vector $\underline{c}_i$ of length N for the ith node. The nth component of $\underline{c}_i$ is given by 1 or 0 depending on whether the nth node is a neighbor node or not. Define a neighbor matrix $C = [\underline{c}_1, \underline{c}_2, \ldots, \underline{c}_N]$. For the ith local map, define an orthogonal matrix $U_i \in \mathcal{R}^{2\times 2}$ and a row vector $T_i \in \mathcal{R}^{1\times 2}$ to represent rotation/reflection (or a combination) and translation, respectively. Define $U \in \mathcal{R}^{2N\times 2}$ and $T \in \mathcal{R}^{N\times 2}$ as

$$U = [U_1; U_2; \ldots; U_N] \text{ and } T = [T_1; T_2; \ldots; T_N], \tag{8}$$

respectively. Let $\mathbf{z}_{ij} \in \mathcal{R}^{1\times 2}$ denote the local coordinates of the ith sensor node in the jth local map. If the ith sensor node is not in the jth local map, then $\mathbf{z}_{ij} = \mathbf{0}$. Define a data matrix $Z_{ij} \in \mathcal{R}^{N\times 2}$, where the jth row of Z_{ij} is $\mathbf{z}_{ij}$. If the ith node is not in the jth local map, then, Z_{ij} is an all-zero matrix. Let $C_i = diag(\underline{c}_i)$ be a diagonal matrix of $N \times N$, where $diag$ puts the elements of $\underline{c}_i$ on its diagonal. For the ith local map, we construct a data matrix X_i

$$X_i = [Z_{i1}, Z_{i2} \ldots, Z_{iN}]. \tag{9}$$

Let Y_i denote an affine transform of X_i given by

$$Y_i = X_i U + C_i T. \tag{10}$$

All Y_i are in a same coordinate system that is referred to as the *global coordinate system*. The global coordinates of the sensor nodes form the *global map*. In MAP, the optimal U is obtained from the following optimization problem [5]

$$\min_U tr\{U^T \Sigma U\}, \tag{11}$$

subject to the constraint that U_i is an orthogonal matrix for $i = 1, 2, \ldots, N$. Denote $\mathcal{M}$ as the manifold that consists of all $U = [U_1, U_2, \ldots, U_N]$ and each U_i

is an orthogonal matrix of 2×2. Then, the constraint implies that the optimal U is in the manifold $\mathcal{M}$. In (11), tr denotes the trace of a square matrix, and

$$\Sigma = \sum_i X_i^T P_i^{\perp} X_i - A_s^T B_s^{-1} A_s \tag{12}$$

$$B_s = \sum_i \tilde{C}_i^T P_i^{\perp} \tilde{C}_i, \quad A_s = \sum_i \tilde{C}_i^T P_i^{\perp} X_i, \quad P_i^{\perp} = I - \frac{1}{N_i} \underline{c}_i \underline{c}_i^T, \tag{13}$$

and $\tilde{C}_i$ is C_i with its first column removed. The translation matrix T is related to the optimal U by $T = [\mathbf{0}; -B_s^{-1} A_s U]$.

The optimization problem (11) involves highly nonlinear criterion function, and analytic solutions are not known to exist. In [5], a gradient projection algorithm is developed for finding the optimal transforms for transforming local maps to a global map. The algorithm is developed based on a general idea by Jennrich in [16,17] and is particularly suitable to the constrained optimization problem of coordinate transformation. The algorithm is iterative, and has the advantages of faster convergence and computationally more efficient than many general numerical optimization techniques [18] for nonlinear programming. The detailed discussion of the GP algorithm can be found in [15].

4 Simulations and Performance Analysis

In this section, we use computer simulations to demonstrate the effectiveness and performance of the proposed relative localization techniques. MLE uses the LLS solution as the initial estimates in each iteration after the initial node selection process. For the MDS method, Dijkstra's algorithm [13] is used to compute the shortest paths to approximate the unavailable internode distances. For MAP, a local maps is constructed for each node, which consists of all direct neighbor nodes within its maximum ranging distance. The root mean square errors (RMSE) of the location estimates are used as a performance metric. In order to compute meaningful RMSEs for relative location estimates, all relative estimates are aligned to best conform to its ground truth node location.

The nodes are assumed to be uniformly distributed in a square area of 100 m by 100 m. All nodes are assumed to have a common maximum ranging distance that can be configured. The maximum ranging distance determines whether the internode distance measurement between a pair of nodes is available or not. The network is assumed to be connected, *i.e.*, each of the nodes of the network is connected to each other either one-hop or *via* multiple hops in terms of internode ranging. The algebraic connectivity of a network is used to check whether the network is connected or not [19]. The distance measurement errors are assumed to be additive and uniformly distributed. The uniform distributed model ensures that the errors are bounded, and leads to the more conservative estimates of uncertainty than the Gaussian error model. Let $\tilde{d}_{ij}$ and d_{ij} denote the actual and measured distances between the ith and the jth node, respectively. Then, the measured distance is given by $d_{ij} = \tilde{d}_{ij} + \epsilon_{ij}$, where ϵ_{ij} is simulated to be uniformly distributed in $[-\sigma\tilde{d}_{ij}, \sigma\tilde{d}_{ij}]$ and $\sigma \in [0, 1]$.

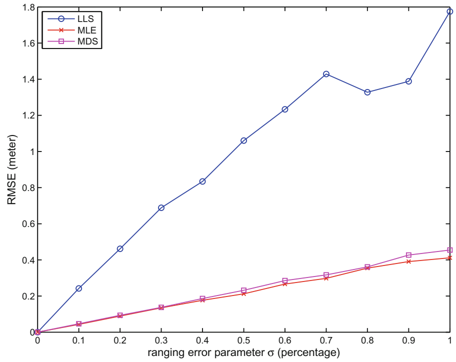

Fig. 3. Variation of RMSE for LLS, MLE and MDS *versus* σ in a fully connected network of $N = 5$ nodes.

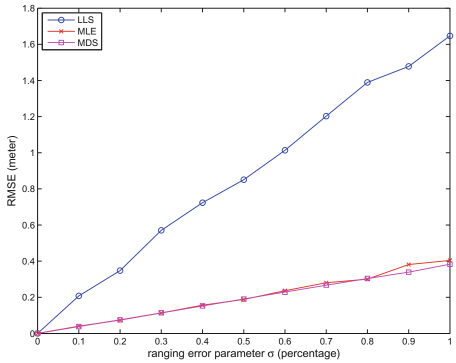

Fig. 4. Variation of RMSE for LLS, MLE and MDS *versus* σ in a fully connected network of $N = 10$ nodes.

Fully Connected Network. By a fully connected network, we mean that the maximum ranging distance for all nodes in the network is sufficiently large such that each node is able to measure its distances to all other nodes in the network. For a fully connected network, distance measurements between all pairs of nodes are available. Figures 3 and 4 show the variation of RMSE for the MLE, LLS and MDS estimates versus the ranging error parameter σ for fully connected networks with 5 and 10 nodes, respectively. The parameter σ is written in the form of percentage. In the simulations, σ varies from 0 to 0.01 (or 1%), and for each value of σ, 1000 tests are repeated to obtain the averaged RMSE results. The averaged internode distances for the networks of 5 and 10 nodes are calculated as 52.39 and 52.25 m, respectively. When $\sigma = 0.01$, it would translate into an averaged internode distance measurement error range of about ± 0.5 m. For each test, all nodes are randomly re-deployed and their random internode distance measurement errors re-generated. Thus, the RMSE results are averaged over both node distribution and random distance measurement errors. For a fully connected network, since all pairs of nodes are within the maximum ranging distance, the Euclidean distance matrix is completely available, and MAP become equivalent to MDS. Thus, only LLS, MLE and MDS are evaluated for fully connected networks. As discussed before, the LLS solutions are sensitive to geometric distribution of the nodes. In order to isolate the impact of node distribution on localization, a condition number of 400 is used to avoid scenarios that would result in ill-conditioned data matrix. In Figs. 3 and 4, it can be observed that the RMSEs of the LLS, MLE and MDS estimates increase as σ increases. The MLE and MDS estimates have similar performance, and both outperform the LLS estimates significantly, especially as σ increases. All approaches have similar performance for networks with 5 and 10 nodes.

Partially Connected Networks. In a partially connected networks, a node may not be able to measure the distances to all other nodes in the network due to the limited ranging distance of the nodes. For the MDS type approaches,

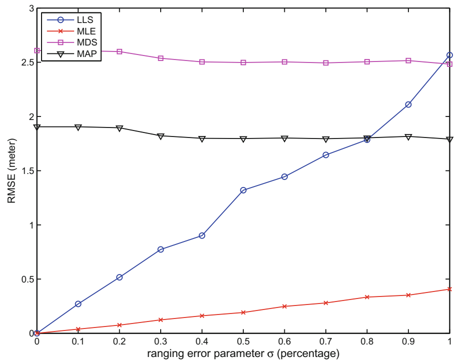

Fig. 5. Variation of RMSE for LLS, MLE, MDS, and MAP *versus* σ in a partially connected network with a maximum ranging distance of 80 m.

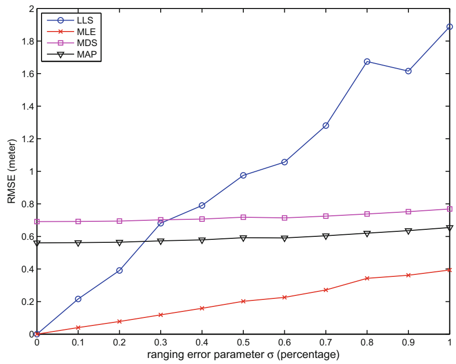

Fig. 6. Variation of RMSE for LLS, MLE, MDS, and MAP *versus* σ in a partially connected network with a maximum ranging distance of 100 m.

this means that the unavailable internode distances will need to be estimated using their corresponding shortest path distances. The shortest path distances are approximate of the Euclidean distances. Partially connected networks with 10 node are simulated. Three scenarios are simulated with the maximum ranging distances being set to 80, 100 and 120, respectively. Figures 5, 6 and 7 show the variations of RMSEs for LLS, MLE, MDS, and MAP *versus* the ranging error parameter σ in those three scenarios. Connectivity level is defined, which is computed as the averaged number of nodes that a node can measure distance to. Connectivity level increases as the maximum ranging distance is increased. For the three scenarios with maximum ranging distances of 80, 100, and 120 m, the connectivity levels are 7.72, 8.77 and 8.99, respectively. In the simulations, σ varies from 0 to 0.01 (or 1% in terms of percentage). For each value of σ, 1000 tests are repeated to obtain the averaged results. In each test, nodes are re-deployed and random ranging errors are re-produced. Similarly, a condition number of 500 is used to avoid the ill-conditioned data matrix for LLS. In Figs. 5, 6 and 7, the top figures show the variations of RMSE of the LLS, MLE, MDS and MAP estimates *versus* σ. In all three scenarios, all approaches in the simulation study show similar performance patterns. The RMSEs of the LLS and MLE estimates increase as σ increases while the RMSEs of the MDS and MAP estimates are relatively constant over the tested range of σ. The accuracy of the MDS and MAP estimates is dominated by the connectivity level of the network rather than the assumed relatively small ranging errors. In all three scenarios, MLE performs the best. The performance of MDS and MAP improves as the network connectivity improves, as can be observed from Figs. 5, 6 and 7. As shown in Fig. 7, MDS and MAP perform as well as MLE when the connectivity level is 8.99 except for small values of σ. LLS outperforms MDS and MAP for small values of σ, and is outperformed by MDS and MAP as σ increases. The demarcation point for LLS moves down as the network connectivity increases as observed from Figs. 5, 6 and 7. It is observed that MDS and MAP produce large

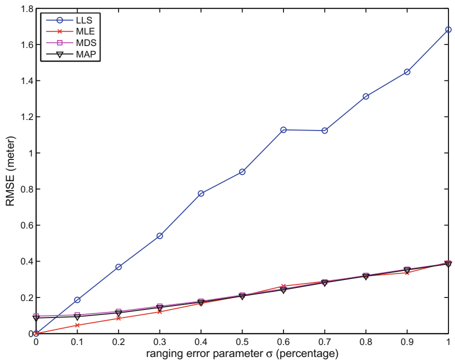

Fig. 7. Variation of RMSE for LLS, MLE, MDS, and MAP *versus* σ in a partially connected network with a maximum ranging distance of 120 m.

RMSEs when the network has low connectivity levels, and improve as the connectivity improves. Although the RMSE for MLE and LLS is less affected by the network connectivity level, MLE and LLS may run into problems in the iteration process due to the problem of insufficient number of anchors for localization in the case of low network connectivity levels.

5 Conclusions

In this paper, the MLE, LLS, MDS, and MAP methods have been formulated for estimating the relative locations of a set of node based on their internode distance measurements. Their performances were discussed and analyzed using computer simulations. Fully and partially connected networks were simulated in the study. From the simulation results, MLE and MAP, among all proposed approaches, are considered the viable solutions to relative localization for small wireless sensor networks. MLE is able to provide the superior localization performance in both fully and partially connected network scenarios. Simulation results showed that, when LLS was used to provide the initial estimates, MLE has converged to the desired optimal estimates almost every time. For partially connected networks with low connectivity, however, MLE may suffer from the problem of not having sufficient numbers of anchors in iterating across the entire network. The performance of MAP is close to that of MLE in fully connected networks and partially connected works with moderate and high connectivity levels, and deteriorates as the network connectivity decreases. In addition, MAP has the advantage of always being able to provide a localization solution in spite of the network connectivity level, although the localization accuracy may be low as in the case of networks with low connectivity levels.

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