Optimizing Power Allocation in Wireless Networks: Are the Implicit Constraints Really Redundant?

Xiuhua Li(✉) and Victor C.M. Leung

Department of Electrical and Computer Engineering,
The University of British Columbia, Vancouver V6T 1Z4, Canada
{lixiuhua,vleung}@ece.ubc.ca

Abstract. The widely considered power constraints on optimizing power allocation in wireless networks, e.g., $p_n \geq 0, \forall n$, and $\sum_{n=1}^{N} p_n \leq P_{\max}$ where N and $P_{\max}$ are given constants, imply the constraints, i.e., $p_n \leq P_{\max}, \forall n$. However, the related implicit constraints are regarded as redundant in the most current studies. In this paper, we explore the question "Are the implicit constraints really redundant?" in the optimization of power allocation especially when using iterative methods that have slow convergence speeds. Using the water-filling problem as an illustration, we derive the structural properties of the optimal solutions based on Karush-Kuhn-Tucker conditions, propose a non-iterative closed-form optimal method, and use subgradient methods to solve the problem. Our theoretical analysis shows that the implicit constraints are not redundant, and their consideration can effectively speed up convergence of the used iterative methods and reduce the sensitivity to the chosen step sizes. Numerical results for the water-filling problem and another existing power allocation problem confirm the effectiveness of considering the implicit constraints.

Keywords: Power allocation · Water-filling · Subgradient method

1 Introduction

Future wireless communication networks are required to support a large number of users with various requirements, especially the large bandwidth demands of multimedia services. To fulfill the requirements, radio resource management (RRM) plays an essential role as the system level control of co-channel interference and other radio transmission characteristics in wireless communication systems [1]. RRM involves strategies and algorithms for controlling parameters such as transmit power, user allocation, beamforming, data rate, handover criteria, modulation scheme and error coding scheme, etc., aiming at maximizing the utilization of the limited radio-frequency spectrum and radio network infrastructure [2]. Among these RRM techniques, optimization of power allocation is an important aspect of wireless communication system design that is well-studied in the past decades [1,2].

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On one hand, as an iterative first-order method, the subgradient method is widely used in many studies [3–13] to solve various power allocation problems or other optimization problems on RRM in wireless systems. In [3,4], the subgradient method was used to solve the problem of maximizing the throughput under the constraints of interference power and individual transmit power in cognitive radio networks. In [5], subgradient methods were utilized based on dual decomposition to solve the simultaneous routing and resource allocation problem. In [6], a subgradient solution was achieved to compute the maximum rate and the optimal routing strategy to solve the maximum multicast rate problem in the general undirected network model. In [7], a distributed subgradient method was used to solve the problems of how to choose opportunistic route for users to optimize the total utility or profit of multiple simultaneous users in wireless mesh networks. In [8], distributed subgradient methods were applied to optimize global performance in delay tolerant networks with limited information. In [9], a subgradient solution was proposed to solve the problem of jointly optimizing channel pairing, channel-user assignment, and power allocation in a single-relay multiple-access system. In [10], an α-approximation dual subgradient algorithm was proposed to optimize the total utility of multiple users in a load-constrained multihop wireless network. Based on the subgradient method, the study in [11] proposed a distributed optimal data gathering cost minimization framework with concurrent data uploading in wireless sensor networks. With the dual subgradient method, the study in [12] focused on convergence analysis of decentralized min-cost subgraph algorithms for multicast in coded networks. In [13], the subgradient method was used for joint power and bandwidth allocation in an improved amplify and forward cooperative communication scheme. Though subgradient methods can be operated in a distributed manner, they usually have **slow convergence speeds** and are very **sensitive to the chosen iteration step sizes** [14,15], which need to be improved to reduce the computation costs and even signaling overhead in wireless networks and to reduce the sensitivity to the chosen step sizes since *(1) the subgradient method may not converge under an improper step size*, and *(2) it is not easy to choose the proper step size, especially when the formulated optimization problem is very complex.*

On the other hand, mathematically, the formulated optimization problems of power allocation in wireless systems are generally subject to at least two inequality constraints [1–13] on p_n, the transmit power allocated at a base station (BS) for the n-th user, e.g., *(1) nonnegative*: $p_n \geq 0, \forall n$, and *(2) limited sum:* $\sum_{n=1}^{N} p_n \leq P_{\max}$, where N and $P_{\max}$ respectively denote the total number of users served by the BS and the BS's maximum transmit power. These two power constraints imply another set of (implicit) constraints, i.e., $p_n \leq P_{\max}, \forall n$. However, in most currently studied power allocation optimization problems or other similar optimization problems with the above two inequality constraints, the implicit constraints are regarded as redundant and useless in the design of strategies and algorithms for solving the problems. From the perspective of mathematics, the implicit constraints obviously hold, but are they really redundant in optimization algorithms? To the best of our knowledge, this question is unexplored.

The above motivates us to answer the question "Are the implicit constraints really redundant?" in power allocation optimization especially when using subgradient methods in the solution algorithms. Specifically, we study the water-filling problem as a typical example of power allocation optimization. Based on Karush-Kuhn-Tucker (KKT) conditions, we derive the structural properties of the optimal solutions to the water-filling problem and evaluate the performance of the proposed methods with and without considering the implicit constraints. Our contributions are summarized below:

- To explore the first studied question, using the water-filling problem as an illustration, our theoretical analysis shows that considering the implicit constraints can effectively speed up the convergence of the subgradient method, reduce the sensitivity to the chosen step size and lead to convergence even when an improper step size is used, while the opposite is true if the implicit constraints are not considered. This finding can be extended to other optimization problems and applied to other iterative methods. Besides, we propose a non-iterative closed-form optimal method.
- Numerical results on the water-filling problem and the power allocation problem for multiuser systems in [16] show that considering the implicit constraints in the algorithm design can effectively improve the performance of the used subgradient methods.

The rest of this paper is organized as follows. Section 2 introduces the water-filling problem as an illustration of power allocation. Section 3 derives the structural properties of the optimal solutions. Section 4 proposes and analyzes the algorithms to solve the optimization problem. Section 5 evaluates the performance of the proposed algorithms. Finally, Sect. 6 concludes this paper.

2 The Water-Filling Problem Typical in Resource Allocation

In this section, we provide a general form of the resource allocation problem and its formulation as the widely studied water-filling problem, to explore whether the implicit constraints are really redundant for optimization.

2.1 General Resource Allocation Problem

Many existing optimization problems for allocation of power or other resource can be formulated or transformed into a general form as

$$\max_{\mathbf{p},\mathbf{y}} f(\mathbf{p},\mathbf{y}) \tag{1a}$$

$$s.t. \sum_{n=1}^{N} p_n \leq P_{\max}; p_n \geq 0, \forall n \in \mathcal{N}, \tag{1b}$$

$$\mathbf{y} \in \mathcal{S}_Y; g_i(\mathbf{p},\mathbf{y}) \leq 0, \forall i \in \mathcal{I} \tag{1c}$$

where N is a given number (e.g., number of users), $\mathcal{N} = \{1, 2, \ldots, N\}$, $\mathcal{I}$ and $\mathcal{S}_Y$ are two given sets about resource constraints; $\mathbf{p} = [p_1, p_2, \ldots, p_N]^T$ and $\mathbf{y}$, respectively, are variable vectors of power and other resource allocations; $f(\mathbf{p}, \mathbf{y})$ and $g_i(\mathbf{p}, \mathbf{y})$ are, respectively, the given objective function (e.g., sum data rate) and constraint functions w.r.t. $\mathbf{p}$ and $\mathbf{y}$; $P_{\max}$ is a positive constant scalar (e.g., maximum sum power). From (1b), we can get the implicit constraints as

$$p_n \leq P_{\max}, \forall n \in \mathcal{N}. \tag{2}$$

In existing studies, the same or similar implicit constraints in (2) are usually overlooked and are regarded as redundant. Besides, whether problem (1) is convex or nonconvex, it can be solved with a family of iterative methods (e.g., subgradient method) to get the optimal or suboptimal solutions.

2.2 Water-Filling Problem

The water-filling problem given below is a typical formulation of the general resource allocation optimization problem described above, in which the sum capacity of users is maximized under transmit power constraints [17].

$$\max_{\mathbf{p}} \sum_{n=1}^{N} \log_2(1 + \alpha_n p_n), \;\; s.t. \; \sum_{n=1}^{N} p_n \leq P_{\max}; \;\; p_n \geq 0, \forall n \in \mathcal{N} \tag{3}$$

where $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_N]^T$ is a strictly positive constant vector. Clearly, (3) is a simple case of (1) without loss of generality.

We incorporate the implicit constraints into problem (3) as

$$\min_{\mathbf{p}} \; z = -\sum_{n=1}^{N} \log_2(1 + \alpha_n p_n) \tag{4a}$$

$$s.t. \;\; -p_n \leq 0, \;\; \forall n \in \mathcal{N}, \tag{4b}$$

$$p_n - P_{\max} \leq 0, \;\; \forall n \in \mathcal{N}, \tag{4c}$$

$$\sum_{n=1}^{N} p_n - P_{\max} \leq 0, \tag{4d}$$

which is a strictly convex optimization problem. Thus, a local optimal solution is also globally optimal and the optimal solution is unique. Moreover, we can get the Lagrangian for problem (4) as $L(\mathbf{p}, \boldsymbol{\lambda}, \boldsymbol{s}, \nu) = -\sum_{n=1}^{N} \log_2(1 + \alpha_n p_n) + \sum_{n=1}^{N} (\nu - \lambda_n + s_n) p_n - \big(\nu + \sum_{n=1}^{N} s_n\big) P_{\max}$, where $\boldsymbol{\lambda} \in \mathbb{R}^N$, $\boldsymbol{s} \in \mathbb{R}^N$ and $\nu \in \mathbb{R}$ are the nonnegative Lagrange multiplier vectors and scalar for constraints (4b), (4c) and (4d), respectively. Thus, we can get the dual objective as $g(\boldsymbol{\lambda}, \boldsymbol{s}, \nu) = \inf_{\mathbf{p}} L(\mathbf{p}, \boldsymbol{\lambda}, \boldsymbol{s}, \nu)$, and then the dual problem as $\max_{\boldsymbol{\lambda}, \boldsymbol{s}, \nu} g(\boldsymbol{\lambda}, \boldsymbol{s}, \nu)$. Since problem (4) is convex, the corresponding duality gap reduces to zero at the optimum.

3 Structural Properties of the Optimal Solutions

According to the KKT conditions [14,15], if a feasible solution $\mathbf{p}^* \in \mathcal{S}_P$ is a local (and global) minimizer of the convex optimization problem (4), then there exist multipliers $(\boldsymbol{\lambda}^*, \boldsymbol{s}^*, \nu^*)$, not all zero, $(\boldsymbol{\lambda}^* \succeq 0, \boldsymbol{s}^* \succeq 0, \nu^* \geq 0)$, such that

$$\frac{\partial L}{\partial p_n} = -\frac{\alpha_n}{(1+\alpha_n p_n^*)\ln 2} - \lambda_n^* + s_n^* + \nu^* = 0,\ \forall n \in \mathcal{N}, \tag{5}$$

$$\lambda_n^* p_n^* = 0,\ \lambda_n^* \geq 0,\ p_n^* \geq 0,\ \forall n \in \mathcal{N}, \tag{6}$$

$$s_n^*(p_n^* - P_{\max}) = 0,\ s_n^* \geq 0,\ p_n^* \leq P_{\max},\ \forall n \in \mathcal{N}, \tag{7}$$

$$\nu^*(\sum_{n=1}^{N} p_n^* - P_{\max}) = 0,\ \nu^* \geq 0,\ \sum_{n=1}^{N} p_n^* \leq P_{\max}. \tag{8}$$

Define $\mathcal{N}_1 \triangleq \{n|s_n^* > 0, n \in \mathcal{N}\}$, $\mathcal{N}_2 \triangleq \{n|s_n^* = 0, n \in \mathcal{N}\}$, and $\boldsymbol{\omega}^* \triangleq \nu^* \cdot \mathbf{1} + \boldsymbol{s}^* \in \mathbb{R}^N$, where $\mathbf{1} = [1, 1, \ldots, 1]^T \in \mathbb{R}^N$. Thus, we have $\mathcal{N}_1 \cup \mathcal{N}_2 = \mathcal{N}$ and $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$. Specifically, if there exists $n \in \mathcal{N}$ such that $p_n^* = P_{\max}$, then denote the index as $k^\#$, i.e., $p_{k^\#}^* = P_{\max}$; otherwise, $k^\#$ does not exist. We give some remarks for the above KKT conditions and derive some structural properties of the optimal solutions via some theorems below.

Remark 1. Specifically, if $\mathcal{N}_1 \neq \emptyset$, then for $\forall n \in \mathcal{N}_1$, we have $p_n^* = P_{\max}$ according to (7), and thus $p_k^* = 0$ and $s_k^* = 0$ for $\forall k \in \mathcal{N}, k \neq n$ according to (7) and (8). Thus, there exists at most one positive element in $\boldsymbol{s}^*$, i.e., $|\mathcal{N}_1| \leq 1$. Besides, if $|\mathcal{N}_1| = 1$, the only element in $\mathcal{N}_1$ is equal to $k^\#$ and we have $\mathcal{N}_2 = \mathcal{N} \setminus \{k^\#\}$. Note that even if $\mathcal{N}_1 = \emptyset$, i.e., $\mathcal{N}_2 = \mathcal{N}$, it is possible that $k^\#$ also exists.

Remark 2. For $\forall n \in \mathcal{N}$, $\lambda_n^* s_n^* \equiv 0$. If there exists any $k \in \mathcal{N}$, such that $\lambda_k^* > 0$ and $s_k^* > 0$, then according to (6) and (7), we can get $p_k^* = 0$ and $p_k^* = P_{\max}$ simultaneously, which is clearly contradictory.

Theorem 1. *With $\boldsymbol{\alpha}$ fixed, if P_{max} is not fixed and can be adjusted, then the optimal objective z^* is strictly decreasing with the increase of P_{max}.*

Theorem 2. *The optimal ν^* satisfies that: if $\mathcal{N}_1 = \emptyset$, $\nu^* = \max_{n\in\mathcal{N}}\{\frac{\alpha_n}{(1+\alpha_n p_n^*)\ln 2}\}$; Otherwise, $\nu^* = \frac{\alpha_n}{(1+\alpha_n P_{max})\ln 2} - s_n^*, n \in \mathcal{N}_1$, and $\nu^* \geq \max_{n\in\mathcal{N}_2}\{\frac{\alpha_n}{\ln 2}\}$.*

Remark 3. Based on *Theorem* 2, we have $\nu^* > 0$, and thus the optimal solution $\mathbf{p}^*$ satisfies $\sum_{n=1}^{N} p_n^* = P_{\max}$ according to (8).

Remark 4. In terms of $\boldsymbol{\omega}^*$, if $\mathcal{N}_1 = \emptyset$, then we have $\boldsymbol{\omega}^* = \nu^* \cdot \mathbf{1} \succ 0$, which indicates that (5) is reduced to the form in existing studies (i.e., $\frac{\partial L_1}{\partial p_n}$) in this case. Otherwise, based on *Remarks* 1 and 3, we have $\omega_{k^\#}^* = \nu^* + s_{k^\#}^* > \nu^* > 0$ for $k^\# \in \mathcal{N}_1$, and $\omega_n^* = \nu^* > 0$ for $\forall n \in \mathcal{N} \setminus \{k^\#\}$, which indicates that $\boldsymbol{\omega}^*$ is divided into two positive parts.

Theorem 3. *The optimal solution* $\boldsymbol{p}^*$ *and the corresponding multiplier scalar* ν^* *satisfy* $p_n^* = min\{\big[\frac{1}{\nu^* \ln 2} - \frac{1}{\alpha_n}\big]^+, P_{max}\}, \forall n \in \mathcal{N}$ *where* $[x]^+ \triangleq max\{x, 0\}$.

Remark 5. From *Theorem* 3, in the optimal solution's closed-form expression, the multiplier vectors $(\boldsymbol{\lambda}^*, \boldsymbol{s}^*)$ can be eliminated, while the the multiplier scalar ν^* is dominating. However, $\boldsymbol{\lambda}^*$ is to operate on $[\mathbf{x}]^+$ such that the optimal solution $\mathbf{p}^* \succeq 0$, while $\boldsymbol{s}^*$ is to operate on $\min\{\mathbf{x}, P_{\max}\}$ such that the optimal solution $\mathbf{p}^* \preceq P_{\max}$. In most works, their corresponding solutions are in the form of $[\mathbf{x}]^+$ and have no operation of $\min\{\mathbf{x}, P_{\max}\}$.

Theorem 4. *If* $\alpha_{n_1} \geq \alpha_{n_2}, n_1 \in \mathcal{N}, n_2 \in \mathcal{N}$, *then* $p_{n_1}^* \geq p_{n_2}^*$ *holds in the optimal solution* $\boldsymbol{p}^*$.

Remark 6. Let $n_1 \in \mathcal{N}$ and $n_2 \in \mathcal{N}$ be two indices such that $\alpha_{n_1} \geq \alpha_{n_2}$. Specifically, for the optimal solution $\mathbf{P}^*$, if $p_{n_1}^* = 0$, then $p_{n_2}^*$ must also be zero based on *Theorem* 4.

Theorem 5. *There exists the only* $k^\# \in \mathcal{N}$ *such that* $p_{k^\#}^* = P_{max}$, *if and only if both* $k^\# = arg\max_{n\in\mathcal{N}}\{\alpha_n\}$ *and* $\max_{n\in\mathcal{N}\setminus\{\mathrm{k}^\#\}}\{\alpha_n\} \leq \dfrac{\alpha_{k^\#}}{1+\alpha_{k^\#}P_{max}}$ *hold.*

Theorem 6. *Let* $\boldsymbol{\pi}$ *be the vector obtained by sorting* $\boldsymbol{\alpha}$ *in a descending order. Then the number of strictly positive elements in the optimal solution* $\boldsymbol{p}^*$ *is*

$$\chi = max\Big\{n \in \mathcal{N} \mid \frac{1}{\pi_n} - \frac{1}{n}\Big(\sum_{r=1}^{n}\frac{1}{\pi_r} + P_{max}\Big) < 0\Big\}, \tag{9}$$

and then the corresponding optimal multiplier ν^* *can be expressed as*

$$\nu^* = \begin{cases} any\ value\ in \Big[\frac{\pi_2}{\ln 2}, \frac{1}{(\frac{1}{\pi_1}+P_{max})\ln 2}\Big], \chi = 1, \\ \frac{\chi}{\big(\sum_{r=1}^{\chi}\frac{1}{\pi_r}+P_{max}\big)\ln 2}, \chi \in \mathcal{N}, \chi \geq 2. \end{cases} \tag{10}$$

Remark 7. From (10) in *Theorem* 6, ν^* can take the value of $\frac{\chi}{\big(\sum_{r=1}^{\chi}\frac{1}{\pi_r}+P_{\max}\big)\ln 2}$ for all the possible values of χ, which holds in most works where the implicit constraints in (2) are not considered. However, if the implicit constraints in (2) is considered, ν^* may take multiple values as shown in *Theorem* 6. Most importantly, *Theorem* 6 provides a simple **non-iterative closed-form method** to get the optimal solution $\mathbf{p}^*$, denoted as Direct Search Method (DSM).

From the above analysis, not considering the implicit constraints in (2) can be regarded as a special case of considering them in this paper, which can be extended to other optimization problems. To get the optimal solution $\mathbf{p}^*$ to problem (3), whether the implicit constraints are considered in this paper or not in most works, it is very important to get the optimal multiplier ν^* by using either non-iterative methods (i.e., the proposed DSM) or iterative methods (e.g., subgradient method). We will show that considering the implicit constraints can greatly improve the convergence speed in iterative methods. In this paper, we only discuss the widely used subgradient method.

4 Algorithms to Solve the Optimization Problem

Based on the above analysis, the subgradient method without considering the implicit constraints and the proposed subgradient method that considers the implicit constraints are described in Algorithm 1 (Alg. 1) and Algorithm 2 (Alg. 2), respectively. Once the convergence condition is given, the only difference between Alg. 1 and Alg. 2 is the updating of $\mathbf{p}^{(t)}$ at each iteration, i.e., the operation of $\min\{\mathbf{x}, P_{\max}\}$ in the proposed Alg. 2 is not found in Alg. 1. Note that these algorithms share a common form of those applying the subgradient method in most works and that $\mathbf{p}^*$ is unique while ν^* may not be unique.

Algorithm 1. Existing Subgradient Method without Implicit Constraints.

Input: $\boldsymbol{\alpha}$, $P_{\max}$.
Initialize $t = 0$, $\mathbf{p}^{(0)} = \mathbf{0}_{N\times 1}$, $\nu^{(0)} = 0.1$, accuracy $\eta = 10^{-5}$.
while not converge **do**
  Update $\mathbf{p}^{(t)}$ as $p_n^{(t)} = \left[\frac{1}{\nu^{(t)}\ln 2} - \frac{1}{\alpha_n}\right]^+, \forall n \in \mathcal{N}$.
  Check convergence condition: $\left|\nu^{(t)}\left(\sum_{n=1}^{N} p_n^{(t)} - P_{\max}\right)\right| < \eta$.
  Set $t \leftarrow t + 1$.
  Update $\nu^{(t)} = \left[\nu^{(t-1)} + \theta^{(t)}\left(\sum_{n=1}^{N} p_n^{(t-1)} - P_{\max}\right)\right]^+$.
end while
Output: $\mathbf{p}$, ν.

Algorithm 2. Proposed Subgradient Method with Implicit Constraints.

Input: $\boldsymbol{\alpha}$, $P_{\max}$.
Initialize $t = 0$, $\mathbf{p}^{(0)} = \mathbf{0}_{N\times 1}$, $\nu^{(0)} = 0.1$, accuracy $\eta = 10^{-5}$.
while not converge **do**
  Update $\mathbf{p}^{(t)}$ as $p_n^{(t)} = \min\left\{\left[\frac{1}{\nu^{(t)}\ln 2} - \frac{1}{\alpha_n}\right]^+, P_{\max}\right\}, \forall n \in \mathcal{N}$.
  Check convergence condition: $\left|\nu^{(t)}\left(\sum_{n=1}^{N} p_n^{(t)} - P_{\max}\right)\right| < \eta$.
  Set $t \leftarrow t + 1$.
  Update $\nu^{(t)} = \left[\nu^{(t-1)} + \theta^{(t)}\left(\sum_{n=1}^{N} p_n^{(t-1)} - P_{\max}\right)\right]^+$.
end while
Output: $\mathbf{p}$, ν.

Besides, note that the achieved nonnegative solution $\mathbf{p}$ may be infeasible in the iteration process with subgradient method. Then we provide the sketch proof that considering the implicit constraints can improve the convergence speed of subgradient method as follow.

(1) If there exists $k < +\infty$ such that $\nu^{(k)} = 0$ in the iteration process, we have

– Without the implicit constraints considered, we have $p_n^{(k)} = \left[\frac{1}{\nu^{(k)}\ln 2} - \frac{1}{\alpha_n}\right]^+ = +\infty, \forall n \in \mathcal{N}$, and $\nu^{(k+1)} = \left[\nu^{(k)} + \theta^{(k+1)}\left(\sum_{n=1}^N p_n^{(k)} - P_{\max}\right)\right]^+ = +\infty$. Then we have $p_n^{(k+i)} = 0, \forall n \in \mathcal{N}$ and $\nu^{(k+i)} = +\infty$ for all $1 \leq i < +\infty$, which means the iteration process will not converge in a limited number of iterations.
– With the implicit constraints considered, we have $p_n^{(k)} = P_{\max}, \forall n \in \mathcal{N}$, and $\nu^{(k+1)} = \theta^{(k+1)}(N-1)P_{\max}$, which can avoid the above bad case and thus guarantee the convergence in a limited number of iterations.

(2) Otherwise, we have $\nu^{(t)} > 0, \forall t$. Thus, we have $\nu^{(t)} = \nu^{(t-1)} + \theta^{(t)}\left(\sum_{n=1}^N p_n^{(t-1)} - P_{\max}\right) > 0$ for $\forall t \geq 1$, and then $|\nu^{(t)} - \nu^{(t-1)}| = \theta^{(t)}\left|\sum_{n=1}^N p_n^{(t-1)} - P_{\max}\right|$, which indicates that the convergence speed of $\nu^{(t)}$ depends on the steps $\theta^{(t)}$ and the value of $\left|\sum_{n=1}^N p_n^{(t-1)} - P_{\max}\right|$. Refer to [14,15] on how to choose a proper specific series of steps $\theta^{(t)}$. Since $\left|\sum_{n=1}^N \min\left\{\left[\frac{1}{\nu^{(t)}\ln 2} - \frac{1}{\alpha_n}\right]^+, P_{\max}\right\} - P_{\max}\right| \leq \left|\sum_{n=1}^N \left[\frac{1}{\nu^{(t)}\ln 2} - \frac{1}{\alpha_n}\right]^+ - P_{\max}\right|$ for $\forall t$, with the same steps $\theta^{(t)}$, the subgradient method considering the implicit constraints can achieve a higher convergence speed and needs fewer iterations than the subgradient method that does not consider the implicit constraints.

In terms of the step size $\theta^{(t)}$, we will use three common categories as: *(1)* C1: constant step size, e.g., $\theta^{(t)} = 0.1$ for $\forall t$; *(2)* C2: nonsummable diminishing step size, e.g., $\theta^{(t)} = \frac{1}{\sqrt{t}}$ for $\forall t$; *(3)* C3: square summable but not summable step size, e.g., $\theta^{(t)} = \frac{100}{t+100N}$ for $\forall t$.

As shown in [14,15], for C1, the subgradient method converges to the optimal value within a small range, i.e., $\lim_{t\to\infty} |\nu^{(t)} - \nu^*| < \varepsilon$. This indicates that the subgradient method finds an $\varepsilon-$suboptimal point within a finite number of iterations. The value ε is a decreasing function of the step size. Moreover, for C2 and C3, the subgradient method is guaranteed to converge to the optimal value if the chosen steps are small enough. With **proper** initialization, both the proposed and existing subgradient methods always converge but need different numbers of iterations. Most importantly, the solution found in each iteration may not be feasible. Considering the implicit constraints in each iteration can accelerate the iterative search of the optimal solution by excluding infeasible solutions from the search subspace, and thus achieve a higher convergence speed.

In terms of the sensitivity of the subgradient method to the chosen step sizes, the existing method may not converge if the step sizes are **improper**, but the proposed method will converge for these step sizes. The detailed theoretical analysis is omitted due to the page limit, but we will provide some numerical examples in Sect. 5 to show that considering the implicit constraints leads to convergence even using step sizes that are **improper** to the existing method.

5 Numerical Results

In this section, we evaluate by simulations the effectiveness of the subgradient methods considering the implicit constraints in solving the water-filling problem and the power allocation problem in [16], in terms of the convergence speed and sensitivity to the chosen step sizes. Since the only difference between the subgradient methods with and without considering the implicit constraints is the operation of $\min\{x, P_{\max}\}$, which computation complexity can be neglected compared to that of the whole algorithm, we use the number of iterations required to satisfy the convergence condition as the convergence speed of the respective algorithms [14,15]. Note that in the following, convergence accuracy refers to the gap between the achieved value (e.g., optimization objective value and multiplier value) and its optimal value, and that we use the ratio of the required numbers of iterations of the subgradient methods without/with considering the implicit constraints for the following comparison of convergence speed x.

5.1 Numerical Examples in Water-Filling Problem

In the water-filling problem in (3), by using the proposed DSM to get the optimal values (i.e., the optimal objective value z^*, the optimal dominating multiplier ν^* and the optimal solution $\mathbf{p}^*$) as the baseline, we mainly evaluate the performance of the subgradient methods with/without considering the implicit constraints and then explore whether the implicit constraints are really redundant in the above three scenarios. We provide two numerical examples as follow.

- **Example 1:** $N = 3$, $P_{\max} = 1$, $\boldsymbol{\alpha} = [0.75,\ 2,\ 3]^T$. DSM gives the optimal solution $\mathbf{p}^* = [0,\ 0.416667,\ 0.583333]^T$, the optimal objective $z^* = -\sum_{n=1}^{N} \log_2(1 + \alpha_n p_n^*) = -2.333901$, and the optimal multiplier $\nu^* = 1.573849$.
- **Example 2:** $N = 20$, $P_{\max} = 2$, $\boldsymbol{\alpha} = [2,\ 1.5,\ \alpha_3,\ \ldots,\ \alpha_{20}]^T$, where α_n is a random value in (0, 2] for $\forall n \in \mathcal{N}, n \geq 3$. DSM can also yield the optimal solution, the optimal objective and the optimal multiplier directly.

Figure 1 compares the values of $|z^{(t)} - z^*|$ and $|\nu^{(t)} - \nu^*|$ versus the iteration number for the subgradient methods in **Example 1** and **Example 2**. Here, the step sizes are set as $\theta^{(t)} = 0.1$, $\theta^{(t)} = \frac{1}{\sqrt{t}}$ and $\theta^{(t)} = \frac{100}{t+100N}$ for C1, C2 and C3, respectively. For **Example 1**, Fig. 1(a) and (b) show that with each of C1, C2 and C3, Alg. 2 has similar convergence accuracy and a higher convergence speed compared with Alg. 1. Specifically, **Alg2-C1** is about 1.3 times as fast as **Alg1-C1** while **Alg2-C2** is about 22.3 times faster than **Alg1-C2**. Besides, **Alg2-C3** is about 2.7 times as fast as **Alg1-C3**. For **Example 2**, Fig. 1(c) and (d) show that with each of C1, C2 and C3, Alg. 2 has higher or since convergence accuracy and higher convergence speed than Alg. 1. Specifically, **Alg2-C1** is about 4.9 times as fast as **Alg1-C1** while **Alg2-C2** is about 35.9 times faster than **Alg1-C2**. Besides, **Alg2-C3** is about 3.9 times as fast as **Alg1-C3**.

Moreover, by setting different step sizes, we show that considering the implicit constraints can reduce the sensitivity to the step sizes and lead to convergence while the existing method fails to converge. Here, if the iteration process does not stop within 10^7 iterations, the method is regarded as not convergent from the perspective of practical engineering implementation. For instance, we set $\theta^{(t)} = \frac{N}{t+N}$ for $\forall t$ in C3 and use the above same convergence conditions. In this

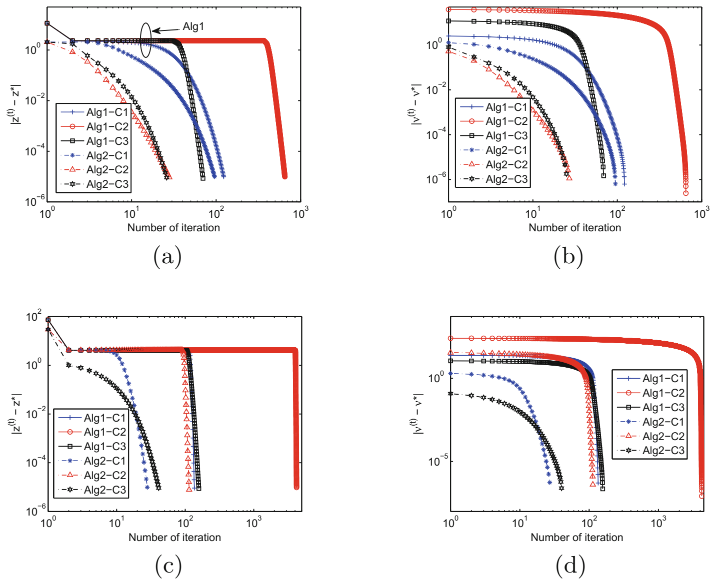

Fig. 1. The values of $|z^{(t)} - z^*|$ and $|\nu^{(t)} - \nu^*|$ versus iteration number, (a) and (b) for **Example 1**, (c) and (d) for **Example 2**.

setting, **Alg1-C3** does not converge in **Example 1** while **Alg2-C3** converges with 29 iterations.

5.2 Numerical Results of Optimizing Power Allocation in [16]

To further evaluate the effectiveness of considering the implicit constraints, we consider the power allocation optimization problem in [16], which maximizes the total system throughput in a multiuser orthogonal frequency-division multiplexing system under the constraints of total power and minimum data rate required by each user. To solve the problem, we use the subgradient method without considering the related implicit constraints (Alg. 3) and with this consideration (Alg. 4). Note that Alg. 3 is the subgradient method used in [16]. In our simulations, we set the subcarrier number $N = 10$, user number $K = 4$, total power $P_{tot} = 5$ Watt and all the users' required minimum data rate is 5 bps/Hz. We also use the same channel model as in [16], the same convergence condition as in Alg. 1 and Alg. 2, and the same step sizes in C1, C2 and C3 as in the previous examples. Note that the convergence condition used in this paper is much stricter than that in [16] and thus the algorithms require more iterations to converge but they achieve more accurate solutions. We use the gap between the achieved objective and the optimal objective as the performance metric. All the results are averaged over 100 channel realizations.

Figure 2 compares the performance of the subgradient methods w.r.t. the values of objective gap versus iteration number. Figure 2 shows that Alg. 4 always

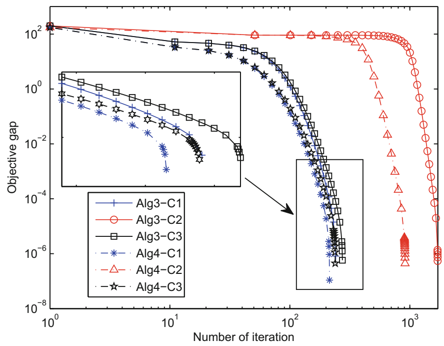

Fig. 2. The values of objective gap versus iteration number.

has a higher convergence accuracy and higher speed than Alg. 3. Specifically, in C1, C2 and C3, Alg. 4 is about 1.13, 1.87 and 1.15 times as fast as Alg. 3. Thus, considering the implicit constraints can effectively speed up the convergence of the subgradient method for optimizing the power allocation in [16].

6 Conclusions

In this paper, we have explored the question "Are the implicit constraints really redundant?" in power allocation optimization especially when using subgradient methods. Specifically, by illustrating the water-filling problem to answer the question, we have derived the structural properties of the optimal solutions based on KKT conditions, proposed a non-iterative closed-form optimal method and applied subgradient methods to solve the water-filling problem. Besides, our theoretical analysis has shown that the implicit constraints are not redundant, and their consideration can effectively improve the subgradient methods' convergence speed and reduce the sensitivity to the chosen step sizes. Numerical results have shown that considering the implicit constraints in the water-filling problem can greatly speed up the methods' convergence speed by up to about 36 times and reduce the sensitivity to the chosen step sizes, and that it can also effectively accelerate the convergence speed by up to 87% in the power allocation problem in [16]. Thus, the implicit constraints are not redundant in the algorithm design. Most importantly, the corresponding theoretical analysis and conclusions about the implicit constraints can be extended to many other resource allocation problems and to other iterative methods.

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