Throughput Capacity Analysis of a Random Multi-user Multi-channel Network Modeled as an Occupancy Problem

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Abstract. In this paper, we model the random multi-user multi-channel access network by using the well known *occupancy problem* from probability theory. Furthermore, we combine this with a network interference model in order to derive the achievable throughput capacity of such networks. The mathematical developments and results are illustrated through various simulations results. The proposed model is particularly relevant in analyzing the performance of networks where the users are not synchronized neither in time nor in frequency as it is often the case in various Internet of Things (IoT) applications.

Keywords: Throughput capacity · Network interference · Occupancy problem

1 Introduction

The issue of transmitting asynchronous signals on a single channel has been studied for several decades [1] and it led to some protocols such as ALOHA (ALOHAnet) proposed by N. Abramson in 1985 [2]. Since then, many solutions have been proposed in the literature to overcome the distortions induced by the collisions among the transmitted signals. [3] and references therein provide an extensive overview of solutions based on packet retransmissions and [4] treats solutions based on signal coding. More recently, the model has been extended to the case of random frequency channel access besides random time channel access [5]. The authors proposed to model the interferences induced by the collisions of ultra narrow-band signals featuring a random frequency channel access in a context of the Internet-of-Things (IoT) applications.

In this paper, we propose to further analyze the issue of signal collisions in the network interference problem [6,7] by considering random multi-channel,

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multi-user access. In other words we extend the analysis to account for random behavior along the frequency dimension as well where each user of the network transmits not only at random times but also on randomly chosen channels within the band. The random frequency access of N_{ut} users into N_c channels can be modeled by the *occupancy problem* used in probability theory [13]. From this model we subsequently derive an analytical expression of the achievable throughput capacity of such a network, i.e. the probability that randomly transmitted signals are properly decoded at the receiver. To the best of our knowledge, the *occupancy problem* is an original approach for modeling a multi-channel, multi-user access network. Moreover, various simulations illustrate the obtained theoretical results.

The remaining of the paper is organized as follows: Sect. 2 presents the model of the considered network, and provides a reminder of the *occupancy problem*. The expression of the throughput capacity is derived in Sect. 3, and simulations are provided in Sect. 4. Finally, Sect. 5 concludes this paper.

2 System Model

2.1 Network Model

In this paper, it is assumed that all the transmitters (called users in this paper) of the network are distributed in the two-dimensional plane according to an homogeneous Poisson point process with a given intensity λ (in users per unit area), whose distribution is given as follows:

$$f_{poi}(k) = \frac{\lambda^k}{k!}e^{-\lambda}. \tag{1}$$

As depicted in Fig. 1, base transceiver stations (BTS) are also located in the plane, in order to pick up the signals from users. For instance considering the BTS at the center of Fig. 1, we assume it services a cell of radius R_c that contains N_{ut} users, indicated as small black circles. The users outside this cell (white circles in Fig. 1) are potential interfering users. It should be noted, however, that the users may interfere with each other as well. According to the defined parameters, the intensity λ is equal to $N_{ut}/(\pi R_c^2)$.

The role of a BTS consists in scanning and sampling the band $\mathcal{B}$ of bandwidth B which is regularly subdivided into N_c channels $\{\mathcal{B}_0, \mathcal{B}_1, .., \mathcal{B}_{N_c-1}\}$ of width B/N_c. The considered random multi-channel multi-user transmission model can be formalized as follows:

- Each user can randomly access, i.e. select, one of the N_c channels with a probability $\mathbb{P}_a = \frac{1}{N_c}$. In that way, a reuse[1] of one or many channels may occur, as shown in Fig. 2-(a). In the following, we denote by $\mathcal{R}$ the reuse factor of the channels, i.e. the number of users sharing the same channel.

[1] Meaning that a channel can be selected by more than one user.

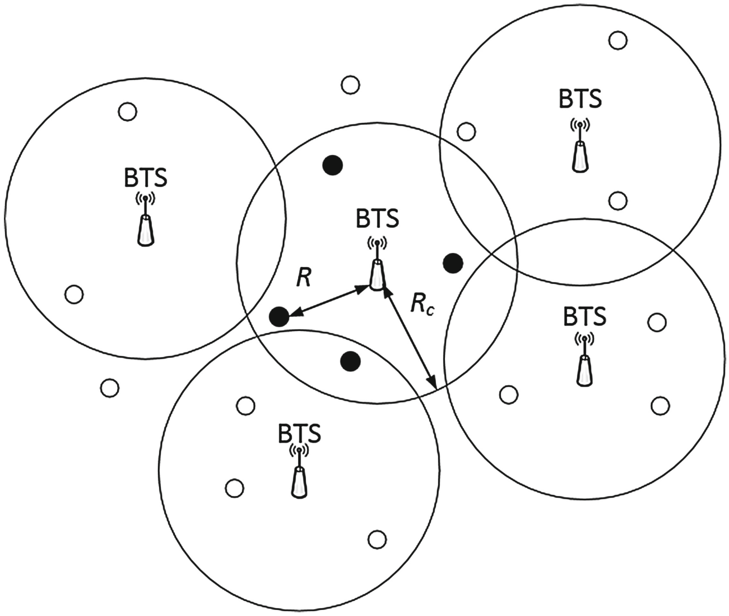


Fig. 1. Poisson point model for the spatial distribution of the users.

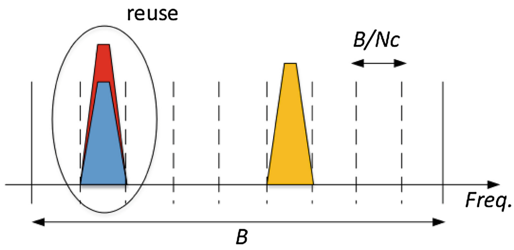
– A slotted traffic scheme is assumed for each user (see Fig. 2-(b)). Therefore, an asynchronous slotted traffic is considered for each channel at the BTS, due to the independence between users. The probability of transmission of a packet (or duty cycle[2]) is denoted by $q \in [0, 1]$, and for simplicity, it is supposed that q is the same for each user. As depicted in Fig. 2-(b), collisions may occur between packets when at least two users transmit at the same time on the same channel.

According to the model in [6,10], the power received at the BTS from a user at a distance R can be defined as

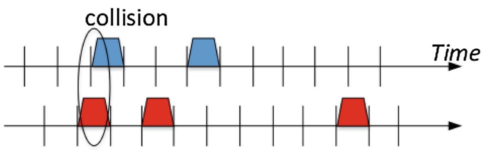
$$P_r = \frac{P_t G}{R^{2c}}, \tag{2}$$

where P_t denotes the transmitted power, and G is a random variable that depends of the propagation environment. Several models can be used to describe G, depending on the shadowing (mainly due to large objects), and the multipath fading (mainly due to the constructive or destructive combinations of the replicas of the transmitted signal). The term $1/R^{2c}$ is defined as the far-field path loss, where c depends on the propagation environment and is in the range of 0.8 to 4 [6]. It should be emphasized that the far-field path loss fits the considered

[2] Note that *duty cycle* is sometimes used to refer to the overlapping factor between to packets, as in [6].



(a) Random frequency access to the channels. A reuse $\mathcal{R} > 1$ occurs when at least 2 users choose the same channel.



(b) Slotted-asynchronous transmission. A collision occurs when to users transmit at the same time in the same channel.

Fig. 2. Frequency and time channel access. (Color figure online)

model, in which the users are supposed to be at least several meters away from the BTS.

2.2 Occupancy Problem

The random access to N_c channels by N_{ut} users is an instance of the *occupancy problem* in probability. This theory provides useful tools to deal with the time-frequency use of the band $\mathcal{B}$ in the considered transmission model. In particular, two theorems will be used in this paper.

Theorem 1. *Let N_{ut} users randomly accessing N_c channel with a probability $\mathbb{P}_a = \frac{1}{N_c}$. Then the probability that b channels are used (at least by one user) is*

$$\mathbb{P}(b) = \binom{N_c}{N_c - b} \sum_{\nu=0}^{b} (-1)^{\nu} \binom{b}{\nu} \left(1 - \frac{N_c - b + \nu}{N_c}\right)^{N_{ut}}. \tag{3}$$

Proof. see [13], Chap. 4.

Theorem 2. *Under the same assumptions as in Theorem 1, the probability that $\mathcal{R}$ users access the same channel ($0 \leq \mathcal{R} \leq N_{ut}$) is*

$$\mathbb{P}(\mathcal{R}) = \binom{N_{ut}}{\mathcal{R}} \left(\frac{1}{N_c}\right)^{\mathcal{R}} \left(1 - \frac{1}{N_c}\right)^{N_{ut}-\mathcal{R}}. \tag{4}$$

Proof. Let $\mathbb{P}(j)$ the probability of the event "the j-th channel is chosen", and $\mathbb{P}(X_j = \mathcal{R})$ the probability that $\mathcal{R}$ users use the j-th channel. As $\mathbb{P}(j) = \frac{1}{N_c}$ and $\mathbb{P}(X_j = \mathcal{R})$ are equal for any j, then we obtain

$$\mathbb{P}(\mathcal{R}) = \sum_{j=1}^{N_c} \mathbb{P}(j)\mathbb{P}(X_j = \mathcal{R}) \tag{5}$$

$$= \sum_{j=1}^{N_c} \frac{\mathbb{P}(X_j = \mathcal{R})}{N_c} \tag{6}$$

$$= \mathbb{P}(X_j = \mathcal{R}). \tag{7}$$

The reuse factor $\mathcal{R}$ can be defined as equal to the sum $X_j = \mathcal{R} = \sum_k x_{j,k}$, where $x_{j,k}$ is a random variable which counts the number of users in the channel j. Since the users access to the channels independently of each other, $x_{j,k}$ are the results of Bernoulli trials with probability $1/N_c$ to access the channel j. Therefore, the sum $X_j = \sum_k x_{j,k}$ has the binomial distribution defined in (4).

3 Deriving the Throughput Capacity

This section deals with the analysis of the throughput capacity, whose definition is given hereafter. In the rest of the paper, we use the so-called physical interference model [7,8], in which the following condition is imposed:

– a message from a user in the cell is successfully decoded if the corresponding signal-to-interference-plus-noise ratio (SINR) exceeds a given threshold γ_p. Thus, successful decoding for a user k in a given channel n (with reuse factor $\mathcal{R}$, and $1 \leq n \leq N_C$) requires that:

$$SINR_{k,n} = \frac{P_{r,k,n}}{I_{k,n} + \sigma_n^2} \geq \gamma_p, \tag{8}$$

where $P_{r,k,n}$ is the received power of the k-th probe user in the n-th channel as defined in (2), and σ_n^2 is the noise power in the n-th channel. The term $I_{k,n}$, using the general definition in [6], is the interference power that can be written as

$$I_{k,n} = \sum_{\substack{i=1 \\ i \neq k}}^{\infty} \frac{P_t \Delta_{i,n} G_{i,n}}{R_{i,n}^{2c}}, \tag{9}$$

where $\Delta_{i,n}$ is the overlapping factor between the i-th interfering signal and the proper signal k. It is assumed that $\Delta_{i,n}$ obeys a uniform distribution in $[0, 1]$.

By convention, we consider that the $\mathcal{R}-1$ first terms of the sum in (9) correspond to the users, and the indexes $i > \mathcal{R}$ point out the interfering users that are located outside the considered cell.

It has been demonstrated in [9] that the interference defined as the superposition of numerous signals from users distributed according to a homogeneous Poisson process on a plane can be modeled as a α-stable distribution [11,12], which can be seen of a generalization of the Gaussian distribution. No analytic expression of the probability density function (pdf) of the α-stable law can be derived, but its characteristic function is expressed as

$$\phi(t) = \exp(-\gamma|t|^{\alpha}(1 - j\beta \mathrm{sign}(t)\omega(t,\alpha))), \tag{10}$$

where

$$\omega(t,\alpha) = \begin{cases} \tan(\frac{\pi\alpha}{2}), & \text{if } \alpha \neq 1 \\ -\frac{2}{\pi}\ln(|t|), & \text{if } \alpha = 1 \end{cases}, \tag{11}$$

and

$$\mathrm{sign}(t) = \begin{cases} -1, & \text{if } t < 0 \\ 0, & \text{if } t = 0 \\ 1, & \text{if } t > 0 \end{cases}. \tag{12}$$

According to [6], the parameters α, β, and γ are defined as

$$\begin{aligned} \alpha &= \frac{1}{c} \\ \beta &= 1 \\ \gamma &= \pi\lambda_{\mathcal{R}} C_{1/c}^{-1} P_t^{1/c} E\{\Delta_{k,n}^{1/c}\} E\{G_{k,n}\}, \end{aligned}$$

where

$$C_\alpha = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}, & \text{if } \alpha \neq 1 \\ \frac{2}{\pi}, & \text{if } \alpha = 1 \end{cases}, \tag{13}$$

with $\Gamma(.)$ the gamma function. It is worth noting that $\lambda_{\mathcal{R}}$ is now a variable function of $\mathcal{R}$, namely $\lambda_{\mathcal{R}} = \mathcal{R}/(\pi R_c^2)$. The throughput capacity denoted by $\mathcal{T}_{\mathcal{R},n,R_{k,n}}$ for a given reuse factor $\mathcal{R}$ in the n-th channel, and located at a distance $R_{k,n}$ from the BTS, can be defined as

$$\mathcal{T}_{\mathcal{R},n,R_{k,n}} = \mathbb{P}(\text{transmit})\mathbb{P}(\text{no outage}), \tag{14}$$

where $\mathbb{P}(\text{transmit}) = q_{\mathcal{R}} = 1-(1-q)^{\mathcal{R}}$ is the probability that a channel is occupied, and $\mathbb{P}(\text{no outage}) = \mathbb{P}(SINR_{k,n} \geq \gamma_p)$. The probability $\mathbb{P}(SINR_{k,n} \geq \gamma_p)$ can be rewritten by substituting (9) into (8), and by using the law of total probability as

$$\mathbb{P}(SINR_{k,n} \geq \gamma_p) = E_{\{G_{k,n}\}}\{\mathbb{P}_{\{I_{k,n}\}}\left(I_{k,n} \leq \frac{P_t G_{k,n}}{\gamma_p R_{k,n}^{2c}} - \sigma_n^2\right)\Big|G_{k,n}\}, \tag{15}$$

and hence

$$\mathbb{P}(SINR_{k,n} \geq \gamma_p) = E_{\{G_{k,n}\}}\{F_{I_{k,n}}\left(\frac{P_t G_{k,n}}{\gamma_p R_{k,n}^{2c}} - \sigma_n^2\right)\}$$
$$= E_{\{G_{i,n}\}}\{\int_{-\infty}^{\frac{P_t G_{k,n}}{\gamma_p R_{k,n}^{2c}} - \sigma_n^2} \int_{-\infty}^{+\infty} \phi_{I_{k,n}}(t) e^{-jtx} dt dx\}, \quad (16)$$

where $F_{I_{k,n}}$ is the cumulative distribution function (cdf) of $I_{k,n}$. Several closed-form of (16) corresponding to different types of shadowing and fading have been derived in [6,10]. In particular, it should be noted that the expectation in (15) disappears if the case *path-loss only* $G_{k,n} = 1$ is considered, and

$$\mathbb{P}(SINR_{k,n} \geq \gamma_p) = F_{I_{k,n}}\Big(\frac{P_t G_{k,n}}{\gamma_p R_{k,n}^{2c}} - \sigma_n^2\Big). \quad (17)$$

The Rayleigh fading leads to the following expression:

$$\mathbb{P}(SINR_{k,n} \geq \gamma_p) = \exp\Big(-\frac{R_{k,n}^{2c} \sigma_n^2 \gamma_p}{P_t}\Big)$$
$$\times \exp\left(-\frac{\lambda_{\mathcal{R}} C_{1/c}^{-1} \Gamma(1 + \frac{1}{c}) E\{\Delta_{k,n}^{1/c}\}}{\cos(\frac{\pi}{2c})} \big(R_{k,n}^{2c} \gamma_p\big)^{1/c}\right). \quad (18)$$

Since the users are homogeneously distributed in the cell, then the distance $R_{k,n}$ obeys the uniform distribution denoted by $\mathcal{U}(0, R_c)$, and defined as:

$$f_u(x) = \begin{cases} \frac{1}{R_c}, & \text{if } x \in [0, R_c] \\ 0, & \text{else} \end{cases}. \quad (19)$$

The throughput capacity is then obtained by averaging $\mathbb{P}(SINR_{k,n} \geq \gamma_p)$ on the interval $[0, R_c]$ as:

$$\mathcal{T}_{\mathcal{R},n} = q_{\mathcal{R}} E_{R_{k,n}}\{\mathcal{T}_{\mathcal{R},n,R_{k,n}}\}$$
$$= q_{\mathcal{R}} \int_0^{R_c} f_u(R_{k,n}) \mathbb{P}(SINR_{k,n} \geq \gamma_p) dR_{k,n}. \quad (20)$$

Note that the bound 0 of the integral in (20) should be replaced by a positive value according to the far-field model (typically ¿1 meter). Besides, this also avoids the division by zero if the *path-loss only* model in (17) is used. Since we consider a slotted-asynchronous packet transmission, then the value $E\{\Delta_{k,n}^{1/b}\}$ can be derived by following the developments in [10], which lead to:

$$E\{\Delta_{k,n}^{1/c}\} = q^2 + 2q(1-q)\frac{c}{1+c}. \quad (21)$$

Since the users are independent and the N_c channels have the same probability of access $\mathbb{P}_a$, the throughput capacity for the given n-th channel with reuse factor $\mathcal{R}$ is given by:

$$\mathcal{T}_{\mathcal{R}} = \mathbb{P}_a \mathbb{P}(\mathcal{R}) \mathcal{T}_{\mathcal{R},n}, \tag{22}$$

and the overall achieved throughput capacity considering all the users of the cell is defined as the following weighted sum:

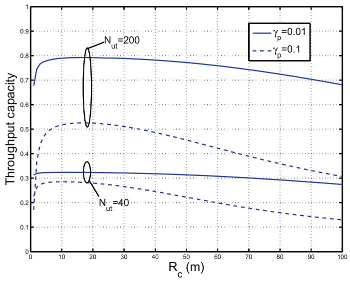
$$\mathcal{T} = \sum_{n=1}^{N_c} \mathbb{P}_a \sum_{\mathcal{R}=0}^{N_{ut}} \mathbb{P}(\mathcal{R}) \mathcal{T}_{\mathcal{R},n} = \sum_{\mathcal{R}=0}^{N_{ut}} \mathcal{T}_{\mathcal{R}}, \tag{23}$$

where $\mathbb{P}(\mathcal{R})$ is given in (4). It can be noticed in the developments from (8) to (23) that at least five parameters have an influence on the throughput capacity value $\mathcal{T}$: the radius of the cell R_c, the number of users N_{ut}, the number of channels N_c, the duty cycle q, the threshold γ_p, and the noise level σ_n^2. Note that we assume $\sigma_n^2 = \sigma^2$ for any channel $1 \leq n \leq N_c$.

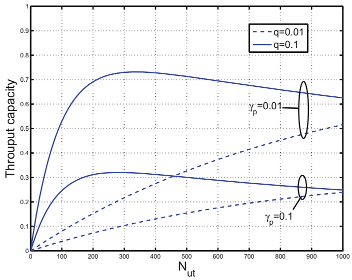
4 Simulations Results

In this section, we present the simulations results related to the throughput capacity defined as in (23). In all the simulations, the value of c has been arbitrarily set equal to 1.2, $N_c = 10$, and a Rayleigh fading is considered. Therefore the "no-outage" probability is modeled by (18). Figure 3 depicts $\mathcal{T}$ versus: (a) the radius of the cell R_c (in m), (b) the number of users N_{ut}, (c) the threshold γ_p, and (d) the duty cycle q. In this first series of simulations, the SNR defined as P_t/σ^2 has been set equal to 30 dB. In Fig. 3-(a), it can be seen that $\mathcal{T}$ values decrease where R_c increases, which reflects the fact that the average signal power $E\{P_{r,k,n}\}$ received at the BTS is lower in large cells than in small cells. Figure 3-(b) shows that the throughput capacity achieves a maximum for a given N_{ut} value that we can denote by N_{ut}^m. We can deduce that in the range $N_{ut} \leq N_{ut}^m$, $\mathcal{T}$ is limited by $\mathbb{P}$(transmit) (i.e. the number of transmitting users induces a low value of $\mathbb{P}$(transmit)), while in the range $N_{ut} \geq N_{ut}^m$, $\mathcal{T}$ is limited by $\mathbb{P}$(no outage) (i.e. the users which transmit induce a large amount of interference). In Fig. 3-(c) it is verified that the throughput capacity decreases where γ_p increases, according to the network interference model in (8). Similarly to Fig. 3-(b), we observe in Fig. 3-(d) that $\mathcal{T}$ increases with q when $N_{ut} = 10$, whereas it reaches a maximum before decreasing for higher numbers of users ($N_{ut} = 100$ and $N_{ut} = 1000$). This phenomenon is due to the presence of the interferences when either or both N_{ut} and q are high-valued.

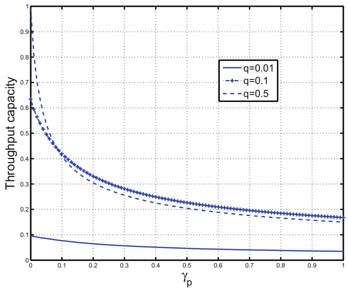
Previous results are completed by Fig. 4,which presents $\mathcal{T}$ versus SNR, in the SNR range from 0 to 60 dB. Note that very high SNR values are consistent with the system model using narrow band signals. The depicted results show the influence of the interferences which lead to an upper bound of the achievable throughput capacity in the proposed system.



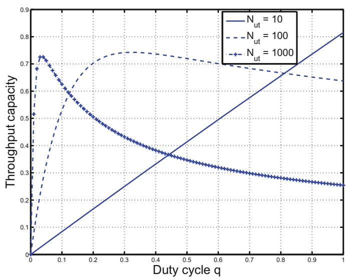
(a) $\mathcal{T}$ versus R_c, for SNR=30 dB, $q = 0.1$.



(b) $\mathcal{T}$ versus N_{ut}, for SNR=30 dB, $R_c = 100$ m.

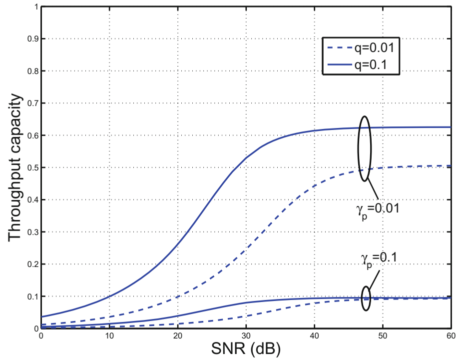


(c) $\mathcal{T}$ versus γ_p, for SNR=30 dB, $R_c = 100$ m.

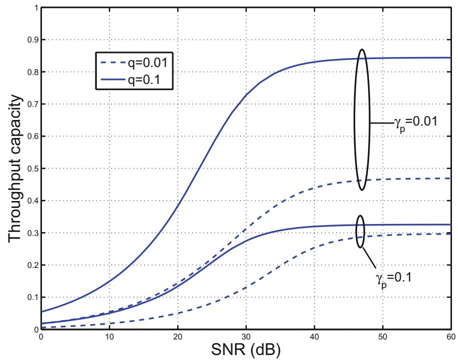


(d) $\mathcal{T}$ versus q, for SNR=30 dB, $R_c = 100$ m, and $\gamma_p = 0.01$.

Fig. 3. Throughput capacity $\mathcal{T}$ versus (a) R_c (in m), (b) N_{ut}, (c) γ_p, and (d) q.



(a) $\mathcal{T}$ versus SNR, for $N_{ut} = 100$.



(b) $\mathcal{T}$ versus SNR, for $N_{ut} = 400$.

Fig. 4. Throughput capacity $\mathcal{T}$ versus SNR, for (a) $N_{ut} = 100$, (b) $N_{ut} = 400$, $R_c = 100$ m.

5 Conclusion

In this paper, we modeled a random multi-channel multi-user network using the, well known in probability, *occupancy problem*. We subsequently derived an analytical expression of the achievable *throughput capacity* of such a system. The mathematical developments and the simulations results revealed that the throughput capacity values largely depend on the cell size, the threshold used in the network interference model, the number of users in the network and the duty cycle of the signals. It is worth noting that the proposed approach can be used as a basis for the performance analysis of specific networks using random multi-channel multi-user access, such as in the context of the IoT applications. Further work will extend the proposed model to a more general case where no channelization of the band is imposed and the user transmissions may occupy different bandwidths within the band.

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