Simple and Accurate Closed-Form Approximation of the Standard Condition Number Distribution with Application in Spectrum Sensing

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Abstract. Standard condition number (SCN) detector is a promising detector that can work effectively in uncertain environments. In this paper, we consider a Cognitive Radio (CR) with large number of antennas (eg. Massive MIMO) and we provide an accurate and simple closed form approximation for the SCN distribution using the generalized extreme value (GEV) distribution. The approximation framework is based on the moment-matching method and the expressions of the moments are approximated using bi-variate Taylor expansion and results from random matrix theory. In addition, the performance probabilities and decision threshold are also considered as they have a direct relation to the distribution. Simulation results show that the derived approximation is tightly matched to the condition number distribution.

Keywords: Standard condition number · Spectrum sensing · Wishart matrix · Massive MIMO

1 Introduction

Cognitive Radio (CR) is being the technology that provides solution for the scarcity and inefficiency in using the spectrum resource. For the CR to operate effectively and to provide the required improvement in spectrum efficiency, it must be able to effectively detect the presence/absence of the Primary User (PU) to avoid interference if it exists and freely use the spectrum in the absence of the PU. Thus, Spectrum Sensing (SS), being responsible for the presence/absence detection process, is the key element in any CR guarantee.

Several SS techniques were proposed in the last decade, however, Eigenvalue Based Detector (EBD) has been shown to overcome noise uncertainty challenges

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and performs adequately even in low SNR conditions. It presents an efficient way for multi-antenna SS in CR [1,2] as it does not need any prior knowledge about the noise power or signal to noise ratio. EBD is based on the eigenvalues of the received signal covariance matrix and it utilises results from random matrix theory. It detects the presence/absence of the PU by exploiting receiver diversity and includes the Largest Eigenvalue detector, the Scaled Largest Eigenvalue detector, and the Standard Condition Number (SCN) detector [1–6].

The SCN is defined as the ratio of maximum to minimum eigenvalues. The SCN detector compares the SCN of the sample covariance matrix with a certain threshold. This threshold was set according to Marchenco-Pastur law (MP) in [1], however, it is not related to any error constraints. In [2], the authors have provided an approximate relation between the threshold and the False-Alarm Probability (P_{fa}) by exploiting the Tracy-Widom distribution (TW) for the maximum eigenvalue while maintaining the MP law for the minimum eigenvalue. This work was further improved in [3,4] by using the Curtiss formula for the distribution of the ratio of random variables. In these two cases, the threshold could not be computed online and Lookup Tables (LUT) should be used instead. The exact distribution of the SCN was, also, derived in [5] for 2 antennas and in [6] for 3 antennas, however, it is very complicated to extend this work for CR with more number of antennas.

In this paper, we are interested in finding a simple approximation for the SCN detector that allows the system to dynamically compute its threshold online. For this purpose, we propose to asymptotically approximate the SCN distribution with the Generalized Extreme Value (GEV) distribution by matching the first three central moments. This approximation yields a simple and accurate closed form expression for the SCN detector. Accordingly, the threshold could be simply computed. The main contributions of this paper are summarized as follows:

- derivation of the asymptotic central moments of the extreme eigenvalues.
- derivation of an asymptotic approximated form of the central moments of the SCN from that of the extreme eigenvalues.
- proposition of a new simple and asymptotic closed form approximation of the SCN detector using the central moments.

The rest of this paper is organized as follows. Section 2 provides the system model, the SCN detector and hypotheses analysis. In Sect. 3, we give the asymptotic mean, variance and skewness of the extreme eigenvalues under $\mathcal{H}_0$ and $\mathcal{H}_1$ hypotheses. The asymptotic mean, variance and skewness of the SCN are derived in Sect. 4. Then, we propose a new asymptotic approximation for the SCN detector. Theoretical findings are validated by simulations in Sect. 5 while the conclusion is drawn in Sect. 6.

2 Standard Condition Number Detector

2.1 System Model

Consider a CR equipped by K receiving antennas. After collecting N samples from each antenna, the received signal matrix, $\boldsymbol{Y}$, is given by:

$$\boldsymbol{Y} = \begin{pmatrix} y_1(1) & y_1(2) & \cdots & y_1(N) \\ \vdots & \vdots & \ddots & \vdots \\ y_K(1) & y_K(2) & \cdots & y_K(N) \end{pmatrix}, \quad (1)$$

where $y_k(n)$ is the baseband sample at antenna $k = 1 \cdots K$ and instant $n = 1 \cdots N$.

Two hypotheses exist: (i) $\mathcal{H}_0$: there is no PU and the received sample is only noise; and (ii) $\mathcal{H}_1$: the PU exists (single PU case is considered in this paper). The received vector, at instant n, under both hypotheses is given by:

$$\mathcal{H}_0: \; y_k(n) = \eta_k(n), \quad (2)$$

$$\mathcal{H}_1: \; y_k(n) = h_k(n)s(n) + \eta_k(n), \quad (3)$$

with $\eta_k(n)$ is a complex circular white Gaussian noise with zero mean and unknown variance σ_η^2, $h_k(n)$ is a the channel coefficient between the PU and antenna k at instant n, and $s(n)$ stands for the primary signal sample modeled as a zero mean Gaussian random variable with variance σ_s^2. Without loss of generality, we suppose that $K \leq N$ and the channel is considered constant during the sensing time for simplicity.

2.2 SCN Detector

Let $\boldsymbol{W} = \boldsymbol{Y}\boldsymbol{Y}^\dagger$, with $\dagger$ denotes the Hermitian notation, be the sample covariance matrix and denote by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_K > 0$ its ordered eigenvalues. Then the SCN of $\boldsymbol{W}$, defined as the ratio of the maximum to minimum eigenvalues, is given by:

$$X = \frac{\lambda_1}{\lambda_K}. \quad (4)$$

Denoting by α the decision threshold, then the probability of false alarm (P_{fa}), defined as the probability of detecting the presence of PU while it does not exist, and the detection probability (P_d), defined as the probability of correctly detecting the presence of PU, are, respectively, given by:

$$P_{fa} = P(X \geq \alpha / \mathcal{H}_0) = 1 - F_0(\alpha), \quad (5)$$

$$P_d = P(X \geq \alpha / \mathcal{H}_1) = 1 - F_1(\alpha), \quad (6)$$

where $F_0(.)$ and $F_1(.)$ are the Cumulative Distribution Functions (CDF) of X under $\mathcal{H}_0$ and $\mathcal{H}_1$ hypotheses respectively. If the expressions of the P_{fa} and/or P_d are known, then a threshold could be set according to a required error constraint. For a given threshold, $\hat{\alpha}$, the SCN detector algorithm could be summarized as follows:

1- compute λ_1 and λ_K of $\boldsymbol{W} = \boldsymbol{Y}\boldsymbol{Y}^\dagger$.
2- evaluate the SCN as $X = \lambda_1 / \lambda_K$.
3- decide according to $X \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \hat{\alpha}$.

2.3 Hypotheses Analysis

$\mathcal{H}_0$ hypothesis: By considering $\mathcal{H}_0$ hypothesis, the received samples are complex circular white Gaussian noise with zero mean and unknown variance σ_η^2. Consequently, the sample covariance matrix is a central uncorrelated complex Wishart matrix denoted as $\boldsymbol{W} \sim \mathcal{CW}_K(N, \sigma_\eta^2 \mathbf{I}_K)$ where K is the size of the matrix, N is the number of Degrees of Freedom (DoF), and $\sigma_\eta^2 \mathbf{I}_K$ is the correlation matrix and $\mathbf{I}$ denotes the identity matrix. The symbol '$\sim$' stands for distributed as.

$\mathcal{H}_1$ hypothesis: By considering $\mathcal{H}_1$ hypothesis, the single PU sample is Gaussian and the channel is constant during sensing time. Consequently, the sample covariance matrix is a non-central uncorrelated complex Wishart matrix denoted as $\boldsymbol{W} \sim \mathcal{CW}_K(N, \sigma_\eta^2 \mathbf{I}_K, \boldsymbol{\Omega}_K)$ where $\boldsymbol{\Omega}_K$ is a rank-1 non-centrality matrix[1].

Let $\widehat{\boldsymbol{\Sigma}}_K$ be the correlation matrix defined as:

$$\widehat{\boldsymbol{\Sigma}}_K = \sigma_\eta^2 I_K + {}^{\boldsymbol{\Omega}_K}/_N, \tag{7}$$

and denote by $\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \cdots, \sigma_K]^T$ its vector of eigenvalues. Then $\boldsymbol{W}$, under $\mathcal{H}_1$, could be modeled as a central semi-correlated complex Wishart matrix denoted as $\boldsymbol{W} \sim \mathcal{CW}_K(N, \widehat{\boldsymbol{\Sigma}}_K)$ [7]. Since $\boldsymbol{\Omega}_K$ is a rank-1 matrix, then $\widehat{\boldsymbol{\Sigma}}_K$ belongs to the class of spiked population model with all but one eigenvalue of $\widehat{\boldsymbol{\Sigma}}_K$ are still equal to σ_η^2 while σ_1 is given by:

$$\sigma_1 = \sigma_\eta^2 + {}^{\omega_1}/_N, \tag{8}$$

with ω_1 is the only non-zero eigenvalue of $\boldsymbol{\Omega}_K$. Denote the channel power by σ_h^2 and the signal to noise ratio by $\rho = \frac{\sigma_s^2 \sigma_h^2}{\sigma_\eta^2}$, then it can be easily shown that:

$$\omega_1 = tr(\boldsymbol{\Omega}_K) = NK\rho. \tag{9}$$

3 Assymptotic Moments of $\boldsymbol{\lambda_1}$ and $\boldsymbol{\lambda_K}$

This section considers the statistical analysis of the extreme eigenvalues (λ_1 and λ_K) of the sample covariance matrix ($\boldsymbol{W}$) by considering both hypotheses. Since SCN is not affected by the noise power, let $\sigma_\eta^2 = 1$ and define the Asymptotic Condition (AC) and the Critical Condition (CC) as follows:

$$\textbf{AC}: \quad (K, N) \to \infty \text{ with } K/N \to c \in (0, 1), \tag{10}$$

$$\textbf{CC}: \quad \rho > \rho_c = \frac{1}{\sqrt{KN}}. \tag{11}$$

[1] The non-centrality matrix is defined as $\boldsymbol{\Omega}_K = \boldsymbol{\Sigma}_K^{-1} \boldsymbol{M}\boldsymbol{M}^\dagger$ where $\boldsymbol{\Sigma}_K$ and $\boldsymbol{M}$ are respectively the covariance matrix and the mean of $\boldsymbol{Y}$ defined as $\boldsymbol{\Sigma}_K = E[(\boldsymbol{Y} - \boldsymbol{M})(\boldsymbol{Y} - \boldsymbol{M})^\dagger]$ and $\boldsymbol{M} = E[\boldsymbol{Y}]$.

3.1 $\mathcal{H}_0$ Hypothesis

Let $\lambda_1^{\mathcal{H}_0}$ and $\lambda_K^{\mathcal{H}_0}$ be the maximum and minimum eigenvalue of $\boldsymbol{W}$ under $\mathcal{H}_0$ respectively, then:

Distribution of $\boldsymbol{\lambda_1^{\mathcal{H}_0}}$: Denote the centered and scaled version of $\lambda_1^{\mathcal{H}_0}$ of the central uncorrelated Wishart matrix $\mathbf{W} \sim \mathcal{CW}_K(N, \mathbf{I}_K)$ by:

$$\lambda_1' = \frac{\lambda_1^{\mathcal{H}_0} - a_1(K,N)}{b_1(K,N)} \tag{12}$$

with $a_1(K,N)$ and $b_1(K,N)$, the centering and scaling coefficients respectively, are defined by:

$$a_1(K,N) = (\sqrt{K} + \sqrt{N})^2 \tag{13}$$

$$b_1(K,N) = (\sqrt{K} + \sqrt{N})(K^{-1/2} + N^{-1/2})^{\frac{1}{3}} \tag{14}$$

then, as AC is satisfied, λ_1' follows a TW distribution of order 2 (TW2) [8].

Distribution of $\boldsymbol{\lambda_K^{\mathcal{H}_0}}$: Denote the centered and scaled version of $\lambda_K^{\mathcal{H}_0}$ of the central uncorrelated Wishart matrix $\mathbf{W} \sim \mathcal{CW}_K(N, \mathbf{I}_K)$ by:

$$\lambda_K' = \frac{\lambda_K^{\mathcal{H}_0} - a_2(K,N)}{b_2(K,N)} \tag{15}$$

with $a_2(K,N)$ and $b_2(K,N)$, the centering and scaling coefficients respectively, are defined by:

$$a_2(K,N) = (\sqrt{K} - \sqrt{N})^2 \tag{16}$$

$$b_2(K,N) = (\sqrt{K} - \sqrt{N})(K^{-1/2} - N^{-1/2})^{\frac{1}{3}} \tag{17}$$

then, as AC is satisfied, λ_K' follows a TW2 [9].

Central Moments of $\boldsymbol{\lambda_1^{\mathcal{H}_0}}$ and $\boldsymbol{\lambda_K^{\mathcal{H}_0}}$: The mean, variance and skewness of λ_1' and λ_K' are that of the TW2. They are given by $\mu_{TW2} = -1.7710868074$, $\sigma_{TW2}^2 = 0.8131947928$ and $\mathcal{S}_{TW2} = 0.2240842036$ respectively [10]. Accordingly, using (12), the mean, variance and skewness of $\lambda_1^{\mathcal{H}_0}$ are, respectively, given by:

$$\mu_{\lambda_1^{\mathcal{H}_0}} = b_1(K,N)\mu_{TW2} + a_1(K,N), \tag{18}$$

$$\sigma^2{}_{\lambda_1^{\mathcal{H}_0}} = b_1^2(K,N)\sigma_{TW2}^2, \tag{19}$$

$$\mathcal{S}_{\lambda_1^{\mathcal{H}_0}} = \mathcal{S}_{TW2}, \tag{20}$$

and using (15), the mean, variance and skewness of $\lambda_K^{\mathcal{H}_0}$ are, respectively, given by:

$$\mu_{\lambda_K^{\mathcal{H}_0}} = b_2(K,N)\mu_{TW2} + a_2(K,N), \tag{21}$$

$$\sigma^2{}_{\lambda_K^{\mathcal{H}_0}} = b_2^2(K,N)\sigma_{TW2}^2, \tag{22}$$

$$\mathcal{S}_{\lambda_K^{\mathcal{H}_0}} = -\mathcal{S}_{TW2}. \tag{23}$$

3.2 $\mathcal{H}_1$ Hypothesis

Let $\lambda_1^{\mathcal{H}_1}$ and $\lambda_K^{\mathcal{H}_1}$ be the maximum and minimum eigenvalue of $\boldsymbol{W}$ under $\mathcal{H}_1$ respectively, then:

Distribution of $\boldsymbol{\lambda_1^{\mathcal{H}_1}}$: Denote the centered and scaled version of $\lambda_1^{\mathcal{H}_1}$ of the central semi-correlated Wishart matrix $\mathbf{W} \sim \mathcal{CW}_K(N, \widehat{\mathbf{\Sigma}}_K)$ by:

$$\lambda_1'' = \frac{\lambda_1^{\mathcal{H}_1} - a_3(K, N, \sigma)}{\sqrt{b_3(K, N, \sigma)}} \tag{24}$$

with $a_3(K, N)$ and $b_3(K, N)$, the centering and scaling coefficients respectively, are defined by:

$$a_3(K, N, \sigma) = \sigma_1(N + \frac{K}{\sigma_1 - 1}) \tag{25}$$

$$b_3(K, N, \sigma) = \sigma_1^2(N - \frac{K}{(\sigma_1 - 1)^2}) \tag{26}$$

then, as AC and CC are satisfied, λ_1'' follows a standard normal distribution ($\lambda_1'' \sim \mathcal{N}(0, 1)$) [11]. However, if CC is not satisfied, then $\lambda_1^{\mathcal{H}_1}$ follows the TW2 distribution of $\lambda_1^{\mathcal{H}_0}$ as AC is satisfied [11]. Accordingly, the PU signal has no effect on the eigenvalues and could not be detected.

Distribution of $\boldsymbol{\lambda_K^{\mathcal{H}_1}}$: As mentioned in [12], when $\widehat{\mathbf{\Sigma}}_K$ has only one non-unit eigenvalue such that CC is satisfied, then only one eigenvalue of $\mathbf{W}$ will be pulled up. In other words, and as could be deduced from [13, Proof of Lemma 2], the rest $K - 1$ eigenvalues of $\mathbf{W}$ $(\lambda_2^{\mathcal{H}_1}, \cdots, \lambda_K^{\mathcal{H}_1})$ has the same distribution of the eigenvalues of $\mathbf{W} \sim \mathcal{CW}_{K-1}(N, \mathbf{I}_{K-1})$ under $\mathcal{H}_0$ hypothesis.

Denote the centered and scaled version of $\lambda_K^{\mathcal{H}_1}$ of the central semi-correlated Wishart matrix $\mathbf{W} \sim \mathcal{CW}_K(N, \widehat{\mathbf{\Sigma}}_K)$ by:

$$\lambda_K'' = \frac{\lambda_K^{\mathcal{H}_1} - a_2(K - 1, N)}{b_2(K - 1, N)} \tag{27}$$

with $a_2(K, N)$ and $b_2(K, N)$ are, respectively, given by (16) and (17). Then, as the AC and CC are satisfied, λ_K'' follows a TW2.

Central Moments of $\boldsymbol{\lambda_1^{\mathcal{H}_1}}$ and $\boldsymbol{\lambda_K^{\mathcal{H}_1}}$: The mean, variance and skewness of $\lambda_1^{\mathcal{H}_1}$ are, due to (24), given respectively by:

$$\mu_{\lambda_1^{\mathcal{H}_1}} = a_3(K, N, \sigma), \tag{28}$$

$$\sigma^2_{\lambda_1^{\mathcal{H}_1}} = b_3(K, N, \sigma), \tag{29}$$

$$\mathcal{S}_{\lambda_1^{\mathcal{H}_1}} = 0, \tag{30}$$

and using (27), the mean, variance and skewness of $\lambda_K^{\mathcal{H}_1}$ are respectively given by:

$$\mu_{\lambda_K^{\mathcal{H}_1}} = b_2(K-1,N)\mu_{TW2} + a_2(K-1,N), \tag{31}$$

$$\sigma^2_{\lambda_K^{\mathcal{H}_1}} = b_2^2(K-1,N)\sigma^2_{TW2}, \tag{32}$$

$$\mathcal{S}_{\lambda_K^{\mathcal{H}_1}} = -\mathcal{S}_{TW2}. \tag{33}$$

As a result, this section provides a simple form for the central moments of the extreme eigenvalues. These moments are used, in the next section, to derive an approximation for the mean, the variance and the skewness of the SCN under both hypotheses.

4 Approximating the SCN Distribution

This section approximates the asymptotic distribution of the SCN by the GEV distribution using moment matching. First, we consider both detection hypotheses and we derive an approximation of the mean, the variance and the skewness of the SCN to be used in the next subsection for the approximation.

4.1 Asymptotic Central Moments of the SCN

The bi-variate first order Taylor expansion of the function $X = g(\lambda_1, \lambda_K) = \lambda_1/\lambda_K$ about any point $\theta = (\theta_{\lambda_1}, \theta_{\lambda_K})$ is written as:

$$X = g(\theta) + g'_{\lambda_1}(\theta)(\lambda_1 - \theta_{\lambda_1}) + g'_{\lambda_K}(\theta)(\lambda_K - \theta_{\lambda_K}) + O(n^{-1}), \tag{34}$$

with g'_{λ_i} is the partial derivative of g over λ_i.

Let $\theta = (\mu_{\lambda_1}, \mu_{\lambda_K})$, then it could be easily proved that:

$$E[X] = g(\theta), \tag{35}$$

$$E\left[(X-g(\theta))^2\right] = g'_{\lambda_1}(\theta)^2 E\left[(\lambda_1-\theta_{\lambda_1})^2\right] + g'_{\lambda_K}(\theta)^2 E\left[(\lambda_K-\theta_{\lambda_K})^2\right] + 2g'_{\lambda_1}(\theta)g'_{\lambda_K}(\theta)E\left[(\lambda_1-\theta_{\lambda_1})(\lambda_K-\theta_{\lambda_K})\right], \tag{36}$$

$$E\left[(X-g(\theta))^3\right] = g'_{\lambda_1}(\theta)^3 E\left[(\lambda_1-\theta_{\lambda_1})^3\right] + g'_{\lambda_K}(\theta)^3 E\left[(\lambda_K-\theta_{\lambda_K})^3\right] + 3g'_{\lambda_1}(\theta)^2 g'_{\lambda_K}(\theta)E\left[(\lambda_1-\theta_{\lambda_1})^2(\lambda_K-\theta_{\lambda_K})\right] + 3g'_{\lambda_1}(\theta)g'_{\lambda_K}(\theta)^2 E\left[(\lambda_1-\theta_{\lambda_1})(\lambda_K-\theta_{\lambda_K})^2\right], \tag{37}$$

where $E[.]$ stands for the expectation. Accordingly, we give the following theorems that formulate an approximation for the central moments of the SCN.

Theorem 1. *Let X be the SCN of $\mathbf{W} \sim \mathcal{CW}_K(N, \sigma_\eta^2 \mathbf{I}_K)$. The mean, the variance and the skewness of X, as AC is satisfied, can be tightly approximated using the mean, the variance and the skewness of the $\lambda_1^{\mathcal{H}_0}$ and $\lambda_K^{\mathcal{H}_0}$ as follows:*

$$\mu_X = \frac{\mu_{\lambda_1^{\mathcal{H}_0}}}{\mu_{\lambda_K^{\mathcal{H}_0}}} \tag{38}$$

$$\sigma_X^2 = \frac{\sigma^2_{\lambda_1^{\mathcal{H}_0}}}{\mu^2_{\lambda_K^{\mathcal{H}_0}}} + \frac{\mu^2_{\lambda_1^{\mathcal{H}_0}} \sigma^2_{\lambda_K^{\mathcal{H}_0}}}{\mu^4_{\lambda_K^{\mathcal{H}_0}}} \tag{39}$$

$$\mathcal{S}_X = \frac{1}{\sqrt{\sigma_X^3}} \cdot \left[\frac{\sqrt{\sigma^3_{\lambda_1^{\mathcal{H}_0}}} \mathcal{S}_{\lambda_1^{\mathcal{H}_0}}}{\mu^3_{\lambda_K^{\mathcal{H}_0}}} - \frac{\sqrt{\sigma^3_{\lambda_K^{\mathcal{H}_0}}} \mu^3_{\lambda_1^{\mathcal{H}_0}} \mathcal{S}_{\lambda_K^{\mathcal{H}_0}}}{\mu^6_{\lambda_K^{\mathcal{H}_0}}} \right] \tag{40}$$

Proof. The result follows (35), (36) and (37) while considering $\lambda_1^{\mathcal{H}_0}$ and $\lambda_K^{\mathcal{H}_0}$ asymptotically independent [14]. The mean, the variance and the skewness of $\lambda_1^{\mathcal{H}_0}$ and $\lambda_K^{\mathcal{H}_0}$ are given in Sect. 3.1.

Theorem 2. *Let X be the SCN of $\mathbf{W} \sim \mathcal{CW}_K(N, \widehat{\mathbf{\Sigma}}_K)$ where $\widehat{\mathbf{\Sigma}}_K$ has only one non-unit eigenvalue. The mean, the variance and the skewness of X, as the AC and CC are satisfied, can be tightly approximated using the mean, the variance and the skewness of the $\lambda_1^{\mathcal{H}_1}$ and $\lambda_K^{\mathcal{H}_1}$ as follows:*

$$\mu_X = \frac{\mu_{\lambda_1^{\mathcal{H}_1}}}{\mu_{\lambda_K^{\mathcal{H}_1}}} \tag{41}$$

$$\sigma_X^2 = \frac{\sigma^2_{\lambda_1^{\mathcal{H}_1}}}{\mu^2_{\lambda_K^{\mathcal{H}_1}}} + \frac{\mu^2_{\lambda_1^{\mathcal{H}_1}} \sigma^2_{\lambda_K^{\mathcal{H}_1}}}{\mu^4_{\lambda_K^{\mathcal{H}_1}}} \tag{42}$$

$$\mathcal{S}_X = - \frac{\sqrt{\sigma^3_{\lambda_K^{\mathcal{H}_1}}} \mu^3_{\lambda_1^{\mathcal{H}_1}} \mathcal{S}_{\lambda_K^{\mathcal{H}_1}}}{\sqrt{\sigma_X^3} \cdot \mu^6_{\lambda_K^{\mathcal{H}_1}}} \tag{43}$$

Proof. The result follows (35), (36) and (37) while considering $\lambda_1^{\mathcal{H}_1}$ and $\lambda_K^{\mathcal{H}_1}$ asymptotically independent [15]. The mean, the variance and the skewness of $\lambda_1^{\mathcal{H}_1}$ and $\lambda_K^{\mathcal{H}_1}$ are given in Sect. 3.2

4.2 Approximating the SCN Using GEV

Generalized Extreme Value (GEV) is a flexible 3-parameter distribution used to model the extreme events of a sequence of i.i.d random variables. These parameters are the location (δ), the scale (β) and the shape (ξ). In the following two propositions, we approximate the distribution of the SCN under $\mathcal{H}_0$ and $\mathcal{H}_1$ hypotheses respectively, however, the proof is omitted due to the lack of space.

Proposition 1. *Let X be the SCN of $\mathbf{W} \sim \mathcal{CW}_K(N, \sigma_\eta^2 \mathbf{I}_K)$ with defined skewness $-0.63 \le \mathcal{S}_X < 1.14$. If AC is satisfied, then the CDF and PDF of X can be asymptotically and tightly approximated respectively by:*

$$F(x;\delta,\beta,\xi) = e^{-\left(1+(\frac{x-\delta}{\beta})\xi\right)^{-1/\xi}} \tag{44}$$

$$f(x;\delta,\beta,\xi) = \frac{1}{\beta}\left(1+(\frac{x-\delta}{\beta})\xi\right)^{\frac{-1}{\xi}-1} e^{-\left(1+(\frac{x-\delta}{\beta})\xi\right)^{-1/\xi}} \tag{45}$$

where ξ, β and δ are defined respectively by:

$$\xi = -0.06393\mathcal{S}_X^2 + 0.3173\mathcal{S}_X - 0.2771 \tag{46}$$

$$\beta = \sqrt{\frac{\sigma_X^2 \xi^2}{g_2 - g_1^2}} \tag{47}$$

$$\delta = \mu_X - \frac{(g_1 - 1)\beta}{\xi} \tag{48}$$

where μ_X, σ_X^2 and $\mathcal{S}_X$ are defined in Theorem 1.

Proposition 2. *Let X be the SCN of $\boldsymbol{W} \sim \mathcal{CW}_K(N, \widehat{\boldsymbol{\Sigma}}_K)$ with defined skewness $-0.63 \le \mathcal{S}_X < 1.14$ and $\widehat{\boldsymbol{\Sigma}}_K$ has only one non-unit eigenvalue. If AC and CC are satisfied, then the CDF and PDF of X can be asymptotically and tightly approximated by (44) and (45) respectively. The parameters ξ, β and δ are defined respectively by (46), (47) and (48) with μ_X, σ_X^2 and $\mathcal{S}_X$ are defined in Theorem 2.*

Now, given (5) and (6), Theorems 1 and 2 and Propositions 1 and 2, the false-alarm probability, the detection probability and the threshold are straightforward. For example, for a target false alarm probability ($\hat{\gamma}$), the threshold is given by:

$$\alpha = \delta + \frac{\beta}{\xi}\left(-1 + \left[-\ln(1-\hat{\gamma})\right]^{-\xi}\right) \tag{49}$$

with δ, β and ξ given in Proposition 1.

5 Numerical Validation

In this section, we discuss the analytical results through Monte-Carlo simulations. We validate the theoretical analysis presented in Sects. 3 and 4. The simulation results are obtained by generating 10^5 random realizations of $\boldsymbol{Y}$. For $\mathcal{H}_0$ case, the inputs of $\boldsymbol{Y}$ are complex circular white Gaussian noise with zero mean and unknown variance σ_η^2 while for $\mathcal{H}_1$ case the channel is considered flat and the PU transmits a BPSK signal.

Table 1 shows the accuracy of the analytical approximation of the mean, the variance and the skewness of the SCN provided by Theorems 1 and 2. It can be easily seen that these Theorems provide a good approximation for the

Table 1. The Empirical and Approximated mean, variance and skewness of the SCN under $\mathcal{H}_0$ and $\mathcal{H}_1$ hypotheses using Theorems 1 and 2 respectively.

	$K \times N$	Empirical			Proposed App.		
		mean	variance	skewness	mean	variance	skewness
$\mathcal{H}_0$	50×500	3.3946	0.0117	0.2639	3.3975	0.0114	0.1652
$\mathcal{H}_1$		12.3363	0.4006	0.1710	12.3139	0.3906	0.0291
$\mathcal{H}_0$	100×500	6.3076	0.0386	0.2992	6.3126	0.0367	0.1740
$\mathcal{H}_1$		34.6345	3.2246	0.1618	34.5387	3.1154	0.0306
$\mathcal{H}_0$	50×1000	2.3386	0.0026	0.2339	2.3396	0.0026	0.1619
$\mathcal{H}_1$		9.6702	0.1205	0.1177	9.6612	0.1184	0.024

statistics of the SCN, however, it could be noticed that the skewness is not perfectly approximated. In fact, the skewness is affected by the slow convergence of the skewness of λ_K that must converge to $-\mathcal{S}_{TW2}$ (i.e. -0.2241) as AC is satisfied. For example, when $K = 50$, the empirical skewness increases from $\mathcal{S}_{\lambda_K} = -0.1504$ to $\mathcal{S}_{\lambda_K} = -0.1819$ as the number of samples increases from $N = 500$ to $N = 1000$. Comparing these results with SCN results in Table 1, one can notice that the empirical and approximated SCN skewness become closer as λ_K skewness converges to that of TW2. Accordingly, Theorems 1 and 2 are good approximations for the mean, the variance and the skewness of the SCN under both hypotheses. It is worth noting that one could approximate the SCN moments using second order bi-variate Taylor series to get a slightly higher accuracy, however, this will cost higher complexity and it is not necessary as shown in Table 1 and Figs. 1 and 2.

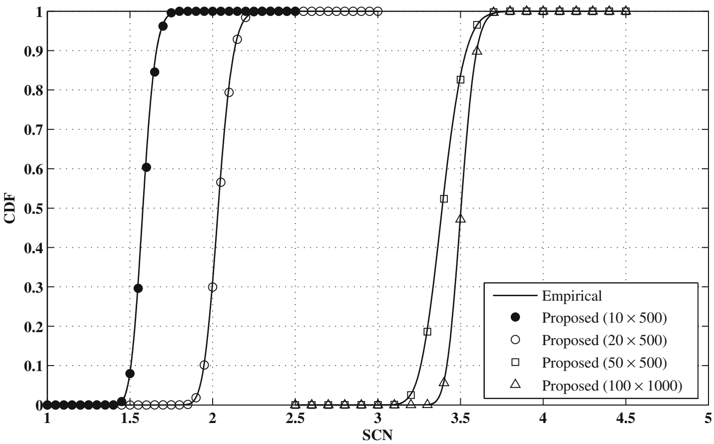

Fig. 1. Empirical CDF of the SCN and its corresponding proposed GEV approximation for different values of K and N under $\mathcal{H}_0$ hypothesis (i.e. false alarm probability).

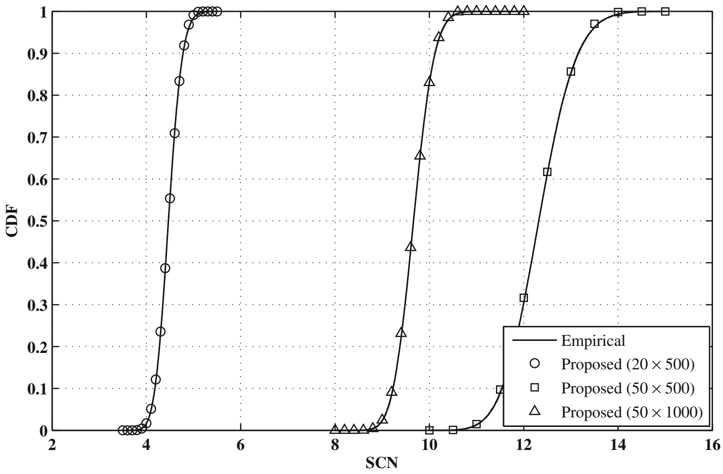

Fig. 2. Empirical CDF of the SCN and its corresponding proposed GEV approximation for different values of K and N under $\mathcal{H}_1$ hypothesis (i.e. detection probability).

Figure 1 shows the empirical CDF of the SCN and its corresponding GEV approximation given by Proposition 1. The results are shown for $K = \{10, 20, 50, 100\}$ antennas and $N = \{500, 1000\}$ samples per antenna. Results show a perfect match between the empirical results and our proposed approximation. Also, it could be noticed that the convergence of the skewness does not affect the approximation and thus the skewness in Theorem 1 holds for this approximation even though the convergence of the skewness of λ_K is slow. From SS perspective, the P_{fa} is in direct relation with this CDF and hence the P_{fa} is perfectly approximated.

Figure 2 shows the empirical CDF of the SCN and its corresponding GEV approximation given by Proposition 2. The results are shown for $K = \{20, 50\}$ antennas and $N = \{500, 1000\}$ samples per antenna and $SNR = -10dB$. Results show high accuracy in approximating the empirical CDF. Also, the difference in the skewness shown in Table 1 does not affect the approximation. Consequently, it could be concluded that the P_d is perfectly approximated.

Finally, it could be noticed that due to the large number of antennas considered in this paper, the proposed SS approximation could be directly applied to the Massive MIMO environment, a potential candidate in 5G.

6 Conclusion

In this paper, we have considered the SCN detector for large number of antennas and/or massive MIMO cognitive radios. We have derived the asymptotic mean, variance and skewness of the SCN using those of the extreme eigenvalues of the sample covariance matrix by means of bi-variate Taylor expansion. A new simple closed form approximation for the false-alarm and the detection probabilities was, also, proposed. This approximation is based on the extreme value theory distributions and uses results from random matrix theory. In addition to its simple form, simulation results show high accuracy of the proposed approximation for different number of antennas.

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