A Simple Formulation for the Distribution of the Scaled Largest Eigenvalue and Application to Spectrum Sensing

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Abstract. Scaled Largest Eigenvalue (SLE) detector stands out as the best single-primary-user detector in uncertain noisy environments. In this paper, we consider a multi-antenna cognitive radio system in which we aim at detecting the presence/absence of a Primary User (PU) using the SLE detector. We study the distribution of the SLE as a large number of samples are used in detection without constraint on the number of antennas. By the exploitation of the distributions of the largest eigenvalue and the trace of the receiver sample covariance matrix, we show that the SLE could be modeled as a normal random variable. Moreover, we derive the distribution of the SLE and deduce a simple yet accurate form of the probability of false alarm. Hence, this derivation yields a very simple form of the detection threshold. The analytical derivations are validated through extensive Monte Carlo simulations.

Keywords: Scaled largest eigenvalue detector · Spectrum sensing · Wishart matrix

1 Introduction

In Cognitive Radio (CR) networks, Spectrum Sensing (SS) is the task of obtaining awareness about the spectrum usage. Mainly it concerns two scenarios of detection: (i) detecting the absence of the Primary User (PU) in a licensed spectrum in order to use it and (ii) detecting the presence of the PU to avoid interference. Hence, SS plays a major role in the performance of the CR as well as the performance of the PU networks that coexist. In this context, an extreme importance for a CR network is to have an optimal SS technique with high probability of accuracy in uncertain environments. The Scaled Largest Eigenvalue detector (SLE) is an efficient technique that is proved to be the optimal

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detector under Generalized Likelihood Ratio (GLR) criterion and noise uncertainty environments [1,2].

SLE is among the detectors that use the eigenvalues of the receiver sample covariance matrix. Such detectors are known as the eigenvalue based detectors and includes, in addition to SLE, other detectors like the Largest Eigenvalue detector (LE) and the Standard Condition Number detector (SCN) [3–7]. In a scenario with perfect knowledge of the noise power, the LE detector is the optimal detector [5]. However, in practical systems the noise power may not be perfectly known. In this case, the SLE and SCN detectors outperform the LE detector due to their blind nature. Moreover, the SLE is proved to be the optimal detector under GLR criterion [1,2] and outperforms the SCN detector.

Even with its importance, existing results on the statistics of the SLE, defined as the ratio of the largest eigenvalue to the normalized trace of the sample covariance matrix, are relatively limited. These results are based on tools from random matrix theory [2,8,9] and Mellin transform [9–11]. SLE was proved, asymptotically, to follow the LE distribution (i.e. Tracy-Widom (TW) distribution) [2]. However, a non-negligible error still exists and new form is derived based on TW distribution and its second derivative [8]. Using Mellin transform, The distribution of the SLE was derived by the exploitation of the distribution of LE and the distribution of the trace [9–11]. The complexity in the form of SLE distribution in these results motivated us to find a simpler form.

In this paper, we are interested in finding a simple form for the Cumulative Density Function (CDF) and Probability Density Function (PDF) of the SLE. We consider the following hypotheses: (i) $\mathcal{H}_0$: there is no primary user and the received signal is only noise; and (ii) $\mathcal{H}_1$: the primary user exists. Our work is concentrated under the $\mathcal{H}_0$ hypothesis which is common to all CR systems, i.e. there are no constraints on the PU signal, number of PUs and the channel conditions. Probability of False-alarm (P_{fa}), defined as the probability of detecting the presence of PU when it does not exist, is also considered. We prove that the SLE can be modeled as a normal random variable and a simple form of the detection threshold is derived. In the following, we summarize the contributions of this paper:

- Derivation of the distribution of the trace of a complex sample covariance matrix.
- Derivation of the distribution of the SLE detector.
- Derivation of a simple form for the correlation coefficient between the largest eigenvalue and the trace.
- Derivation of a simple form for the P_{fa} and the threshold for detection.

The rest of this paper is organized as follows. Section 2 studies the system model. In Sect. 3, we recall the distribution of the LE and we derive the distribution of the trace of sample covariance matrix. The distribution of the SLE is considered in Sect. 4. We derive its distribution and formulate the correlation coefficient between the LE and the trace. The false alarm probability and the threshold are also addressed. Theoretical findings are validated by simulations in Sect. 5 while the conclusion is drawn in Sect. 6.

2 System Model

Consider a multi-antenna cognitive radio system and denote by K the number of received antennas. Let N be the number of samples collected from each antenna, then the received sample from antenna $k = 1 \cdots K$ at instant $n = 1 \cdots N$ under the two hypotheses is given by

$$\mathcal{H}_0 : \; y_k(n) = \eta_k(n), \tag{1}$$

$$\mathcal{H}_1 : \; y_k(n) = s(n) + \eta_k(n), \tag{2}$$

with $\eta_k(n)$ is a complex circular white Gaussian noise with zero mean and unknown variance σ_η^2 and $s(n)$ is the received signal sample including the channel effect.

After collecting N samples from each antenna, the received signal matrix, $\boldsymbol{Y}$, is given by:

$$\boldsymbol{Y} = \begin{pmatrix} y_1(1) & y_1(2) & \cdots & y_1(N) \\ y_2(1) & y_2(2) & \cdots & y_2(N) \\ \vdots & \vdots & \ddots & \vdots \\ y_K(1) & y_K(2) & \cdots & y_K(N) \end{pmatrix}, \tag{3}$$

Without loss of generality, we suppose that $K \leq N$ then the sample covariance matrix is given by $\boldsymbol{W} = \boldsymbol{Y}\boldsymbol{Y}^\dagger$ where $\dagger$ is the Hermitian notation. Denote the eigenvalues of $\boldsymbol{W}$ by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_K > 0$.

Under $\mathcal{H}_0$, the received samples are complex circular white Gaussian noise with zero mean and unknown variance σ_η^2. Consequently, the sample covariance matrix is a central uncorrelated complex Wishart matrix denoted as $\boldsymbol{W} \sim \mathcal{CW}_K(N, \sigma_\eta^2 \mathbf{I}_K)$ where K is the size of the matrix, N is the number of Degrees of Freedom (DoF), and $\sigma_\eta^2 \mathbf{I}_K$ is the correlation matrix. $\mathbf{I}$ and '$\sim$' denote the identity matrix and 'distributed as' respectively.

3 Distributions of the Largest Eigenvalue and of the Trace

This section considers the distributions of the LE and of the trace under $\mathcal{H}_0$ hypothesis. We prove that the LE and the trace follow Gaussian distributions for which the mean and variance are formulated. Since the SLE does not depend on the noise power, we suppose, in this section, that $\sigma_\eta^2 = 1$. Based on results of this section, we derive the distribution of the SLE in the next section.

3.1 Distribution of the LE

Let λ_1 be the maximum eigenvalue of $\boldsymbol{W}$ under $\mathcal{H}_0$ and denote the centered and scaled version of λ_1 of the central uncorrelated Wishart matrix $\mathbf{W} \sim \mathcal{CW}_K(N, \mathbf{I}_K)$ by:

$$\lambda_1' = \frac{\lambda_1 - a(K,N)}{b(K,N)} \tag{4}$$

with $a(K, N)$ and $b(K, N)$, the centering and scaling coefficients respectively, are defined by:

$$a(K, N) = (\sqrt{K} + \sqrt{N})^2 \tag{5}$$

$$b(K, N) = (\sqrt{K} + \sqrt{N})(K^{-1/2} + N^{-1/2})^{\frac{1}{3}} \tag{6}$$

then, as $(K, N) \rightarrow \infty$ with $K/N \rightarrow c \in (0, 1)$, λ_1' follows a Tracy-Widom distribution of order 2 (TW2) [12]. However, it was shown that, for a fixed K and as $N \rightarrow \infty$, λ_1 follows a normal distribution [13]. The mean and the variance of λ_1 could be approximated using TW2 and they are, respectively, given by:

$$\mu_{\lambda_1} = b(K, N)\mu_{TW2} + a(K, N), \tag{7}$$

$$\sigma^2{}_{\lambda_1} = b^2(K, N)\sigma^2_{TW2}, \tag{8}$$

where $\mu_{TW2} = -1.7710868074$ and $\sigma^2_{TW2} = 0.8131947928$ are, respectively, the mean and variance of TW distribution of order 2. This approximation is very efficient and it achieves high accuracy for K as small as 2 [13].

3.2 Distribution of the Trace

For a fixed K, as $N \rightarrow \infty$ the LE converges to a Gaussian distribution. On the other hand, let $T = \sum \lambda_i$ be the trace then the following theorem holds:

Theorem 1. *Let T be the trace of $\mathbf{W} \sim \mathcal{CW}_K(N, \mathbf{I}_K)$. Then, as $N \rightarrow \infty$, T follows a Gaussian distribution as follows:*

$$P(\frac{T - NK}{\sqrt{NK}} \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du, \tag{9}$$

Proof. Let us write:

$$T = tr(\boldsymbol{YY}^{\dagger}) = \sum_{i=1}^{K} \left[\sum_{j=1}^{N} |y_{i,j}|^2 \right] \tag{10}$$

with y_{ij} are independent circularly symmetric complex standard normal random variables ($y_{i,j} \sim \mathcal{CN}(0, 1)$). Accordingly, the square of the norm, $|y_{i,j}|^2$, is exponentially distributed with unit mean and unit variance. Hence, by Central Limit Theorem (CLT), as $N \rightarrow \infty$ the term in the square bracket of (10) follows Gaussian distribution and T is the sum of Gaussians.

To the best of the authors' knowledge, the result in Theorem 1 is new. Let $T_n = \frac{1}{K}T$ be the normalized trace, then T_n, following Theorem 1, is Normally distributed with mean and variance given respectively by:

$$\mu_{T_n} = N, \tag{11}$$

$$\sigma^2{}_{T_n} = N/K, \tag{12}$$

4 SLE Detector

Let $\boldsymbol{W}$ be the sample covariance matrix at the CR receiver, then the SLE of $\boldsymbol{W}$ is defined by:

$$X = \frac{\lambda_1}{\frac{1}{K}\sum_{i=1}^{K}\lambda_i} = \frac{\lambda_1}{T_n} \tag{13}$$

Denoting by α the decision threshold, then the P_{fa} is given by:

$$P_{fa} = P(X \geq \alpha/\mathcal{H}_0) = 1 - F_X(\alpha), \tag{14}$$

where $F_X(.)$ is the CDF of X under $\mathcal{H}_0$ hypothesis. If the expression of the P_{fa} is known, then a threshold could be set according to a required error constraint. Hence, it is important to have a simple and accurate form for the distribution of X.

4.1 SLE Distribution

Under $\mathcal{H}_0$, both the LE and the normalized trace follow the Gaussian distribution as $N \to \infty$ which is realistic in practical spectrum sensing scenarios. Herein, we show that the SLE is normally distributed when both the LE and the normalized trace follows the normal distribution as stated by the following theorem:

Theorem 2. *Let X be the SLE of $\mathbf{W} \sim \mathcal{CW}_K(N, \sigma_\eta^2\mathbf{I}_K)$. Then, for a fixed K and as $N \to \infty$, X follows a normal distribution with CDF and PDF, respectively, given by:*

$$F_X(x) = \Phi\left(\frac{x\mu_{T_n} - \mu_{\lambda_1}}{\sqrt{\sigma^2{}_{\lambda_1} - 2xc + x^2\sigma^2{}_{T_n}}}\right) \tag{15}$$

$$f_X(x) = \frac{\mu_{T_n}\sigma^2{}_{\lambda_1} - c\mu_{\lambda_1} + (\mu_{\lambda_1}\sigma^2{}_{T_n} - c\mu_{T_n})x}{(\sigma^2{}_{\lambda_1} - 2xc + x^2\sigma^2{}_{T_n})^{\frac{3}{2}}}\phi\left(\frac{x\mu_{T_n} - \mu_{\lambda_1}}{\sqrt{\sigma^2{}_{\lambda_1} - 2xc + x^2\sigma^2{}_{T_n}}}\right) \tag{16}$$

with

$$\Phi(v) = \int_{-\infty}^{v}\phi(u)du \quad \textbf{\textit{and}} \quad \phi(u) = \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}} \tag{17}$$

where μ_{λ_1}, μ_{T_n} and $\sigma^2{}_{\lambda_1}$, $\sigma^2{}_{T_n}$ are, respectively, the mean and the variance of λ_1 and T_n given by (7), (11) and (8), (12) respectively. The parameter $c = \sigma_{\lambda_1}\sigma_{T_n}\rho$ where ρ is the correlation coefficient between λ_1 and T_n.

Proof. Let λ_1 and T_n be two normally distributed random variables with μ_{λ_1}, μ_{T_n}, $\sigma^2{}_{\lambda_1}$ and $\sigma^2{}_{T_n}$ their means and variances and let ρ be their correlation coefficient. Denote by $g(\lambda, t)$ the joint density of λ_1 and T_n then the PDF of X is $f_X(x) = \int_{-\infty}^{+\infty}|t|g(xt, t)dt$ and the result is found in [14], however, since $\boldsymbol{W}$ is positive definite then $Pr(T_n > 0) = 1$ and the CDF of X could be written as:

$$F_X(x) = Pr(\lambda/t < x) = Pr(\lambda_1 - xt < 0) \tag{18}$$

and thus, CDF is given by (15) and the PDF is its derivative in (16) [15].

4.2 Correlation Coefficient ρ

Theorem 2 gives the form of the distribution of the SLE as a function of the mean and the variance of λ_1 and T_n as well as the correlation coefficient ρ usually not negligible especially for small K. In this section, we will give a simple analytical form to calculate the correlation coefficient, ρ, between the largest eigenvalue and the trace of Wishart matrix based on the mean of the SLE. In the following, we calculate the mean of SLE in two different ways such that a simple form for ρ could be derived.

Mean of SLE Using Independent Property: Using (13) and the property that the SLE and the trace are independent as proved in [16], then the mean of λ_1 could be written as:

$$E[\lambda_1] = E\big[X \times T_n\big] = E[X] \cdot E[T_n] \tag{19}$$

where $E[.]$ stands for expectation operator.

Recall that the mean of λ_1 and the mean of T_n are given respectively by (7) and (11), then based on (19), the mean of the SLE is given by:

$$\mu_X = \frac{\mu_{\lambda_1}}{\mu_{T_n}} = \frac{b(K,N) \cdot \mu_{TW2} + a(K,N)}{N} \tag{20}$$

Mean of SLE Using Its Distribution: Using SLE distribution, it is difficult to find numerically the mean of the SLE, however, it turns out that a simple and accurate approximation could be found.

An approximation of the mean of the ratio $(u + Z_1)/(v + Z_2)$ could be found where u and v are positive constants and Z_1 and Z_2 are two independent standard normal random variables. It is based on approximating formula for $E[1/(v + Z_2)]$ when $v + Z_2$ is normal variate conditioned by $Z_2 > -4$ and $v + Z_2$ is not expected to approach zero as follows [15]:

$$E\left[\frac{1}{v + Z_2}\right] = \frac{1}{1.01v - 0.2713} \tag{21}$$

By using the transformation of the general ratio of two normal random variable λ_1/T_n into the ratio $(u+Z_1)/(v+Z_2)$, which has the same distribution, we have:

$$\frac{\lambda_1}{T_n} \sim \frac{1}{q}(\frac{u + Z_1}{v + Z_2}) + s \tag{22}$$

with $s = \rho\frac{\sigma_{\lambda_1}}{\sigma_{T_n}}$, $v = \frac{\mu_{T_n}}{\sigma_{T_n}}$ and

$$u = \frac{\mu_{\lambda_1} - \rho\frac{\mu_{T_n} \cdot \sigma_{\lambda_1}}{\sigma_{T_n}}}{(\pm\sigma_{\lambda_1}\sqrt{1-\rho^2})} \tag{23}$$

$$q = \frac{\sigma_{T_n}}{(\pm\sigma_{\lambda_1}\sqrt{1-\rho^2})} \tag{24}$$

where one chooses the $\pm$ sign so that u and v have the same sign (i.e. positive). Consequently, the left-side and the right-side of (22) have the same mean. Therefore the mean of the SLE could be approximated as follows:

$$\mu_X = \frac{\mu_{\lambda_1} - \delta\ \mu_{T_n}}{\theta} + \delta \tag{25}$$

with $\delta = \rho\frac{\sigma_{\lambda_1}}{\sigma_{T_n}}$ and $\theta = 1.01\mu_{T_n} - 0.2713\sigma_{T_n}$.

Correlation Coefficient $\boldsymbol{\rho}$: Using (25), then ρ, after some algebraic manipulation, is given by:

$$\rho = \frac{\sigma_{T_n}}{\sigma_{\lambda_1}} \cdot \frac{\theta\ \mu_X - \mu_{\lambda_1}}{\theta - \mu_{T_n}} \tag{26}$$

where μ_{λ_1}, μ_{T_n} and μ_X are respectively the means of the LE, the normalized trace and the SLE given by (7), (11) and (20) respectively. σ_{λ_1} and σ_{T_n} are respectively the standard deviations of the LE and the normalized trace and are the square root of (8) and (12) respectively.

4.3 Performance Probabilities and Threshold

Using (14) and (15), then P_{fa} is given by:

$$P_{fa}(\alpha) = Q(\frac{\alpha\mu_{T_n} - \mu_{\lambda_1}}{\sqrt{\sigma^2{}_{\lambda_1} - 2\alpha c + \alpha^2\sigma^2{}_{T_n}}}) \tag{27}$$

where $Q(.)$ is the Q-function. Based on (27), we can derive a simple and accurate form for the threshold as a function of the means and variances of the LE and T_n and the correlation coefficient between them as well as the false alarm probability. That is, for a target false alarm probability, $\hat{P}_{fa}$, the equation of the threshold of the SLE detector will be:

$$\alpha = \frac{\mu_{12} - \beta^2\rho\sigma_{12} + \beta\sqrt{m_v - 2\rho\mu_{12}\sigma_{12} + \beta^2\sigma_{12}^2(\rho^2-1)}}{\mu_{T_n}^2 - \beta^2\sigma_{T_n}^2} \tag{28}$$

where $\mu_{12} = \mu_{\lambda_1}\mu_{T_n}$, $\sigma_{12} = \sigma_{\lambda_1}\sigma_{T_n}$, $m_v = \mu_{T_n}^2\sigma_{\lambda_1}^2 + \mu_{\lambda_1}^2\sigma_{T_n}^2$ and $\beta = Q^{-1}(\hat{P}_{fa})$ with $Q^{-1}(.)$ is the inverse Q-function.

5 Numerical Validation

In this section, we discuss the analytical results through Monte-Carlo simulations. We validate the theoretical analysis presented in Sects. 3 and 4. The simulation results are obtained by generating 10^5 random realizations of $\boldsymbol{Y}$. The inputs of $\boldsymbol{Y}$ are complex circular white Gaussian noise with zero mean and unknown variance σ_η^2.

Table 1 shows the accuracy of the analytical approximation of the correlation coefficient (ρ) of the SLE in (26). The results are shown for $K = \{2, 4, 50\}$

Table 1. The empirical and approximated value of the correlation coefficient ρ for different values of K and N.

$K \times N$	2×500	4×500	2×1000	4×1000	50×1000
ρ-Emperical	0.849	0.6974	0.839	0.6915	0.3353
ρ-Analytical	0.8548	0.6957	0.8623	0.6967	0.3356

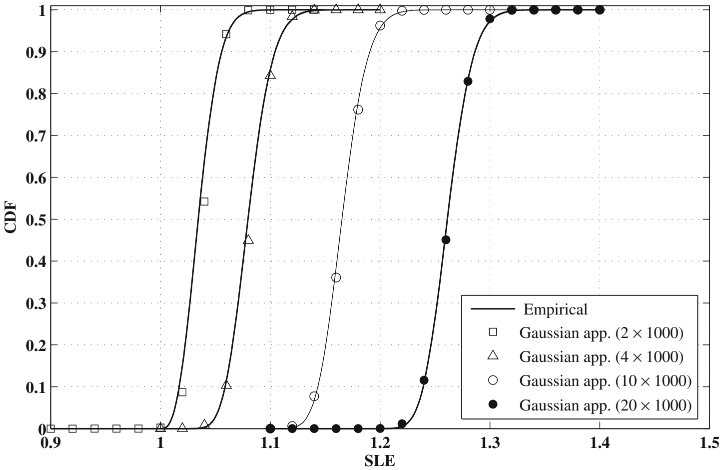

Fig. 1. Empirical CDF of the SLE and its corresponding Gaussian approximation for different values of K with $N = 1000$.

antennas and $N = \{500, 1000\}$ samples per antenna. Table 1 shows that the accuracy of this approximation is higher as the number of antenna increases, however, we can also notice that we have very high accuracy even when $K = 2$ antennas. Also, as expected, it is easy to notice that the correlation between the largest eigenvalue and the trace decreases as the number of antenna increases, however, this correlation could not be ignored even if the number of antenna is large.

Figure 1 shows the empirical CDF of the SLE and its corresponding Gaussian approximation given by Theorem 2. The results are shown for $K = \{2, 4, 10, 20\}$ antennas and $N = 1000$ samples per antenna. Results show a perfect match between the empirical results and our Gaussian formulation. The slight difference in the case $K = 2$ is due to the use of an approximation for the mean and variance of the largest eigenvalue as mentioned in Sect. 3.1. If the exact mean and variance of the LE are used, better results would be expected. However, the results in this paper combine between accuracy and simplicity.

Figures 2 shows the accuracy of the proposed false alarm form proposed in (27). We have considered multi-antenna CR with different number of antennas that aim to sense the spectrum for a time corresponding to $N = 500$ samples.

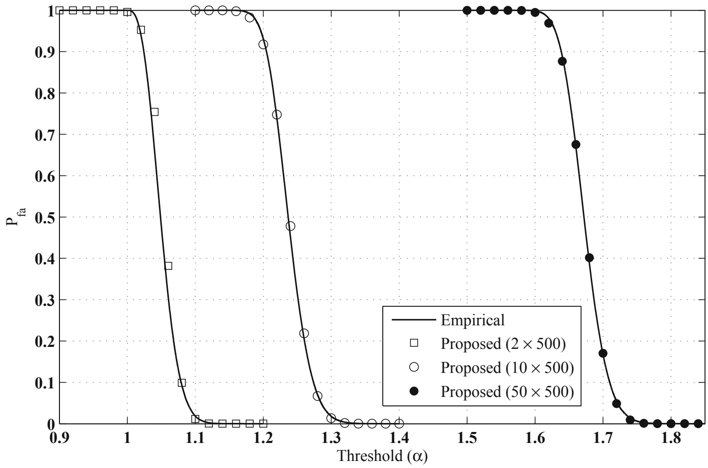

Fig. 2. Empirical probability of false alarm for the SLE detector and its corresponding proposed form in (27) for different values of K with $N = 500$ samples.

The considered number of antennas is as small as $K = 2$ and as large as $K = 50$. Simulation results show a high accuracy in our proposed form which increases as K increases. In addition to the accuracy, the form given in (27) is a simple Q-function equation.

6 Conclusion

In this paper, we have considered the SLE detector due to its optimal performance in uncertain environments. We proved that the SLE could be modeled as a normal random variable and we derive its CDF and PDF. The false alarm probability and the threshold were also considered as we derive new simple and accurate forms. These forms are simple function of the means and variances of the LE and the trace as well as the correlation function between them. Simple forms for the mean, the variance and the correlation coefficient are provided. As a result, this paper provides a simple form for the false alarm probability and the threshold for the SLE detector under relatively large number of samples. However, this constraint is always satisfied in spectrum sensing. Simulation results have shown that the proposed simple forms achieve high accuracy. In addition, results have shown that the correlation between the largest eigenvalue and the trace decreases as the number of antenna increases but it could not be ignored even for large number of antennas. However, the approximation of the correlation coefficient, derived in this paper, shows high accuracy.

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