TST: A New Randomness Test Method Based on Coupon Collector's Problem

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Abstract. In this paper we find that a random sequence is expected to obey a new interesting distribution, and the coefficient of variation of this distribution approximates the value of **golden section ratio**, the difference between these two numbers is only 0.000797. As this interesting property, this newfound distribution is derived from Coupon Collector's Problem and founded by the uniformity of frequency. Based on this distribution a new method is proposed to evaluate the randomness of a given sequence. Through the new method, the binary and decimal expansions of e, π, $\sqrt{2}$, $\sqrt{3}$ and the bits generated by Matlab are concluded to be random. These sequences can pass NIST tests and also pass our test. At the same time, we test some sequences generated by a physical random number generator WNG8. However, these sequences can pass the NIST tests but cannot pass our test. In particular, the new test is easy to be implemented, very fast and thus well suited for practical applications. We hope this test method could be a supplement of other test methods.

Keywords: Randomness tests · Cryptography · Golden section ratio · Coefficient of variation

1 Introduction

The random sequence is very important and it serves two common purposes [1–4]. One is that most encryption algorithms require a source of random data, even some symmetric ciphers (where the secret is shared), either to generate new private/public key pairs, for session keys, for padding, or for other reasons [5]. For instance, if the random number is not well selected, the secure system based on RSA is not secure anymore [6]. Another important usage of random number is that random number generators ("RNGs") are basic tools of stochastic modeling. If the bad random is used in simulation, it will ruin a simulation.

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At present there are many test suites to evaluate the randomness of binary bit sequences, such as Diehard Crypt-XS [7] and NIST test suites [8]. Typically, the test method usually defines a test statistic whose theoretical distribution is known, and the randomness property of a given sequence can be evaluated by hypothesis testing. Because there are so many tests for judging whether a sequence is random or not, usually the test result is part of the picture of the randomness [9].

In this paper we construct a new randomness test method based on Coupon Collector's Problem. The main inspiration of this paper comes from the martingale betting system [10]. The martingale betting system has a long and interesting history. Suppose that a gambler is betting on red to turn up in roulette, in which the probability of hitting either red or black is close to 50 %. Every time the gambler wins, bets 1 dollar next time. Every time the gambler loses, doubles the previous bet. In this betting system, the gambler will always win, because the gambler is sure that the red must turn up in some time.

In the old martingale betting system, there are only two states. In this paper, we define a new martingale betting system, and in the new betting system, there are ten states. The frequency of each state's occurrence is the same. In this new betting system, when the gambler gathers all ten states, he will win the game. We find an interesting random variable in this new betting system and the coefficient of variation(CV) [11] of it approximates the value of **golden section ratio** [12] and the difference between these two numbers is only 0.000797.

Based on this similar Coupon Collector's Problem, we construct a new randomness test method, name the newfound test Traversal Sequence Test (TST) and calculate the theoretical distribution of this random variable, then use the chi-square test [13] which is of great importance in testing whether observed data fits a given probability distribution to decide the randomness of the sequences. At the same time, the proposed test is easy to be implemented, very fast and thus well suited for practical applications. We hope this test method could be a supplement of the other test methods.

In order to evaluate our test method, we evaluate the randomness of the binary and decimal expansions of e, π, $\sqrt{2}$, $\sqrt{3}$, log2 and random binary sequences from Matlab by our method and NIST test suite respectively. Compared with the reports of NIST tests, our method can also give right decision. At the same time, we also test some sequences which can not pass the NIST tests and our test gives the same decision as NIST does. In particular, we evaluate some sequences generated by WNG8 which is physical random number generator, and these sequences can pass NIST tests but can not pass our test. For these reasons, we hope our method can be a supplement of the present test suites.

The main contributions in our paper:

1. Compared with the existing test statistics, the coefficient of variation of the newfound test statistic based on Coupon Collector's Problem approximates the value of golden section ratio and the difference between these two numbers is only 0.000797.
2. Based on our newfound random variable, we proposed a new randomness test method, which can be a supplement of the present test suites.

2 The Design of Our Statistical Test

In the old martingale betting system, there are only two states. In our new martingale betting system, there are ten states, the frequency of each state's occurrence is the same. Consider the following situation: there is a long decimal sequence, such as 34985023667419087123904697530686 9$\cdots\cdots$, and we define a new random variable X, X is the number of the elements needed traversing from '0' to '9'. In this new betting system, when the gambler gathers all ten numbers, he will win the game. In the above string of decimal numbers, for the first time to finish traversing from '0' to '9', it needs this small segment '3498502366741', so $x_1 = 13$; for the second time to traverse from '0' to '9', it needs the next small segment '90871239046975', so $x_2 = 14$; and so on, we can get $x_3, x_4, \cdots, x_n, \cdots$.

Based on the random variable defined above, there comes a question:

What is the probability distribution of the random variable X?

In the old martingale betting system, the distribution of this random variable is obviously Geometric Distribution [14]. The next subsections will give the process that how to calculate the probability of the new random variable in our new martingale betting system.

2.1 The Preliminary to Calculate the Probability

There is a question for putting the balls into the boxes. In this question, we have ten boxes without serial numbers and k balls, and the probability for every ball into any a box is 1/10. When we put the k^{th} ball into one box, each box at least has one ball at that moment. In other word, when we put the $(k-1)^{th}$ ball into one box, only one of the boxes has no balls. So there is how many kinds of such combination.

Stirling number [15] can easily give the number of such combination. Stirling number S(P,K) denotes the number of the combination to put P elements into K nonempty sets. Therefore, with the help of Stirling number, the number of the combination for k balls is $\mathrm{S(k-1,9)}$.

2.2 To Calculate the Probability

If all the ten boxes have serial numbers from '0' to '9', what is the number of the combination that when the k^{th} ball is put into one box, every box has just at least one ball at that moment? Because of the Stirling number, the number of combination is $10! * S(k-1,9)$. Now this question that k balls are put into ten boxes is similar to the above question that k numbers are traversed from '0' to '9'. So, $\mathrm{P(X = k)} = 10! * S(k-1,9)/10^k = 9! * S(k-1,9)/10^{k-1}$. In order to calculate the probability for $k \geqslant 10$, substitute the Stirling number's formula into $\mathrm{P(X = k)}$. The Stirling number is calculated by the formula (1):

$$S(n,k) = (1/k!) * (\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n) \tag{1}$$

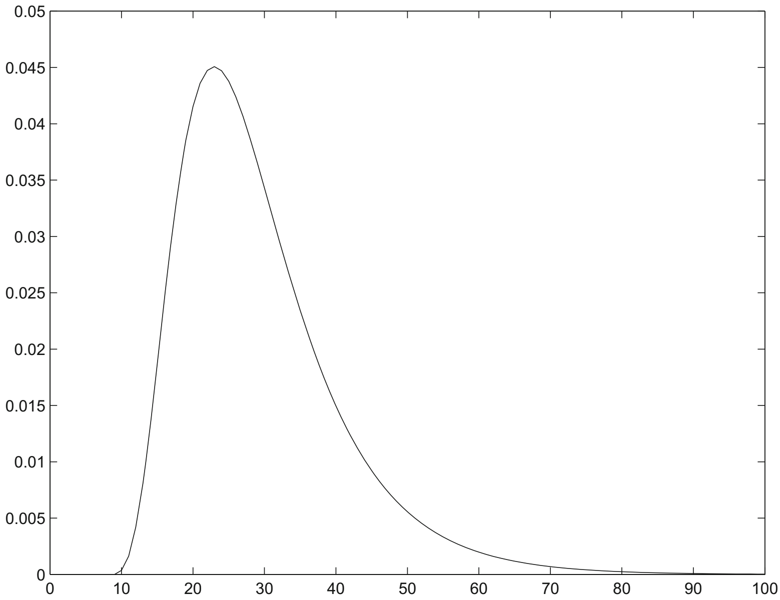

Fig. 1. The probability distribution for the traversal from '0' to '9'

Then we can calculate the P(X = k):

$$P(X = k) = 9! * S(k-1, 9)/10^{k-1} \quad (2)$$

By the formula (1) and (2),

$$P(X = k) = 9! * 1/9! * (\sum_{j=0}^{9}(-1)^{9-j}\binom{9}{j} * j^{k-1})/10^{k-1} \quad (3)$$

$$= (\sum_{j=0}^{9}(-1)^{9-j}\binom{9}{j} * j^{k-1})/10^{k-1} \quad (4)$$

Suppose i = 9 − j,

$$P(X = k) = \sum_{i=0}^{9}(-1)^{i}\binom{9}{i} * ((9-i)/10)^{k-1} \quad (5)$$

Through the formula (5), we can calculate the probability distribution and Fig. 1 shows the probability distribution for the traversal from '0' to '9'.

2.3 The Newfound CV Closes to Golden Section Ratio

The expectation of the random variable X which is the number of the elements needed traversing from '0' to '9' can be calculated by the formula (6):

$$E(X) = \sum_{k=10}^{\infty} k * \sum_{i=0}^{9} (-1)^i \binom{9}{i} * ((9-i)/10)^{k-1} \quad (6)$$

Although we get the probability distribution for the random variable X, we can not easily calculate the expectation and standard deviation of X. However, we can divide one traversal from '0' to '9' into ten parts. The first part is the first time to get one element a_0 from '0' to '9', and the probability P_0 for this event is 1. The second part is the first time to get one element a_1 which is not equal to a_0, and the probability P_1 for this event is 9/10. The same method for the other parts. The 10^{th} part is the first to get one element a_9 which is not equal to any element among $a_0, a_1, \cdots\cdots, a_8$, and the probability P_9 for this event is 1/10. Based on the above description, since these ten parts are independent geometric distribution, the expectation of each part is $E_i = 1/P_i$ and the variance is $\mathrm{Var}_i = (1 - P_i)/P_i^2$. According to the additivity of independent events' probability, we can calculate the E(X) and Var(X) by the formula (7)–(9):

$$P_i = (10 - i)/10 \quad (7)$$

$$E(X) = \sum_{i=0}^{9} E_i = \sum_{i=0}^{9} 1/P_i = 29.2897 \quad (8)$$

$$Var(X) = \sum_{i=0}^{9} Var_i = \sum_{i=0}^{9} (1 - P_i)/P_i^2 = 125.6871 \quad (9)$$

A random variable – the coefficient of variation [11] measures the variability of a series of numbers independently of the unit of measurement used for these numbers. The coefficient of variation eliminates the unit of measurement of the standard deviation of a series of numbers by dividing it by the mean of these numbers. In the above test for the traversal from '0' to '9', we can calculate the CV by the formula (10):

$$CV = \sigma(X)/E(X) = Var(X)^{1/2}/E(X) = 0.38276 \quad (10)$$

The golden section ratio is 0.38196601 and the computed CV of the traversal test for decimal numbers is 0.38276363. The difference between the golden section ratio and the computed CV is 0.00079762. It is attractive that the CV of the random variable is so close to the golden section ratio.

2.4 One Example for Traveral Test

Here we take π as an example and the test traverses from '0' to '9'.

$$\pi = 3.14159265358979323846264338327950288419716939\cdots\cdots$$

Begin the test with the part after the decimal point. It needs '14159265358979 323846264338327950' for the first time to finish one traveral from '0' to '9' and the number of decimal elements is 32. It needs '288419716939937510' for the second time to finish one traveral from '0' to '9' and the number of decimal numbers is 18. Then we finish 10000 traverals from '0' to '9' and we can get the statistical histogram for the number of decimal numbers needed to finish one traveral. Figure 2 shows the comparison of the theoretical probability distribution and the statistical probability distribution.

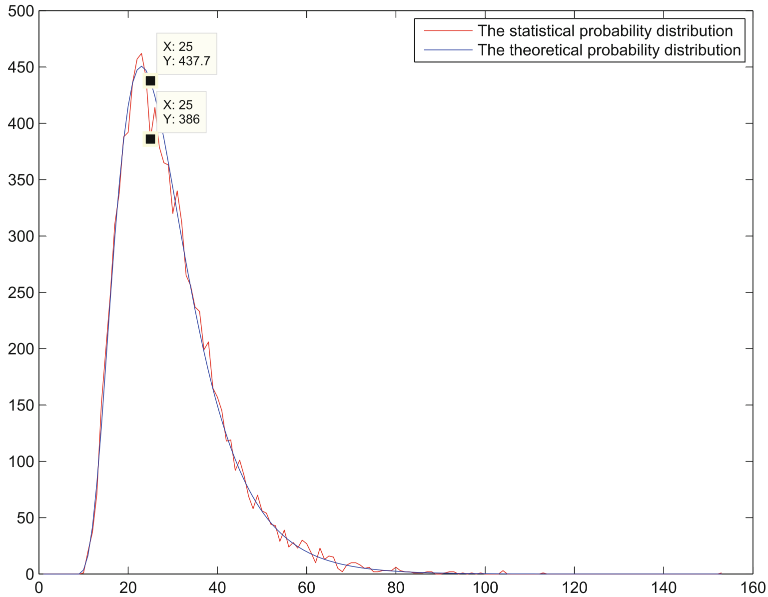

Fig. 2. The comparison of the theoretical probability distribution and the statistical probability distribution

Through the theoretical probability distribution and the statistical probability distribution, we can calculate the chi-square for the test to evaluate the randomness of the sequence as follows:

n denotes the number of traverals, pp_i denotes the probability that it needs i elements to finish one traversal and f_i denotes the number of times that it needs i elements to finish one traveral in the n traverals.

if $n * pp_i < 5$, we can calculate one part of the chi-square as follow:

$$\chi_1^2 = (\sum_{\{i|n*pp_i<5\}} pp_i * n - \sum_{\{i|n*pp_i<5\}} f_i)^2/(\sum_{\{i|n*pp_i<5\}} pp_i * n) = 0.8893 \quad (11)$$

if $n * pp_i \geq 5$, we can calculate the other part of the chi-square as follow:

$$\chi_2^2 = \sum_{\{i|n*pp_i\geq5\}} (pp_i * n - f_i)^2/(pp_i * n) = 67.2301 \quad (12)$$

Then calculate the chi-square χ^2 by adding up χ_1^2 and χ_2^2.

$$\chi^2 = \chi_1^2 + \chi_2^2 = 68.1194 \tag{13}$$

Then compute P_value = **igamc**(N/2, $\chi^2/2$) = **igamc**(64/2, 68.1194/2) = 0.3390. We select the significance level $\alpha = 0.01$. If the computed P_value is less than 0.01, then conclude that the sequence is non-random. Otherwise, conclude that the sequence is random.

2.5 Traversal Sequence Test

This section describes the procedure of the proposed test — Traversal Sequence Test (TST). In [3], A Statistical Test Suite for Random and Pseudorandom Number Generators for Cryptographic Applications (Revised: April 2010) consists of 15 tests and mentions that there are an infinite number of possible statistical tests and each of them is applied in a necessary condition for the randomness in probabilistic terms. Namely, no specific finite set of tests is deemed "complete". For example, among the NIST tests, the frequency (monibit) test focuses on the proportion of zeros and ones for the entire sequence; the frequency test within a block determines whether the number of ones is approximately m/2 in a m-bit block. In order to test the randomness of a sequence, these are necessary but not sufficient.

Here is a sequence n = 10000 as follows:

$$0, 1, 0, 1, 0, 1, 0, 1, \ldots, 0, 1, 0, 1$$

The sequence has 5000 ones and 5000 zeros. However, the sequence is obviously non-random, but the frequency test and the frequency within a block test (in which the length of the test block is even) would accept the sequence.

The focus of the TST is the proportion of the number of the needed element for one traversal. The purpose of this test is to determine whether the frequency of the traversal is similar with the theoretical probability distribution. The TST is suitable for many bases. For a bit sequence, the traversal can be from '0' to '2^m' and m can be $1, 2, \cdots, 7, 8, \cdots$. The TST also can be applied for the decimal sequence and the traversal is from '0' to '9'. The test process can be summarized as follows:

- **Step 1:** For a given bit sequence, the length is n and select the radix 2^m for the traversal from '0' to '2^m'.
- **Step 2:** Based on the radix selected in step 1, finish one thousand traversals and record the value of the random variable X which is the number of the needed elements for each traversal.
- **Step 3:** Analyze statistical result from step 2 to get the statistical distribution for the traversal test.
- **Step 4:** According to the statistical distribution of the random variable X, apply the chi-square test to compute P_value.
- **Step 5:** Decide the significance level α to determine whether to accept the sequence based on the P_value.

3 The Simulation Results

The proposed test (TST) was applied to different series of numbers (π, e, $\sqrt{2}$, $\sqrt{3}$, log2, some which are generated by WNG8, some which are generated by the random function of Matlab2012a, some which are the result of SHA-256, some other series which are concluded to be random by the NIST tests and some which are concluded to be non-random).

π, e, $\sqrt{2}$, $\sqrt{3}$, log2 and the sequences are generated by WNG8 are concluded to be random by the NIST tests. Table 1 shows that the result of the NIST tests for the binary expansion of π, e, $\sqrt{2}$. Meanwhile, WNG8 and the result of Hash-256 are also concluded to be random by the NIST tests. Sequence A is a bit sequence with the probability 48 % of '0' and 52 % of '1' and sequence B is a periodic sequence. Seq. A and Seq. B are both concluded to be non-random by the NIST tests.

3.1 The General TST Test

In the TST, based on the binary sequences of the above series, do traversal tests in different bases. Regard 1000 traversals as one test and do one thousand times in total for each base. Then record *P_value* which is computed in each test. If in any radix a sequence is concluded to be non-random, conclude this sequence non-random. Table 2 records the number of the computed *P_value* which is greater than 0.01 in 1000 tests for each radix, and it shows that the TST test can distinguish these random and non-random sequences well.

3.2 The TST Test for Simple Periodic Sequence

The traversal sequence test is based on the normality of the sequence. However, the TST test can distinguish simple-constructed periodic sequences. Generally, π is considered to be a random sequence, so construct the simple periodic sequences

Table 1. The result of NIST for the binary expansion of some entities

Statistical test	π	e	$\sqrt{2}$	WNG8	Hash-256	Seq.A	Seq.B
Frequency	1.000	0.989	0.989	0.993	0.986	0.000	0.921
BlockFrequency	0.989	1.000	0.989	0.997	0.989	0.000	0.968
CumulativeSums (forward)	0.989	0.989	0.989	0.993	0.985	0.000	0.928
CumulativeSums (backward)	1.000	0.989	1.000	0.990	0.987	0.000	0.928
Runs	0.978	1.000	1.000	0.987	0.987	0.000	0.926
LongestRuns	1.000	0.989	0.989	0.991	0.990	0.000	0.801
Rank	0.978	1.000	0.989	0.989	0.995	0.988	0.993
FFT	0.956	0.967	1.000	0.984	0.985	0.128	0.000
ApproximateEntropy	1.000	0.989	0.989	0.988	0.993	0.000	0.000
Serial (∇^1)	1.000	0.989	1.000	0.989	0.992	0.000	0.000
Serial (∇^2)	1.000	0.989	0.989	0.989	0.991	0.981	0.000

Table 2. The number of the computed *P_value* which is greater than 0.01

Statistical test	Radix = 2^1	Radix = 2^2	Radix = 2^3	Radix = 2^4
π	995	996	985	990
e	990	993	990	996
$\sqrt{2}$	993	992	990	990
$\sqrt{3}$	989	987	990	991
log2	993	997	995	993
One sequence from WNG8	999	994	991	991
One sequence from WNG8	993	986	992	992
Random sequence from Matlab	990	992	994	991
Random sequence from Matlab	997	994	992	991
One non-random sequence (Seq.A)	970	914	673	201

Table 3. The computed *P_value* of periodic sequences

Statistical test	The length of the sequence	Radix = 2^1	Radix = 2^2	Radix = 2^3	Radix = 2^4
The constructed sequence 1	512000	2.2328e-12	1.3609e-20	0.0044	1.4128e-41
Sequence 1 from π	512000	0.6862	0.9936	0.8445	0.9532
The constructed sequence 2	1024000	1.0888e-25	1.6914e-52	6.7067e-15	1.7901e-127
Sequence 2 from π	1024000	0.7721	0.8013	0.8117	0.8664
The constructed sequence 3	1536000	8.9484e-42	6.0224e-93	2.3053e-30	5.5521e-235
Sequence 3 from π	1536000	0.8546	0.8652	0.9048	0.2797

based on some segments of π. Consider the constructed sequence 1 consisting of 102400 random bit string which is one segment of π and repeated 5 times, the constructed sequence 2 consisting of 102400 random bits which are copied from π and repeated 10 times and the constructed sequence 3 consisting of 102400 random bit string which is one segment of π and repeated 15 times. Meanwhile, get three referential sequences from π, then apply the TST test to these sequences. Table 3 shows the computed *P_values* and the result indicates that these constructed sequence is non-random and the TST test can discover the periodicity in sequences.

3.3 The Large Sample TST Test

On the observation of the above TST tests, the sample size for one test is not so large. Here consider some large sample data and the sample data is 800 million

Table 4. The result of thest large sample data

Statistical test	Radix = 2^1	Radix = 2^2	Radix = 2^3	Radix = 2^4	Radix = 2^5	Radix = 2^6
π	0.5514	0.8637	0.1637	0.329	0.1402	0.9334
e	0.8007	0.5293	0.6132	0.4867	0.2054	0.3358
$\sqrt{2}$	0.9156	0.2361	0.5448	0.8475	0.5290	0.8608
$\sqrt{3}$	0.3393	0.7141	0.8947	0.6648	0.4416	0.1803
log2	0.6930	0.2128	0.9143	0.6373	0.3637	0.8688
One sequence from linear shift register generator	0.1928	4.8312e-11	0.1389	0.2490	0.0798	0.0652
Sequence 1 from WNG8	0.7607	0.5343	0.5387	9.6747e-06	8.7977e-08	0.9162
Sequence 2 from WNG8	0.1347	0.5297	0.9111	9.4943e-08	1.5921e-04	0.2702
Sequence 3 from WNG8	0.2674	0.9789	0.0941	0.0767	0.0070	0.8107
Sequence 4 from WNG8	0.5532	0.9311	0.0116	2.0995e-14	0.2988	4.5243e-04
Sequence 5 from WNG8	0.3597	0.4902	0.9273	5.5788e-06	0.4842	0.1365
Result 1 of SHA-256	0.4599	0.5631	0.0133	0.9423	0.5207	0.968
Result 2 of SHA-256	0.7785	0.3862	0.8919	0.8775	0.4998	0.1264
Sequence 1 from Matlab	0.5669	0.5542	0.0230	0.2175	0.9939	0.4087
Sequence 2 from Matlab	0.8980	0.4024	0.8892	0.7337	0.8179	0.7217
Sequence 3 from Matlab	0.3928	0.7429	0.5018	0.1946	0.8881	0.9147
Sequence 4 from Matlab	0.2219	0.6064	0.5295	0.8075	0.6041	0.3401
Sequence 5 from Matlab	0.7840	0.7269	0.6235	0.4041	0.7974	0.2971

bits. In this section, the large samples contain one sequence which is from linear shift register generator, five sequences which are generated by WNG8, two sequences which are the results of SHA-256, five sequences that are generated by the random function of Matlab2012a and other sequences that are from π, e, $\sqrt{2}, \sqrt{3}, \log 2$.

Table 4 records the computed *P_value* of these sequences in each base. From the result of Table 4, when the sample size is large, π, e, $\sqrt{2}, \sqrt{3}, \log 2$ are still concluded to be random, the result of SHA-256 and the sequences generated by matlab are also concluded to be random. However, to the sequence from

Table 5. The computed *P_value* for the random variable CV

Name	π	e	$\sqrt{2}$	$\sqrt{3}$
P_value	0.5952	0.8715	0.8858	0.6784
Name	Periodic sequence 1	Periodic sequence 2	Periodic sequence 3	Periodic sequence 4
P_value	4.3679e-90	4.1967e-20	1.9650e-04	1.9788e-05

WNG8 in some radix the sequence is non-random, so the large sample TST test can distinguish some non-randomness of the sequences from WNG8. To the sequence from linear shift register generator, in the radix of 2^2 it is concluded to be non-random.

3.4 The Simulation of the Random Variable CV

In the above section, we notice that the theoretical value of random variable CV for decimal sequences is extremely close to the Golden Section Ratio. Then in this section, we will apply the T test [16] to distinguish sequences' randomness. Here these sequences are decimal and the length of each test block is 90000. For each sequence, calculate 100 CVs. The calculated CV is conformed to normal distribution with the mean is the theoretical value of CV. We can assume that the calculated CVs have the mean which is the theoretical value of CV and the variance δ^2 which is unknown. Then calculate the statistical variable t by the formula (14), where $\overline{X}$ is the mean of the calculated CVs, CV_0 is the theoretical value of CV, S is the standard deviation of these CVs and n is the number of these CVs.

$$t = (\overline{X} - CV_0)/(S/\sqrt{n}) \sim t(n-1) \tag{14}$$

Then through the computed t, make use of the **ttest** function in Matlab to get the *P_value*. Table 5 shows that the *P_value* of the decimal sequences for π, e, $\sqrt{2}$, $\sqrt{3}$ and some periodic sequences whose periods are 48000, 120000, 240000, 480000. The result indicates that the random variable CV can efficiently distinguish the non-random sequences.

4 Conclusion

In this paper, we propose a new randomness test method. First, we calculate the probability distribution for the number to traversal the binary sequence from different bases. Then apply chi-square test to evaluate the randomness of the binary sequences. An amazing discovery of this paper is that we find that the Coefficient of Variation of the test statistic defined in this paper approaches the value of golden section ratio and the difference between these two number is only about 0.000797. As the test result shown, our new test method can find that some physical random number generator is not so good as the pseudorandom, such as the binary expansions of e, π, $\sqrt{2}$, $\sqrt{3}$ etc. We hope that our new test method can be a supplement of the existing test suites.

References

1. Doganaksoy, A., Calık, C., Sulak, F., Turan, M.S.: New randomness tests using random walk (2006)
2. Hamano, K., Yamamoto, H.: A randomness test based on t-codes, pp. 1–6 (2008)
3. Rukhin, A.L., et al.: Approximate entropy for testing randomness. J. Appl. Probab. **37**(1), 88–100 (2000)
4. Maurer, U.M.: A universal statistical test for random bit generators. J. Cryptology **5**(2), 89–105 (1992)
5. Katos, V.: A randomness test for block ciphers. Appl. Math. Comput. **162**(1), 29–35 (2005)
6. Lenstra, A.K., Hughes, J.P., Augier, M., Bos, J.W., Kleinjung, T., Wachter, C.: Ron was wrong, Whit is right. IACR Cryptology ePrint Archieve **2012**, 64 (2012)
7. Alcover, P.M., Guillamon, A., del Carmen Ruiz, M.: A new randomness test for bit sequences. Informatica **24**(3), 339–356 (2013)
8. Rukhin, A., Soto, J., Nechvatal, J., Smid, M., Barker, E.: A statistical test suite for random and pseudorandom number generators for cryptographic applications (2001)
9. Soto, J.: Statistical testing of random number generators **10**(99), 12 (1999)
10. Grinstead, C.M., Snell, J.L.: Introduction to Probability. American Mathematical Society, Providence (1998)
11. Abdi, H.: Coefficient of variation. In: Salkind, N. (ed.) Encyclopedia of Research Design, pp. 169–171. SAGE Publications Inc., Thousand Oaks (2010)
12. Svensson, L.T.: Note on the golden section. Scand. J. Psychol. **18**(1), 79–80 (1977)
13. Moore, D.S.: Chi-square tests (1976)
14. Philippou, A.N., Georghiou, C., Philippou, G.N.: A generalized geometric distribution and some of its properties. Stat. Probab. Lett. **1**(4), 171–175 (1983)
15. Loeb, D.E.: A generalization of the stirling numbers. Discrete Math. **103**(3), 259–269 (1992)
16. William, L.T.: Null hypothesis testing: problems, prevalence, and an alternative. J. Wildl. Manage. **64**(4), 912–923 (2000)