# Impact of the Railway Centerline Geometry Uncertainties on the Train Velocity Estimation by GPS

Guoliang Zhu[1,⋆], Lionel Fillatre[2], and Igor Nikiforov[1]

[1] ICD - LM2S - UTT, UMR STMR - CNRS, Troyes, France
{guoliang.zhu,igor.nikiforov}@utt.fr
[2] Laboratoire I3S, UMR7271 - UNS CNRS, Sophia-Antipolis, France
lionel.fillatre@i3s.unice.fr

**Abstract.** The railway centerline is defined by a polygonal line with some level of uncertainty in the train onboard database. The goal of this paper is to estimate the train speed by GPS and to study the impact of railway centerline uncertainty on the speed estimation. The equations for first two moments of the estimated speed are obtained and compared with the results of Monte-Carlo simulations.

**Keywords:** GPS train positioning, Train track model, Least square method, Model nonlinearity.

## 1 Introduction and Motivation

The estimation of train speed and distance to target plays an important role in the management of modern railways. For safe and efficient railway operations, these parameters should be estimated with a high level of accuracy [1,2] and integrity [3].

This paper is devoted to the GPS train positioning by using a low-cost receiver. Two limit cases of the train speed estimation can be considered. The first approach is based on the exactly known three-dimensional train-track model (this is so-called one-dimensional navigation). The second approach does not use any information about the train track (hence, it is the classical two or three-dimensional navigation). The first case provides the user with the best precision but it is unrealistic because the exact three-dimensional train-track model needs an enormous effort of geodesic measuring and the onboard train database preparation. The second case is rather pessimistic, some information about the train track is always available, at least by using electronic maps for large public. A crucially important question is the impact of such a map imprecision on the estimation of train speed and distance to target.

This paper is organized as follows. Section 2 is devoted to the problem statement. Section 3 provides the geometric model of railway track, the train dynamical model and the method of train speed estimation. The impact of train-track model imprecision on the speed estimation is discussed in section 4. Simulation results are shown in section 5. Finally, some conclusions are drawn in section 6.

⋆ The authors would like to thank the China Scholarship Council and the University of Technology of Troyes (UTT), France, for supporting their research.

R. Dhaou et al. (Eds.): PSATS 2013, LNICST 123, pp. 156–161, 2013.

## 2 Problem Statement and Contribution

Let us consider that the train runs along the track with a constant (unknown) speed. The goal of this paper is twofold : first, to calculate the train speed by using GPS and an imprecise geometric model of the railway centerline; second, to estimate a negative impact of the railway centerline uncertainty on the mean error and on the second order moment of the estimated speed. The equations for first two moments of the estimated speed are obtained and compared with the results of Monte-Carlo simulations.

## 3 Description of Models

### 3.1 Train Track Model

Let us assume that the railway centerline is approximated by a polygonal line (piecewise linear curve), which represents a connected series of line segments in the Earth-centered, Earth-fixed coordinates. More formally, the railway centerline is defined by a sequence of vertices $Z_0, Z_1, Z_2, \ldots, Z_n$, $Z_i \in \mathbb{R}^3$, so that the curve consists of the line segments connecting the consecutive vertices. It is assumed that the errors related with such an approximation of the vector function $\ell \mapsto X(\ell)$, $\ell \in \mathbb{R}$, $X \in \mathbb{R}^3$, defining the railway centerline is negligible for our study. Here and in the rest of the paper, $\ell$ denotes the curvilinear abscissa, or the covered distance, and $m = \|Z_{j+1} - Z_j\|_2 = \text{const}$ is the distance between two adjacent vertices, respectively. Unfortunately, the on-board database uses an

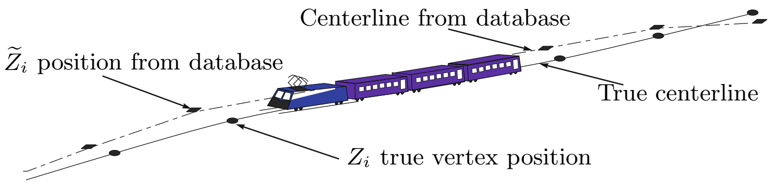


**Fig. 1.** Train track model

imprecise information about the positions of vertices, namely : $\widetilde{Z}_0, \widetilde{Z}_1, \widetilde{Z}_2, \ldots, \widetilde{Z}_n$. The quantity $\xi_i = Z_i - \widetilde{Z}_i$ defines the knowledge uncertainty concerning the train track. This situation is illustrated by Fig. 1. To simplify the presentation, a two-dimensional train trajectory is considered.

### 3.2 Train Dynamical Model

Traditionally, the train dynamical model is described by an equation formulated in term of the covered distance, speed and acceleration. To simplify the estimation problem, it is assumed for this study that the acceleration is negligible for some short periods. Let us suppose that the train runs along the above railway track with an unknown constant speed $v$. Hence, the true train position is defined as follows: $X_k = X_{k-1} + A_{j(k)} \cdot v \cdot \Delta t$, $k = 1, 2, \ldots$, where $X_k = (x_k, y_k, z_k)^T$ is the train position at the $k$-th GPS measurement (GPS epoch), $t_k$ denotes the instant of the $k$-th measurement, $\Delta t = t_k - t_{k-1}$ represents the GPS sampling interval, $A_j = (a_x^j, a_y^j, a_z^j)^T = \dfrac{1}{m}(Z_{j+1} - Z_j)$ is the directional vector corresponding to the segment number $j$, $\|A_j\|_2 = 1$. The current segment number $j = j(k)$ is calculated as a function of $k$ by using the following equation

$$j(k) = \min \{ j \in \mathbb{N} | j \geq \lfloor (v \cdot \Delta t \cdot k)/m \rfloor \}, \tag{1}$$

where $\mathbb{N}$ is the set of natural numbers. The train position $X_k$ can be rewritten as

$$X_k = X_0 + v\Delta t \sum_{t=1}^{k} A_{j(t)} \tag{2}$$

where $X_0 = (x_0, y_0, z_0)^T$ is the starting point.

### 3.3 Exact and Imprecise Pseudo-range Measurement Model

Suppose that there are $n$ satellites located at the known positions $X_i^s = (x_i, y_i, z_i)^T$, $i = 1, \ldots, n$. The pseudo-range $r_i$ from the satellite $i$ to the train can be written as:

$$r_i^k = d_i^k + cb_r^k + \varepsilon_i^k = \left\| X_0 + v\Delta t \sum_{t=1}^{k} A_{j(t)} - X_i^s \right\|_2 + cb_r^k + \varepsilon_i^k, \quad \varepsilon_i^k \sim \mathcal{N}(0, \sigma^2),$$

where $b_r^k$ is a user clock bias, $c \simeq 2.9979 \cdot 10^8 m/s$ is the speed of light and $\varepsilon_i^k$ is a pseudo-range noise. By linearizing the pseudo-range equation around the working point $V_0 = (v_0, cb_0)^T$, we get

$$r_i^k - r_{i,0}^k \simeq h_{i,0}^k (v - v_0) + c(b_r^k - b_0) + \varepsilon_i^k, \quad i = 1, \ldots, n, \tag{3}$$

where $r_{i0}^k = d_{i0}^k + cb_0$, $d_{i0}^k = \left\| X_0 + \left( \sum_{t=1}^{k} A_{\widehat{\jmath}(t)} \right) \cdot v_0 \Delta t - X_i^s \right\|_2$ and

$$h_{i0}^k = \frac{1}{d_{i0}^k} \left[ X_0 + \left( \sum_{t=1}^{k} A_{\widehat{\jmath}(t)} \right) \cdot v_0 \Delta t - X_i^s \right]^T \left( \sum_{t=1}^{k} A_{\widehat{\jmath}(t)} \right) \Delta t.$$

Because the true train speed $v$ is unknown, the current segment number $\widehat{\jmath} = \widehat{\jmath}(t)$ is calculated as a function of the previously calculated speed $\widehat{v}_t$ by using (1) with $v = \widehat{v}_t$. The above mentioned linearized measurement equation (3) can be rewritten in the following matrix form

$$R^k - R_0^k \simeq H_0^k \cdot (V_k - V_0) + \Xi^k, \tag{4}$$

where $V_k = (v, cb_r^k)^T$ and the working point at step $k$ is equal to the previously calculated estimation : $V_0 = \widehat{V}_{k-1}$.

Let us discuss now an unprecise measurement model. Since the true vertex position $Z_j$ is unknown and only its imprecise estimation $\widetilde{Z}_j$ is available, the linearized measurement equation (4) cannot be used to compute the train speed. To estimate the impact of this uncertainty, let us define the directional vector $\widetilde{A}_j = A_j + \delta_j$, where the random vector $\delta_j = (\delta_x^j, \delta_y^j, \delta_z^j)^T$ is assumed to be uniformly distributed in the cube $[-b, b]^3$ with $b > 0$. Finally, the pseudo-range measurement model (4) is defined for the imprecise directional vectors $\widetilde{A}_j$ in the following manner

$$R^k - \widetilde{R}_0^k \simeq \widetilde{H}_0^k \cdot (V_k - V_0) + \Xi^k \tag{5}$$

where $\widetilde{R}_0^k$ and $\widetilde{H}_0^k$ are calculated exactly as in equation (3) but with the vector $\widetilde{A}_j$ instead of $A_j$.

## 4 The Impact of Track Uncertainty on the LS Estimator

The goal of this section is to study the impact of the train track uncertainty $\delta_j$ on the first and second moments of the least square (LS) estimator $\widehat{v}_k$. To seek simplicity, let us assume that the track entirely belongs to the local tangent plane. We follow here the analysis of the regression model uncertainties and their impact on the LS estimators developed in [4]. First, the measurement equation (5) can be rewritten as follows:

$$Y^k + \Delta Y^k \simeq (H_0^k + \Delta H^k) \cdot \beta_k + \Xi^k \tag{6}$$

where $Y^k = R^k - R_0^k$, $\Delta Y^k = R_0^k - \widetilde{R}_0^k$, $\Delta H^k = \widetilde{H}_0^k - H_0^k$ and $\beta_k = V_k - V_0$. It is assumed that the second column of $\Delta H^k$ is equal to zero because the impact on the clock bias estimation is of no interest for this study. The LS estimator is given by

$$\widehat{\beta}_k = \left[ \left( H_0^k + \Delta H^k \right)^T \left( H_0^k + \Delta H^k \right) \right]^{-1} \left( H_0^k + \Delta H^k \right)^T \left( Y^k + \Delta Y^k \right). \tag{7}$$

After expanding $\left[ \left( H_0^k + \Delta H^k \right)^T \left( H_0^k + \Delta H^k \right) \right]^{-1}$ around $H_0^k$ (see appendix of [4]) and computing the expectation of equation (7), the mean error is

$$\mathbb{E}(\widehat{V}_k - V) = B_0^{-1} \left[ (H_0^k)^T \Sigma_H C - F + G \right] \beta \tag{8}$$

where $\Sigma_H$ denotes the covariance matrix of $\Delta H^k$, $F = \begin{pmatrix} \mathrm{tr}(\Sigma_H) & 0 \\ 0 & 0 \end{pmatrix}$, $G = \begin{pmatrix} \mathrm{tr}\left[ H_0^k B_0^{-1} (H_0^k)^T \Sigma_H \right] & 0 \\ 0 & 0 \end{pmatrix}$, $B_0 = \left( H_0^k \right)^T H_0^k$, the first column of a $(n \times 2)$ matrix $C$ is equal to the first column of $H_0^k B_0^{-T}$ and its second column is equal to zero.

Since the random vector $\Delta Y^k$ acts in the same way as the pseudo-range noise $\Xi^k$, the two errors can be considered together. After expanding and ignoring the terms of order $\left(\Delta H^k\right)^2$ and under the assumption that the errors $\Delta H^k$ are reasonably small, the second order moment of $\widehat{V}_k - V$ is given by

$$\mathbb{E}(\widehat{V}_k - V)(\widehat{V}_k - V)^T = B_0^{-1}(H_0^k)^T\left[\sigma^2 I_n + \Sigma_Y - \beta_1(\Sigma_{HY} + \Sigma_{YH}) + \beta_1^2\Sigma_H\right]H_0^k B_0^{-1}; \quad (9)$$

where $\beta_1 = v - v_0$, $\Sigma_Y$ denotes the covariance matrix of $\Delta Y^k$, $\Sigma_{HY} = \mathbb{E}\left[\Delta H^k(\Delta Y^k)^T\right]$ and $\Sigma_{YH} = \Sigma_{HY}^T$. When the expectation (8) of $\widehat{V}_k - V$ is almost zero, this second order moment corresponds to the variance of $\widehat{V}_k$.

## 5 Numerical Simulations

The comparison of the theoretical mean error and second order moment given by (8) and (9), respectively, with the results of a $10^4$-repetition Monte-Carlo simulation, is shown in Fig. 2-7. The standard GPS constellation has been used with $n = 6$ visible satellites and $\sigma = 2$ (m). The distance between two adjacent vertices has been chosen $m = 50$ (m). Different values of railway centerline

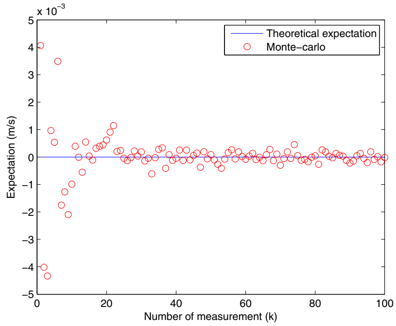


**Fig. 2.** The estimated speed mean error for $\delta_j = 0$

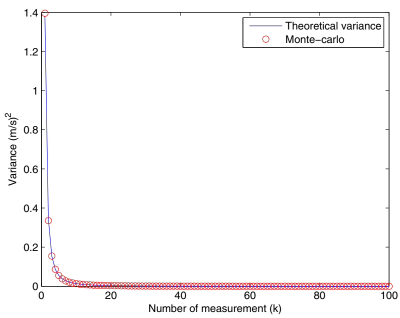


**Fig. 3.** The estimated speed second order moment for $\delta_j = 0$

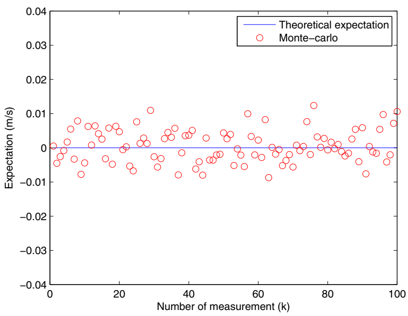


**Fig. 4.** The estimated speed mean error for $\delta_j \in [-0.01, 0.01]^2$

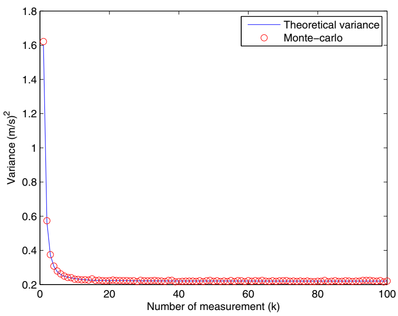


**Fig. 5.** The estimated speed second order moment for $\delta_j \in [-0.01, 0.01]^2$

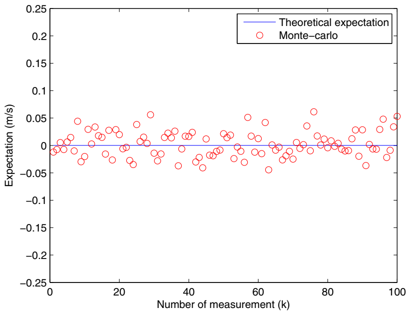


**Fig. 6.** The estimated speed mean error for $\delta_j \in [-0.05, 0.05]^2$

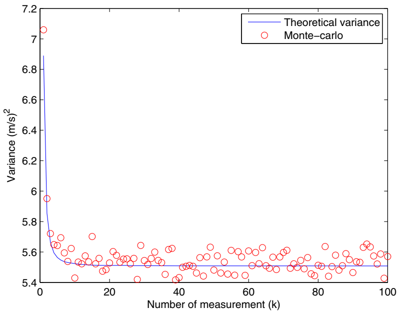


**Fig. 7.** The estimated speed second order moment for $\delta_j \in [-0.05, 0.05]^2$

uncertainty have been tested : $b = 0$ (no uncertainty); $b = 0.01$ (uncertainty $\simeq \pm 0.5$ (m)); $b = 0.05$ (uncertainty $\simeq \pm 2.5$ (m)).

## 6 Conclusions

The comparison of the train speed estimation by GPS with an imprecise geometric model of the railway centerline and by using odometric measurements [1,2] shows that the second order moments of the estimated speed are comparable but the speed mean error obtained by GPS is better. It is practically unbiased, even with an imprecise geometric model of the railway centerline. A hybrid estimation connecting the odometer, accelerometer and GPS measurements seems to be very promising for the train speed estimation.

## References


1. Malvezzi, M., Allotta, B., Rinchi, M., Bruzzo, M., De Bernardi, P.: Odometric Estimation for Automatic Train Protection and Control Systems. In: 8th World Congress on Railway Research, Korea (2008)
2. Malvezzi, M., Allotta, B., Rinchi, M.: Odometric Estimation for Automatic Train Protection and Control Systems. Vehicle System Dynamics 49(5), 723–739 (2011)
3. Nikiforov, I.V., Choquette, F.: Integrity Equations for Safe Train Positioning Using GNSS. Istituto Italiano di Navigazione 171, 52–77 (2003)
4. Hodges, S.D., Moore, P.G.: Data Uncertainties and Least Squares Regression. JRSS, Series C (Applied Statistics) 21, 185–195 (1972)